ALMOST SURE CONVERGENCE OF THE CLUSTERING FACTOR IN α-MIXING PROCESSES

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Abstract. Abadi and Saussol (2011) have proved that the first time a dynamical system, starting from its equilibrium measure, hits a target set A has approximately an exponential law. These results holds for systems satisfying the α-mixing condition with rate function α decreasing to zero at any rate. The parameter of the exponential law is the product $\lambda(A)\mu(A)$, where the later is the measure of the set A; only bounds for $\lambda(A)$ were given. In this note we prove that, if the rate function α decreases algebraically and if the target set is a sequence of nested cylinders sets $A_n(x)$ around a point x, then $\lambda(A_n)$ converges to one for almost every point x. As a byproduct, we obtain the corresponding result for return times.

1. Introduction

In the statistical analysis of Poincaré's Recurrence, a major effort was done in the last two decades to determine the behavior of several quantities in the evolution of a stochastic processes and/or measure preserving dynamical system. For instance, the time τ_A elapsed until a fixed target set A is observed, called the hitting time, the above quantity but observed in the restricted space A, called return time and the subsequent returns, the return time to the first now randomly generated set, the number of visits to both, a fixed or random set in a given interval. For references of results and methodologies we mention the three review papers [7, 3, 13] and references therein.

This paper is the natural complement of the author's previous one [4], in which the problem of the approximation of the hitting time distribution of a general sequence of target sets $A_n, n \in \mathbb{N}$ was addressed. The technique used there was the one introduced by Galves and Schmitt in [12]. This technique provides results that apply to limiting exponential law for hitting time distribution along sequence of cylinder sets nested around *every* point of the phase space, in contrast with other techniques that allow to prove results which hold only for *almost* every point.

At this level of generality the parameter of the exponential law must be re-scaled not just by the measure of the target set $\mu(A)$, but an extra factor of existentialist kind $\lambda(A)$, related to some specific quantile of the hitting time distribution, must be added. That is

$$\mu(\tau_A > t) \approx e^{-\lambda(A)\mu(A)t} \ .$$

The motivation for these notes is to shed some light on the behavior of the re-scaling factor $\lambda(A)$. In our previous paper only some bounds for $\lambda(A)$ where given. Here we prove that this parameter converges to one almost surely for α-mixing

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systems with rate function α decreasing to zero, at least, at an algebraic rate. Our results holds for sequences of cylinder sets nested around a given point x. This is done in section 2.

We recall the reader's attention to a recent paper by Abadi, Cardeño and Gallo [2] where it is described the spectrum of values of this parameter in the whole phase space for positive recurrent renewal processes. The authors prove that, in that case, the parameter $\lambda(A)$ represents the probability the process has to escape the set A soonly after having entered A. This was first proved for particular n-cylinders in exponential α-mixing systems in [1] and later on for ϕ-mixing systems in [5]. Specifically $\lambda(A)$ is determined by the first possible return to A, which for n-cylinders, is typically of order n [6, 16] and of course, for cylinders around a periodic point, is given by the period. This makes easy to compute in practice $\lambda(A)$, since it depends on very short values of the time scale. It is not known if this is possible in more general settings like the α-mixing case.

What can be interpreted about $\lambda(A)$ is that it is a clustering factor. It determines if, once the process hits the set A, if either this hit will appear isolated or if there will be a cluster or clumps[1] of occurrences of A. This can be deduced by the result by Haydn et al. [14] and then extended by Abadi and Saussol in [4] where the return time distribution is determined by the hitting time one (or vice-versa). Thus, under the α-mixing condition and applying the result in the later article we get that the return time distribution verifies

$$\mu_A(\tau_A > t) \approx \lambda(A)\ e^{-\lambda(A)\mu(A)t}\ .$$

Thus, we also show here that for almost every point, the return time distribution actually behaves typically like $e^{-\mu(A)t}$.

To bring also some more information in the case of non typical cylinders (in the sense of Ornstein-Weiss's theorem), we prove also a number of general properties for $\lambda(A)$. In section 2 we give to $\lambda(A)$ not an existentialistic expression (as was done in our previous paper [4]) but rather a constructivist one. It shows that $\lambda(A)$ may depend on values of the time scale smaller than the inverse of the measure of A but they could be also very large, which may render its explicit computation difficult. Therefore, some more information is needed.

In Section 3 we show that, given an escape time m (not so large), $\lambda(A)$ has the same order than the escape probability $\mu_A(\tau_A > m)$ times $\lambda(A')$, where A' is the portion of A that does not return before the escape time. Namely $A' = \{A \cap \tau_A > m\}$. This automatically says that $\lambda(A)$ is asymptotically bounded by $\mu_A(\tau_A > m)$. This is of particular interest for using with a small value of m. In particular one may take m equal to the period of a periodic point x when A is the n-cylinder around it, say $T(A)$. Thus, we generalize the results on [1, 5, 2] where under more restrictive conditions, it was proved that $|\lambda(A_n) - \mu_A(\tau_A > T(A))| \to 0$ for $A = A_n$, n-cylinder nested around a point x, for *any* x. This shows that in particular that our upper bound is sharp.

Part of the argument of this result relies on a fact that we also prove which is interesting by itself: the hitting time distributions of A and A' are close to each other whenever m is not so large. This idea already appeared in the context of extreme

[1]In the literature, the words clustering and clumping are both used to refer to observations that appear in groups. Cluster is more used in the extreme Value Theory. Clump is used in Poisson analysis of such observations. See for instance Aldous, D. (1989.) "Probability Approximations via the Poisson Clumping Heuristic", Applied Mathematical Sciences, 7, Springer.

value theory in [9] (see Proposition 1 and Theorem 1) and in [11] (see Proposition 2.7), where the extremal index is shown to characterize also the clustering of maxima in a stochastic process.

We recall that the connection between hitting/return time laws and Extreme Value Theory was already well studied. Besides the two aforementioned papers, [8] establishes some links between the two subjects in the context of non-uniformly hyperbolic systems in the case of generic points. Furthermore, [10] does the same in the case of periodic points, considering the extremal index as a quantification of the clustering properties around a periodic point the process has. So that the extremal index plays a similar role to the one that the the clustering factor plays in our case.

We also prove a lower an upper bound for $\lambda(A)$. Following [2] we show that it can be well approximated by the arithmetic mean of escape time probabilities. These escape times range between two appropriate explicit scales. Since the escape rate are monotonic on the scale values, we obtain upper and lower bounds as functions of the endpoint of this scale. Typically, the smaller endpoint is larger or equal than the first possible return to A and the bigger one is much smaller that $\mu(A)^{-1}$.

In Section 4 we consider systems with an appropriate decay of the mixing function α. Under this condition, we show what is the precise scale that must be taken for m in order to have $\lambda(A)$ replaced by the escape probability.

2. Result

Let $\mathcal{A}$ be a finite or countable set and let $\Sigma = \mathcal{A}^{\mathbb{N}}$ be the set of sequences. We endow Σ with the shift map T. Given non negative integers $m \leq n$ and a point $x \in \Sigma$ we denote by $[x_m \dots x_n]$ the cylinder of rank (m, n) containing x, that is

$$[x_m \dots x_n] := \{y \in \Sigma \colon y_m = x_m, \dots, y_n = x_n\}.$$

$A_n^x = [x_1 \dots x_n]$ denotes the n-cylinder about x. We denote by $\mathcal{F}_m^n$ the σ-algebra generated by the collection of cylinders of rank (m, n). Let $\mathcal{F}$ be the σ-algebra generated by the $\mathcal{F}_m^n$'s and μ be a T-invariant probability measure on $(\Sigma, \mathcal{F})$. Let

$$\alpha(g) = \sup_{m,n} \sup_{A \in \mathcal{F}_0^n, B \in \mathcal{F}_{n+g}^{m+g}} |\mu(A \cap B) - \mu(A)\mu(B)|$$

for any integer g. We assume that the system (Σ, T, μ) is α-mixing, that is $\alpha(g) \to 0$ as $g \to \infty$. We say that the rate of α-mixing is algebraic if for some $p > 0$ one has

$$\lim_{n \to \infty} \alpha(n) n^p = 0 \ . \tag{1}$$

Let $A \in \Sigma$ be a measurable set. We define the hitting time to A by

$$\tau_A(x) = \inf\{k \geq 1 \colon T^k x \in A\}, \quad x \in \Sigma.$$

We are interested in the distribution of the hitting time τ_A on the probability space (Σ, μ), and the return time, defined with the same formula but on the probability space (A, μ_A) where μ_A denotes the conditional measure on A.

We recall the following result valid for any α-mixing system

Theorem 1 (Theorem 1 and Example 3 in [4])**.** *For any sequence $A_n \in \mathcal{F}_0^n$ such that $\mu(A_n) > 0$ and $\mu(\tau_{A_n} \leq n) \to 0$ there exists a normalizing constant $\lambda(A_n)$ such that*

$$\sup_{k \in \mathbb{N}} |\mu(\tau_{A_n} > k) - \exp(-k\lambda(A_n)\mu(A_n))| \to 0 \tag{2}$$

as $n \to \infty$. Specializing to cylinders we get that for any $x \in X$

$$\sup_{k\in\mathbb{N}} \left|\mu(\tau_{A_n^x} > k) - \exp(-k\lambda(A_n^x)\mu(A_n^x))\right| \to 0. \tag{3}$$

Theorem 2. *If the rate of α-mixing is algebraic the exponential law for return and hitting time holds, with parameter one, for typical cylinders.*

Proof. Indeed we prove that for μ-a.e. x the normalizing constant $\lambda(A_n^x)$ in Theorem 1 can be chosen so that it converges to one as $n \to \infty$. To do this, instead of using the parameter s given by [4, Lemma 10], we define it constructively and then do the proof of [4, Theorem 8] with it.

Let h be the entropy of the process. Let $p > 0$ be such that (1) holds.

Take $a < h$ such that $(1+p)a > h$ and choose $\epsilon > 0$ so small that $a + \epsilon < h$ and $-(1+p)a + h + (3p+1)\epsilon < 0$. Let $b = a - 3\epsilon$.

Let us fix some x such that the (7) in Lemma 4 holds and such that for n sufficiently large

$$e^{-n(h+\epsilon)} \le \mu(A_n^x) \le e^{-n(h-\epsilon)}.$$

Let $N_n = \lfloor e^{nb} \rfloor$ and $S_n = \lfloor e^{na} \rfloor$. We claim that as $n \to \infty$,

$$\mu(\tau_{A_n^x} \le S_n) \to 0 \quad \text{and} \quad \frac{\mu(\tau_{A_n^x} \le 2N_n) + \alpha(N_n)}{\mu(\tau_{A_n^x} \le S_n - 2N_n)} \to 0. \tag{4}$$

The first statement is simply because by invariance one has

$$\mu(\tau_{A_n^x} \le S_n) \le S_n \mu(A_n^x) \le e^{na} e^{-n(h-\epsilon)} \to 0.$$

For the second statement, the numerator is bounded above by

$$\mu(\tau_{A_n^x} \le 2N_n) + \alpha(N_n) \le 2e^{bn} e^{-n(h-\epsilon)} + (e^{bn} - 1)^{-p}$$

while for the denominator is bounded from below by

$$\begin{aligned}
\mu(\tau_{A_n^x} \le S_n - 2N_n) &= \sum_{j=1}^{S_n - 2N_n} \mu(A_n^x)\mu_{A_n^x}(\tau_{A_n^x} \ge j) \\
&\ge (S_n - 2N_n)\mu(A_n^x)\mu_{A_n^x}(\tau_{A_n^x} \ge S_n - 2N_n) \\
&\ge \frac{1}{2}(e^{an} - 2e^{bn})e^{-n(h+\epsilon)}
\end{aligned}$$

where in the last expression we used (7).

Following the proof of [4, Theorem 8], with s replaced by S_n, n replaced by N_n (observe that $A_n^x \in \mathcal{F}_0^{N_n - 1}$ since $N_n \ge n$) and the constant $\lambda(A_n^x)$ defined accordingly by

$$\lambda(A_n^x) = -\frac{\log \mu(\tau_{A_n^x} \ge S_n - 2N_n)}{S_n \mu(A_n^x)}$$

we get that

$$\sup_{k\in\mathbb{N}} \left|\mu(\tau_{A_n^x} > k) - e^{-\lambda(A_n^x)\mu(A_n^x)k}\right| \to 0 \tag{5}$$

as $n \to \infty$.

It remains to show that $\lambda(A_n^x) \to 1$ as $n \to \infty$. Since $\mu(\tau_{A_n^x} \le S_n - 2N_n) \to 0$ it suffices to show that

$$\frac{\mu(\tau_{A_n^x} \le S_n - 2N_n)}{S_n \mu(A_n^x)} \to 1.$$

Using previous computations we get the double inequality

$$\mu_{A_n^x}(\tau_{A_n^x} \geq S_n - 2N_n)\frac{S_n - 2N_n}{S_n} \leq \frac{\mu(\tau_{A_n^x} \leq S_n - 2N_n)}{S_n \mu(A_n^x)} \leq \frac{S_n - 2N_n}{S_n} \tag{6}$$

and the conclusion follows by our choice of S_n and N_n and (7). □

Remark 3. *The speed of convergence to zero in* (5) *is exponentially small in n. Indeed, it is directly related to the speed of convergence in* (4) *(See the proof of* [4, Theorem 8]*) which is clearly exponential.*

Recall that $h > 0$ is the entropy of the system.

Lemma 4. *Let $a < h$. For μ-a.e. x we have*

$$\lim_{n\to\infty} \mu_{A_n^x}(\tau_{A_n^x} \leq e^{an}) = 0. \tag{7}$$

Proof. By Ornstein-Weiss theorem about the repetition time R_n, we have

$$\lim_{n\to\infty} \frac{1}{n} \log R_n = h \quad \mu - a.e.$$

Consider for some integer n_0 the set $Y(n_0) = \{x \in X : \forall n \geq n_0, R_n(x) > e^{an}\}$. Let $X(n_0) \subset Y(n_0)$ be the set of Lebesgue density points of $Y(n_0)$, that is for any $x \in X(n_0)$

$$\lim_{n\to\infty} \mu_{A_n^x}(Y(n_0)) = 1.$$

Whenever $y \in A_n^x$ we have $R_n(y) = \tau_{A_n^x}(y)$, therefore

$$A_n^x \cap \{\tau_{A_n^x} \leq e^{an}\} \subset Y(n_0)^c$$

as soon as $n \geq n_0$. Thus, the conclusion holds for any $x \in X(n_0)$. By Lebesgue density theorem, this set has the same measure as $Y(n_0)$. The latter, taking n_0 arbitrarily large, has a measure arbitrarily close to one. □

3. Presence of cluster

Given a measurable set A with $\mu(A) > 0$ and an integer m we can define a coefficient related to the probability of escape from A before time m:

$$\theta_m(A) := \mu_A(\tau_A > m). \tag{8}$$

Proposition 5. *Let (A_n) be a sequence of sets $A_n \in \mathcal{F}_0^n$ such that $\mu(\tau_{A_n} \leq n) \to 0$. Let (m_n) be a sequence of integers $m_n \leq n$. Then the scaling factors satisfy the relation*

$$\lambda(A_n) \sim \theta_{m_n}(A_n)\lambda(A_n \cap \{\tau_{A_n} > m_n\}).$$

In particular we have the estimate

$$\limsup_n \frac{\lambda(A_n)}{\theta_{m_n}(A_n)} \leq 1.$$

Proof. Since $A_n \in \mathcal{F}_0^n$ and $\mu(\tau_{A_n} \leq n) \to 0$, by Theorem 1 there exists a factor $\lambda(A_n)$ such that

$$\sup_k |\mu(\tau_{A_n} \leq k) - 1 + e^{-k\lambda(A_n)\mu(A_n)}| \to 0 \ .$$

Let us write for simplicity $A_n' = A_n \setminus \{\tau_{A_n} \leq m_n\}$. Since $A_n' \in \mathcal{F}_0^{2n}$ and

$$\mu(\tau_{A_n'} \leq 2n) \leq \mu(\tau_{A_n} \leq 2n) \leq 2\mu(\tau_{A_n} \leq n) \to 0,$$

by Theorem 1 there exists $\lambda(A'_n)$ such that $\limsup \lambda(A'_n) \leq 1$ and

$$\sup_k |\mu(\tau_{A'_n} \leq k) - 1 + e^{-k\lambda(A'_n)\mu(A'_n)}| \to 0.$$

By definition of $\theta_{m_n}(A_n)$ we have

$$\mu(A'_n) = \mu(A_n \cap \{\tau_{A_n} > m_n\}) = \theta_{m_n}(A_n)\mu(A_n)$$

from which follows

$$\sup_k |\mu(\tau_{A'_n} \leq k) - 1 - e^{-k\lambda(A'_n)\theta_{m_n}(A_n)\mu(A_n)}| \to 0.$$

We now show that the distributions of τ_{A_n} and $\tau_{A'_n}$ are close. We first claim that

$$U_k = \{\tau_{A_n} \leq k\} \setminus \{\tau_{A'_n} \leq k\} \subset T^{-k}\{\tau_{A_n} \leq m_n\}.$$

Indeed, if $x \in U_k$ then there exists $j_1 \leq k$ such that $T^{j_1}x \in A_n \setminus A'_n$. Hence $\tau_{A_n}(T^{j_1}x) \leq m_n$. Hence there exists $j_2 > j_1$, $j_2 - j_1 \leq m_n$ such that $T^{j_2}x \in A_n$. If $j_2 \leq k$ then $T^{j_2}x \notin A'_n$, hence one can construct j_3, ... At the end there exists $j_{r-1} \leq k < j_r \leq k + m_n$ such that $T^{j_r}x \in A_n$, proving the claim. This gives that

$$\sup_k |\mu(\tau_{A'_n} \leq k) - \mu(\tau_{A_n} \leq k)| \leq \mu(\tau_{A_n} \leq m_n) \to 0.$$

Combining this estimate with the previous one, taking $k \approx 1/(\lambda(A'_n)\theta_{m_n}(A_n)\mu(A_n))$, we get

$$\exp(-\frac{\lambda(A_n)}{\theta_{m_n}(A_n)\lambda(A'_n)}) - \exp(-1) \to 0$$

which proves the first assertion. The conclusion follows since $\limsup_n \lambda(A'_n) \leq 1$. □

We now specialize to sequence of cylinder sets. Denote the period of the cylinder A_n^x by

$$\tau(A_n^x) = \inf\{\tau_{A_n^x}(y) \colon y \in A_n^x\}$$

and the potential well by

$$\rho(A_n^x) = \mu_{A_n^x}(\tau_{A_n^x} > \tau(A_n^x)) = \theta_{\tau(A_n^x)}(A_n^x).$$

Corollary 6. *For any x we have $\lambda(A_n^x) \leq (1 + o(1))\rho(A_n^x)$.*

We emphasize that this formula is sharp for the binary renewal stochastic process [2]. In particular this holds for the Markov chain model of the Pommeau-Manneville intermittent map.

Let now

$$\bar\theta(A_n^x) = \sum_{N_n}^{S_n - 2N_n} \mu_{A_n^x}(\tau_{A_n^x} > j)$$

be the mean escape time between N_n and $S_n - 2N_n$ (we avoid the dependence on N_n and S_n to simplify notation). The next proposition shows that $\lambda(A_n^x)$ can be replaced by the arithmetic mean of the escape times of some neighborhood smaller that the inverse of $\mu(A_n^x)$.

Proposition 7. *Suppose the rate of α-mixing is algebraic. For every x*

$$\frac{\lambda(A_n^x)}{\bar\theta(A_n^x)} - 1 \to 0\ .$$

Proof. The proof is identical to that of Theorem 3.1 in [2] changing both $T(A_n^x)$ and n by N_n, and s by S_n, as we did here in Theorem 2. Actually we only need to show the approximation between $\lambda(A_n^x)$ and $\lambda_3(A_n^x)$ there defined. By stationarity $\bar\theta(A_n^x) = \lambda_3(A_n^x)$. To get our result we only need, with these choices of N_n and S_n, to show that $\mu(\tau_{A_n^x} \le S_n) \to 0$ and $\mu(\tau_{A_n^x} \le N_n)/\mu(\tau_{A_n^x} \le S_n - 2N_n) \to 0$. These two conditions were proved right after (4). □

Observe in the definition of $\bar\theta(A_n^x)$ that $\mu_{A_n^x}(\tau_{A_n^x} > j)$ are decreasing. Therefore we get that there exists (ϵ_n) such that

$$(1-\epsilon_n)\theta_{N_n} \le \lambda(A_n^x) \le (1+\epsilon_n)\theta_{S_n - 2N_n} \ .$$

4. Estimation of the clustering factor

In principle the clustering factor λ appears only for macroscopic times, of the order $\mu(A_n^x)^{-1}$. For example from (2) we get that $\lambda(A_n^x) \approx -\log \mu(\tau_{A_n^x} > \mu(A_n^x)^{-1})$. The result below shows that when the mixing is sufficiently fast, the factor λ can be computed with the distribution of return times of much smaller orders. This could be of practical interest when one wants to estimate the clustering factor for the occurrence of a rare event A, since it suffices to know the distribution of return times until a time of order much less than the inverse of the probability of the event A.

Proposition 8. *We suppose that $\alpha(n) = O(1/n^\beta)$ for some $\beta > 1$. Let $\gamma_* = 1 - 1/\beta > 0$. For any $\gamma \in (0, \gamma_*)$ we have*

$$\lambda(A_n^x) - \mu_{A_n^x}(\tau_{A_n^x} > \mu(A_n^x)^{-1+\gamma}) \to 0.$$

We recall here again the connection between the extremal index of the maxima distribution which also measure the amount of clustering and our clustering factor for hitting times. Consider for instance the paper [15] about Extreme Value Theory. Specifically, in formula (1.2), the exponent of the approximating law of the maxima distribution resembles $\mu_{A_n^x}(\tau_{A_n^x} > \mu(A_n^x)^{-1})$. This is because both parameters measure the probability the process has to run along the complement of the starting set.

Proof. Let $A \in \mathcal{F}_0^n$, m, q integers and set $A' = A \cap \{\tau_A > m\}$. Note that $\{\tau_{A'} \le m\}$ is the disjoint union $T^{-i}(A')$ for $i = 1, \ldots, m$, therefore $\mu(\tau_{A'} \le m) = m\mu(A')$. We have by Poincaré inequality

$$\begin{aligned}
\mu(\tau_{A'} \le mq) &= \mu\Big(\bigcup_{i=0}^{q-1} T^{-im}\{\tau_{A'} \le m\}\Big) \\
&\ge \sum_{i=0}^{q-1} \mu(\tau_{A'} \le m) - \sum_{i \ne j} \mu(T^{-im}\{\tau_{A'} \le m\} \cap T^{-jm}\{\tau_{A'} \le m\}).
\end{aligned}$$

The first term is equal to $mq\mu(A')$ and we will find some conditions which ensure that this is the leading term. The contribution of each couple (i,j) such that $|i-j| > 2$ is

$$\mu(T^{-im}\{\tau_{A'} \le m\} \cap T^{-jm}\{\tau_{A'} \le m\}) \le (m\mu(A'))^2 + \alpha(m-n)).$$

Their total contribution is thus bounded by

$$m^2q^2\mu(A')^2 + q^2\alpha(m-n) \tag{9}$$

[rem: one could get $\sum_{i=1}^{q}\alpha(im-k)$ instead of $q\alpha(m-n)$]. The contribution of each couple (i,j) such that $|i-j|=s=1,2$ is

$$\mu(\{\tau_{A'}\le m\}\cap T^{-sm}\{\tau_{A'}\le m\}).$$

This event implies an entrance in A at a time $r=1,\dots,m$, followed by an entrance in A' at a time $\ge r+m$, which has measure bounded by

$$\sum_{r=1}^{m}\mu(T^{-r}(A\cap\{\tau_{A'}\le m\}))\le m(\mu(A)\mu(\tau_{A'}\le m)+\alpha(m-n))\ .$$

Therefore the contribution of these couples is bounded by

$$2mq\mu(A')m\mu(A)+2mq\alpha(m-n). \tag{10}$$

We now apply this with a cylinder A_n and choose the parameters $m=m_n$, $q=q_n$ such that the conditions $m_n\mu(A_n)\to 0$, $\alpha(m_n-n)=o(\mu(A'_n))$ and $q_n\le m_n$ are satisfied[2]. We suppose for the moment that $\theta_{m_n}(A_n)$ is bounded from below.

This gives by (9) and (10) that

$$\mu(\tau_{A'_n}\le p_n)\sim p_n\mu(A'_n), \tag{11}$$

for any integers $p_n\le m_nq_n$ (it follows directly when $p_n=m_nq_n$ and by a convexity argument it has to be true for any smaller integer). Now we take $S_n=m_nq_n$ and $N_n=m_n+n$ and proceed as in the proof of Theorem 2, but with the sets A'_n. Using (11) we see that if $q_n\to\infty$ sufficiently slowly then we have

$$\mu(\tau_{A'_n}\le S_n)\to 0 \quad\text{and}\quad \frac{\mu(\tau_{A'_n}\le 2N_n)+\alpha(N_n)}{\mu(\tau_{A'_n}\le S_n-2N_n)}\to 0. \tag{12}$$

Finally, by (6) again we get that

$$\frac{\mu(\tau_{A'_n}\le S_n-2N_n)}{S_n\mu(A'_n)}\to 1,$$

proving that $\lambda(A'_n)\to 1$. The first assertion of Proposition 5 (applied with N_n instead of n) then gives

$$\lambda(A_n)\sim\theta_{m_n}(A_n).$$

We emphasize that if we only have a subsequence for which $\theta_{m_n}(A_n)$ is bounded from below then the equivalence above is still true for that subsequence.

To finish, we remark that if $\theta_{m_n}(A_n)\to 0$ along a subsequence, it follows immediately from the second assertion of Proposition 5 that also $\lambda(A_n)\to 0$ along that subsequence. □

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[2]This is possible since if $\theta_{m_n}(A_n)\ge a>0$, then $\mu(A'_n)\ge a\mu(A_n)$ so $p_n=\mu(A_n)^{-1+\gamma}$ is fine.

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