AN INTRODUCTION TO QUANTITATIVE POINCARÉ RECURRENCE IN DYNAMICAL SYSTEMS

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Abstract. We present some recurrence results in the context of ergodic theory and dynamical systems. The main focus will be on smooth dynamical systems, in particular those with some chaotic/hyperbolic behavior. The aim is to compute recurrence rates, limiting distributions of return times, and short returns. We choose to give the full proofs of the results directly related to recurrence, avoiding as possible to hide the ideas behind technical details. This drove us to consider as our basic dynamical system a one-dimensional expanding map of the interval. We note however that most of the arguments still apply to higher dimensional or less uniform situations, so that most of the statements continue to hold. Some basic notions from the thermodynamic formalism and the dimension theory of dynamical systems will be recalled.

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1. Classical recurrence results in ergodic theory

In this section we briefly present some classical results on recurrence in the general context of Ergodic Theory. Most of them are of qualitative nature, and the main purpose of the lecture will be to give some quantitative refinement of them. From now on we have a measure preserving dynamical system $(X, \mathcal{A}, T, \mu)$: X is a space, $\mathcal{A}$ is a σ-algebra on X, $T\colon X \to X$ is a measurable map and μ a probability measure on $(X, \mathcal{A})$, such that $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{A}$. We say that the system is ergodic if the invariant sets are trivial: $T^{-1}A = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$.

1.1. **Some examples of dynamical systems.** 1) on the unit circle, the angle α-rotation $Tx = x + \alpha \mod 1$;
2) on the unit circle, the doubling map $Tx = 2x \mod 1$;
3) on the 2-torus, the cat map $Tx = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x$.
Good thing with these maps : they all preserve the Lebesgue measure.

4) full shift on two symbols, $X = \{0,1\}^{\mathbb{N}}$, $Tx = (x_{n+1})_n$. preserves the infinite product of a Bernoulli measure.

5) shift of a stationnary process : $W = (W_n)$ a stationary real valued process and $X = \mathbb{R}^{\mathbb{N}}$ with the shift map again, and the probability measure $\mathbb{P}_W$.

1.2. **Hitting and return time.** Given a point $x \in X$ the sequence of iterations $x, Tx, T^2x, \ldots, T^nx, \ldots$ is called its (forward) orbit.

Given a set A and an initial point x, the basic object of study here will be the (first) hitting time of the orbit of x to the set A. We denote it by $\tau_A(x)$, defined by

$$\tau_A(x) = \min\{n \colon T^n x \in A, n = 1, 2, \ldots\}$$

or $\tau_A(x) = +\infty$ if the (forward) orbit never enters in A. When $x \in A$ we usually call $\tau_A(x)$ the (first) return time.

The first theorem of this lecture could reasonably not be something else than the famous Poincaré recurrence theorem itself:

Theorem 1. *Let $A \in \mathcal{A}$ be a measurable set. Then for μ-almost all $x \in A$, the forward orbit $T^n x$, $n = 1, 2, \ldots$ belongs to A infinitely often.*

We will call these points A-recurrent.

Proof. Let $n \geq 1$ be an integer. The disjoint union

$$\{\tau_A \leq n\} = \{\tau_A \circ T \leq n-1\} \cup T^{-1}\big(A \cap \{\tau_A > n-1\}\big).$$

gives using the invariance of the measure

$$\mu(\tau_A = n) = \mu(A \cap \{\tau_A \geq n\}) \tag{1}$$

In particular we have $\mu(\tau_A = n) \geq \mu(A \cap \{\tau_A = +\infty\})$. The sets $\{\tau_A = n\}$, $n = 1, 2, \ldots$ are disjoints in the finite measure space, thus the left hand side is summable with n. So is the right hand side, hence $A \cap \{\tau_A = +\infty\}$ is a null set. Therefore

$$\bigcup_{n \geq 0} T^{-n}(A \cap \{\tau_A = +\infty\})$$

is again a null set. □

Poincaré recurrence theorem tells in particular that $\tau_A(x) < +\infty$ for μ-almost every $x \in A$. We emphasize that this theorem is valid for any finite measure preserving dynamical system and any measurable set. Obviously, for zero measure sets the statement is empty. Note that this statement only concerns return times, since the initial point x needs to be in the set A.

When X is a separable metric space (i.e. there is a dense countable subset) we obtain a corollary of topological nature, somewhat more concrete:

Theorem 2. *Assume (X, d) is a separable metric space and that μ is a Borel T-invariant measure. Then the orbit of almost any initial point returns arbitrarily close to the initial point:*
For μ-almost every x, there exists a subsequence n_k such that $T^{n_k}x \to x$.

Proof. Let $\{B, B \in \mathcal{B}\}$ be a countable basis of X (for example balls of rational radius centered at a dense sequence). By Poincaré recurrence theorem, for each set B there exists a negligible $N_B \subset B$ such that any point in $B \setminus N_B$ is B-recurrent. The set $N = \cup_B N_B$ is negligible. Let $x \in X \setminus N$. Let $B_i \in \mathcal{B}$ be a sequence of sets with diameter going to zero and containing x. Since x is B_i-recurrent, there exists an integer n_i such that $T^{n_i}x \in B_i$. Therefore $T^{n_i}x \to x$. $\square$

1.3. **Mean behavior of return times.** We just have seen that the function τ_A is almost surely finite on A. If we denote by $\mu_A = \frac{1}{\mu(A)}\mu|_A$ the conditional measure on A, then we can look at the expectation of τ_A:

Theorem 3 (Kač's lemma [37])**.** *Let $A \in \mathcal{A}$ be such that $\mu(A) > 0$. We have*

$$\int_A \tau_A d\mu = \mu(\{\tau_A < +\infty\}).$$

In particular, when the system is ergodic we have $\displaystyle\int_A \tau_A d\mu_A = \frac{1}{\mu(A)}$, i.e. the mean return time is equal to the inverse of the measure.

An elegant proof uses towers, however it requires the map to be bi-measurable.

Proof. We recall the relation between hitting and return times (1)

$$\mu(\tau_A = n) = \mu(A \cap \{\tau_A \geq n\}).$$

Summing up over n yields

$$\mu(\tau_A < +\infty) = \sum_{n=1}^{\infty} \mu(A \cap \{\tau_A \geq n\}) = \int_A \tau_A d\mu.$$

For the last statement, observe that the set $\{\tau_A < +\infty\}$ is invariant by Poincaré recurrence theorem. $\square$

If $x \in A$ is such that $\tau_A(x) < +\infty$, then the iterate $T_A(x) = T^{\tau_A(x)}x$ is well defined and belongs to A again. This defines (almost everywhere) an induced map on A, called the first return map to A.

Theorem 4. *The system (A, T_A, μ_A) is a well defined measure preserving dynamical system. It is ergodic if the original system is ergodic.*

Proof. Let $B \subset A$ be a measurable set. To prove the invariance of μ_A it is sufficient to prove that $\mu(T_A^{-1}B) = \mu(B)$. First,

$$\mu(T_A^{-1}B) = \sum_{n=1}^{\infty} \mu(A \cap \{\tau_A = n\} \cap T^{-n}B).$$

We then refine equation (1) starting from the disjoint union

$$\{\tau_A \leq n\} \cap T^{-n-1}B = T^{-1}(\{\tau_A \leq n-1\} \cap T^{-n}B) \cup T^{-1}(A \cap \{\tau_A = n\} \cap T^{-n}B).$$

This gives by invariance of the measure

$$\mu(A \cap \{\tau_A = n\} \cap T^{-n}B) = \mu(B_n) - \mu(B_{n-1})$$

where $B_n = \{\tau_A \leq n\} \cap T^{-n-1}B$. We have $\mu(B_n) \to \mu(B)$ as $n \to \infty$, thus

$$\mu(T_A^{-1}B) = \lim \mu(B_n) = \mu(B).$$

Let us assume now the ergodicity of the original system. Let $B \subset A$ be a measurable T_A-invariant subset. For any $x \in B$, the first iterate $T^n x$ $(n \geq 1)$ that belongs to A also belongs to B, which means that $\tau_B = \tau_A$ on B. But if $\mu(B) \neq 0$, Kač's lemma gives that

$$\int_B \tau_B d\mu = 1 = \int_A \tau_A d\mu,$$

which implies that $\mu(B \setminus A) = 0$, proving ergodicity. □

We will invoke several time the classical Birkhoff ergodic theorem [8], that we recall without proof.

Theorem 5. *Let φ be an integrable function. The time average $\frac{1}{n}S_n\varphi$ converges pointwise and in L^1 to some function $\tilde{\varphi}$. If the measure μ is ergodic then $\tilde{\varphi}$ a.e. equal to the space average $\int \varphi d\mu$.*

In an ergodic system, the ergodic theorem gives a quantitative information on the recurrence property in a given set. More precisely, if $A \in \mathcal{A}$ then we get that

$$\frac{\mathrm{card}\{1 \leq k \leq n \colon T^k x \in A\}}{n} \to \mu(A) \quad \text{for } \mu\text{-a.e. } x.$$

If we define inductively the nth return time by $\tau_A^{(n)}(x) = \tau_A^{(n-1)}(x) + \tau_A(T_A^{n-1}(x))$, we get by the Birkhoff ergodic theorem again (but on the induced map) that

$$\frac{1}{n}\tau_A^{(n)}(x) \to \frac{1}{\mu(A)} \quad \text{for } \mu_A\text{-a.e. } x$$

when the system is ergodic and $\mu(A) > 0$.

2. Thermodynamic formalism for expanding maps of the interval.

Sensitivity to initial conditions, i.e. separation of nearby orbits at an exponential speed, is at the origin of deterministic chaos. A possible mathematical formalisation of this exponential separation is the hyperbolic dynamic. This geometric property implies some randomness on the statistical properties of the system, which behaves much like an i.i.d. process.

Some results about recurrence that we want to present in this review are only known in the low dimensional case, or in sufficiently strong mixing conditions. To give a unified presentation of these results, we decided to work with a class of dynamical systems which possess all these features. The choice is thus to consider Markov maps of the interval together with an equilibrium state.

Remark 6. *Our aim is to consider the dynamical system with its natural metric. For example we will be mainly interested by return time to sets which are natural (e.g. balls). Therefore, the connection with symbolic dynamics is on purpose maintained relatively low. We warn the reader that the existence of a Markov partition is not essential. Roughly speaking, it makes many geometric and measure theoretic estimates uniform, which makes our life easier. This simplifying assumption allows us to give a self-contained proof of Ruelle-Perron-Frobenius theorem, from which we get precise estimates on decorrelation.*

Finally, the choice to consider expanding maps instead of real hyperbolic maps (with expanding and contracting directions, e.g. Anosov, or Axiom A) is made on purpose to keep the technicity at a low level.

We refer the reader to [38] for a complete presentation of hyperbolic dynamics, and also to [48] for the dimensional theory of conformal dynamics.

2.1. **Coding and geometry.** We assume that X is the interval $[0,1]$ and that T is a piecewise $C^{1+\alpha}$ expanding map on X:

(E) there exists some constant $\beta > 1$ such that $|T'(x)| \geq \beta$ for every $x \in X$.

There exists a collection $\mathcal{J} = \{J_1, \ldots, J_p\}$ such that each J_i is a closed interval and

(M1) T is a $C^{1+\alpha}$ diffeomorphism from $\operatorname{int} J_i$ onto its image;

(M2) $X = \cup_i J_i$ and $\operatorname{int} J_i \cap \operatorname{int} J_j = \emptyset$ unless $i = j$;

(M3) $T(J_i) \supset J_j$ whenever $T(\operatorname{int} J_i) \cap \operatorname{int} J_j \neq \emptyset$.

$\mathcal{J}$ is called a Markov partition.

Remark 7. *For real hyperbolic systems the notion of Markov partition involves stable and unstable manifolds. The definition here is considerably simpler, although it is consistent with the general one. The simplification is due to the fact that the local stable manifold is trivially reduced to a point for expanding maps.*

Such Markov maps of the interval can be modeled by symbolic systems as follows. Define a $p \times p$ matrix $A = (a_{ij})$ by $a_{ij} = 1$ if $T(J_i) \supset J_j$ and $a_{ij} = 0$ otherwise. Let $\mathcal{A} = \{1, \ldots, p\}$ and $\Sigma_A \subset \mathcal{A}^{\mathbb{N}}$ be the set of sequences ω such that $a_{\omega_i, \omega_{i+1}} = 1$ for any $i \in \mathbb{N}$. Denote by $\sigma = \Sigma_A \to \Sigma_A$ the shift map defined by $\sigma(\omega)_i = \omega_{i+1}$ for any $i \in \mathbb{N}$. Setting $\chi(\omega) = \cap_{i=0}^{\infty} T^{-i} J_{\omega_i}$ gives the symbolic coding of the interval map (X, T) by (Σ_A, σ):

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \chi \downarrow & & \downarrow \chi \\ X & \xrightarrow{T} & X \end{array}$$

Let $\partial\mathcal{J} := \cup_i \partial J_i$. The map χ is one-to-one except on the set $S := \cup_{n=0}^{\infty} T^{-n} \partial\mathcal{J}$, where it is at most p^2-to-one.

For $\omega \in \Sigma_A$ we denote by $C_n(\omega)$ the nth cylinder of ω, that is the set of sequences $\omega' \in \Sigma_A$ such that $\omega_i = \omega'_i$ for any $i = 0, \ldots, n-1$. When $x = \chi(\omega) \notin S$ we let $\mathcal{J}_n(x) = \chi(C_n(\omega))$.

Lemma 8. *Let $\psi\colon X \to \mathbb{R}$ be α-Holdër continuous. For any x, y in the same n-cylinder, we have*

$$|S_n\psi(x) - S_n\psi(y)| \leq |\psi|_\alpha \delta^\alpha \frac{1}{\beta^\alpha - 1}.$$

Proof. For any $k = 0, \ldots, n-1$, $T^k x$ and $T^k y$ are in the same element of the partition. By the expanding property and an immediate recurrence we get that

$$d(T^k x, T^k y) \le \beta^{k-n} d(T^n x, T^n y) \le \beta^{k-n}. \tag{2}$$

Therefore

$$\begin{aligned} S_n\psi(x) - S_n\psi(y) &= \sum_{k=0}^{n-1} \psi(T^k x) - \psi(T^k y) \\ &\le \sum_{k=0}^{n-1} |\psi|_\alpha d(T^k x, T^k y)^\alpha \le |\psi|_\alpha \sum_{k=0}^{n-1} (\delta\beta^{k-n})^\alpha. \end{aligned}$$

□

Proposition 9. *There exist two constants c_0, c_1 such that for any $x \notin S$, any integer n,*

$$c_0 |(T^n)'(x)|^{-1} \le \operatorname{diam} \mathcal{J}_n(x) \le c_1 |(T^n)'(x)|^{-1}.$$

Proof. The function $x \mapsto \log |T'(x)|$ is α-Hölder continuous on X. Thus by Lemma 8 there exists some constant D such that for each $n \in \mathbb{N}$ and x, y in the same n-cylinder,

$$\frac{|(T^n)'(x)|}{|(T^n)'(y)|} \le D.$$

The restriction of T^n to the interval $\mathcal{J}_n(x)$ is a diffeomorphism, so we can apply the mean value theorem: there exists $y \in \mathcal{J}_n(x)$ such that

$$\operatorname{diam} T^n(\mathcal{J}_n(x)) = |(T^n)'(y)| \operatorname{diam} \mathcal{J}_n(x).$$

This together with the distortion estimate proves the upper bound with $c_1 = D \operatorname{diam} X$.

We now prove the lower bound. Let $\rho = \min \operatorname{diam} J_i > 0$. Since $T^n(\mathcal{J}_n(x))$ is an union of some of the J_i's, $\operatorname{diam} T^n(\mathcal{J}_n(x)) \ge \rho$. This together with the distortion estimate proves the second statement with $c_0 = \rho D^{-1}$ □

Remark 10. *We emphasize that this picture is true in the more general situation of conformal repellers. X is a compact invariant subset of a $C^{1+\alpha}$ map of a Riemaniann manifold M, such that (i) T is expanding on X: $\|d_x T v\| \ge \beta \|v\|$ for all $v \in T_x M$, for all $x \in X$; (ii) there exists an open set $V \subset M$ such that $X = \cap_n T^{-n} V$; In this case there exist Markov partitions of arbitrariy small diameter. Under the additional assumption (iii) T is conformal: $d_x T$ is a multiple of an isometry, for each cylinder $\mathcal{J}_n(x)$ the inner and outer diameter are in some range $\|d_x T^n\|^{-1}[c_0, c_1]$.*

We **assume for simplicity that the map is topologically mixing**, that is for any two open sets A and B, there exists N such that for all $n > N$, $A \cap T^{-n} B \ne \emptyset$. This is equivalent to assume that Σ_A is irreducible and aperiodic, i.e. there exists m_0 such that A^{m_0} has only nonzero entries.

2.2. **Dimension, entropy and Lyapunov exponent.** We now review briefly some essential notion coming from the thermodynamic formalism of expanding maps, as well as its relation with dimensions and Lyapunov exponents. We emphasize that most of the results are known in much more general situations. It is out of the scope of this note to present them in full generality, with the weakest hypothesis.

2.2.1. *Dimensions.* We now introduce some dimensions for sets and measures. The ambient space is $\mathbb{R}^N$.

Let $\alpha \geq 0$ and define a set function $\mathcal{H}^\alpha(\cdot)$ by

$$\mathcal{H}^\alpha(A) = \lim_{\delta\to 0} \inf \sum_i (\operatorname{diam} V_i)^\alpha$$

where the infimum is taken among all countable covers of A by sets V_i with $\operatorname{diam} V_i \leq \delta$ (the limit exists by monotonicity). $\mathcal{H}^\alpha(A)$ is called the Hausdorff measure of dimension α of the set A.

We define the Hausdorff dimension of the set A, denoted by $\dim_H A$, as the unique number such that $\mathcal{H}^\alpha(A) = +\infty$ if $\alpha < \dim_H A$ and $\mathcal{H}^\alpha(A) = 0$ if $\alpha > \dim_H A$.

Recall that for all $\alpha \geq 0$, $\mathcal{H}^\alpha$ is an exterior measure, for which Borel sets are measurable. We recover (a multiple of) the Lebesgue measure when $\alpha = N$.

For a countable collection of set (A_n) we have $\dim_H \cup_{n\in\mathbb{N}} A_n = \sup_{n\in\mathbb{N}} \dim_H A_n$.

Given a Radon measure (Borel measure finite on compact sets) μ its Hausdorff dimension is defined by

$$\dim_H \mu = \inf\{\dim_H A\colon A^c \ \mu\text{-negligeable}\}.$$

The pointwise dimensions of a measure μ are defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

Theorem 11. *For any Radon measure μ we have $\dim_H \mu = \operatorname{esssup} \underline{d}_\mu$.*

Proof. Fix $\alpha > \operatorname{esssup} \underline{d}_\mu$ and an integer $n \geq 1$. Let

$$A_n = \{x \in B(0,n) : \mu(B(x,r)) \geq r^\alpha,\ \forall r < 1/n\}.$$

Let $\delta \in (0, 1/n)$. By Vitali's lemma there exists a countable family (x_i, r_i) with $x_i \in A_n$ and $r_i < \delta$ such that $B(x_i, r_i)$ are disjoints and $V_i := B(x_i, 3r_i)$ covers A_n. Moreover,

$$\sum_i \operatorname{diam}(V_i)^\alpha \leq \sum_i (6r_i)^\alpha \leq 6^\alpha \sum_i \mu(B(x_i, r_i))^\alpha \leq \mu(B(0, n+1/n)).$$

This gives $\mathcal{H}^\alpha(A_n) < \infty$, therefore $\dim_H A_n \leq \alpha$. Since $\{\underline{d}_\mu < \alpha\} = \cup_n A_n$ has full measure, we get $\dim_H \mu \leq \alpha$.

We prove the upper bound. Let $\alpha < \operatorname{esssup} \underline{d}_\mu$. Let Y be a measurable set with $\mu(Y^c) = 0$. There exists $n \geq 1$ such that

$$Z := \{x \in Y : \mu(B(x,r)) \leq r^\alpha,\ \forall r < 1/n\}$$

has positive measure. Let $\delta < 1/2n$ and consider a δ-cover (V_i) of Z. Let $x_i \in V_i \cap Z$ (if the intersection is empty we simply discard this set). We have

$$\sum_i (\operatorname{diam} V_i)^\alpha \geq \sum_i \mu(B(x_i, \operatorname{diam} V_i)) \geq \mu(\cup_i V_i) \geq \mu(Z).$$

This proves that $\dim_H Y \geq \dim_H Z \geq \alpha$. Therefore $\dim_H \mu \geq \alpha$. $\square$

2.2.2. *Entropy.* From now on and until the end of Section 2.2 μ will denote an invariant measure. Its entropy with respect to a finite measurable partition ξ is defined by

$$h_\mu(T,\xi) = \lim -\frac{1}{n}\int \log\mu(\xi_n(x))d\mu(x),$$

where $\xi_n = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-n+1}\xi$ and $x \in \xi_n(x) \in \xi_n$. We will invoke several time the Shannon-McMillan-Breiman theorem, that we recall without proof (but see Proposition 17).

Theorem 12. *The limit of $-\frac{1}{n}\log\mu(\xi_n(x))$ exists for μ-a.e. x. It is called the local entropy at x, denoted by $h_\mu(x)$. If the measure μ is ergodic then the limit is a.e. equal to the entropy $h_\mu(T,\xi)$.*

The entropy $h_\mu(T)$ of the map is defined by the supremum of the metric entropies $h_\mu(T,\xi)$, taken among all finite measurable partitions. It can be proved that any generating partition achieves this supremum. In particular for our Markov partition $\mathcal{J}$ the entropy is maximal.

In the case of the full shift on m symbols endowed by the Bernoulli measure μ_p with weights $p = (p_1,\dots,p_m)$, the entropy is

$$h_{\mu_p}(\sigma) = -\sum_{i=1}^m p_i\log p_i,$$

which is maximal for the uniform measure $p_i = \frac{1}{m}$ for all i's. The supremum $\log m$ is equal to the topological entropy. This is a special case of the variational principle (see Section 2.3 below).

In the case of the shift on m symbols with a Markov measure $\mu_{P,\pi}$, where P is the stochastic matrix giving the transition probabilities and π is the left eigenvector $\pi P = \pi$, the entropy is simply

$$h_{\mu_{P,\pi}} = -\sum_{i,j=1}^m \pi_i P_{i,j}\log P_{i,j}.$$

2.2.3. *Lyapunov exponents.* Let $x \notin S$. A small interval $I \ni x$ is mapped by the map T^n to some larger interval $T^n(I)$; as long as $I \subset \mathcal{J}_n(x)$, T^n expands the length of I by a factor $|(T^n)'(x)|$ up to a multiplicative correction $\mathrm{e}^{\pm D}$. The average expansion factor of T is thus

$$|(T^n)'(x)|^{1/n} = \exp\frac{1}{n}\log|(T^n)'(x)|.$$

up to a correction $\mathrm{e}^{\pm D/n}$. The limit

$$\lambda(x) = \lim\frac{1}{n}\log|(T^n)'(x)| = \lim\frac{1}{n}S_n\log|T'|(x),$$

if it exists, is called the Lyapunov exponent of T at the point x.

Proposition 13. *For μ-a.e. $x \in X$ the Lyapunov exponent exists and if the measure is ergodic then*

$$\lim\frac{1}{n}\log|(T^n)'(x)| = \int_X \log|T'|d\mu.$$

Proof. By the ergodic theorem this limit exists μ-a.e. when μ is an invariant measure. If furthermore this measure is ergodic then it is constant and equal to the Lyapunov exponent of the measure

$$\lambda_\mu = \int_X \log |T'| d\mu.$$

□

Remark 14. *In higher dimension: the image of a ball in $\mathbb{R}^d$ by a linear map is in general an ellipsoid, with axes of different length and directions. This picture remains in a loose sense in the nonlinear case: approximation of T^n in a vicinity of x by its differential $d_x T^n$ shows that the iterate by the map T^n of a small ball $B(x,r)$ looks like an ellipsoid, with axes $E_{n,i}(x)$ and length $\mathrm{e}^{\lambda_{n,i}(x)n} r$, $i=1,\dots,d$. By Oseledet's theorem* [45], *for μ-a.e. x the directions $E_{n,i}(x)$ and the exponents $\lambda_{n,i}(x)$ converge to some asymptotic value $E_i(x)$ and $\lambda_i(x)$; moreover these values are constant a.e. if the measure μ is ergodic.*

2.2.4. *Their relation.* These three quantities attached to a measure preserving map of the interval are linked through the following relation:

Theorem 15. *The pointwise dimension $d_\mu(x)$ exists a.e. and $d_\mu(x) = \dfrac{h_\mu(x)}{\lambda(x)}$ a.e. If the measure is ergodic, for μ-a.e. x we have*

$$d_\mu(x) = \frac{h_\mu(T)}{\lambda_\mu(T)} = \dim_H \mu.$$

Proof. Let $\varepsilon > 0$. Let $x \in X$ be such that $\mu(\mathcal{J}_n(x)) \geq \mathrm{e}^{-n(h_\mu(x)+\varepsilon)}$ and $\operatorname{diam} \mathcal{J}_n(x) \leq \mathrm{e}^{-n(\lambda(x)-\varepsilon)}$ for any n sufficiently small. This concerns a.e. points by Shannon-McMillan-Breiman theorem (Theorem 12), Proposition 9 and Proposition 13. Given $r > 0$ sufficiently small, we take n the smallest integer such that $\mathrm{e}^{-n(\lambda(x)-\varepsilon)} < r$. Since $\mathcal{J}_n(x) \subset B(x,r)$ we have

$$\frac{\log \mu(B(x,r))}{\log r} \leq \frac{\log \mu(\mathcal{J}_n(x))}{\log r}$$

This proves $\overline{d}_\mu(x) \leq \frac{h_\mu(x)+\varepsilon}{\lambda(x)-\varepsilon}$.

Let us define the set

$$G_\varepsilon(m) = \{x \in X \colon \forall n > m, \mu(\mathcal{J}_n(x)) \leq \mathrm{e}^{-n(h_\mu(x)-\varepsilon)} \text{ and } \operatorname{diam}(\mathcal{J}_n(x)) \geq \mathrm{e}^{-n(\lambda(x)+\varepsilon)}\}.$$

Let x be a density point of $G_\varepsilon(m)$. Given $r > 0$ we take the largest $n = n(r)$ such that $\mathrm{e}^{-n(\lambda(x)+\varepsilon)} > r$. For any r sufficiently small (so that $n > m$) we have

$$\mu(B(x,r)) \leq 2\mu(G_\varepsilon(m) \cap B(x,r)) \leq 4\mathrm{e}^{-n(h_\mu(x)-\varepsilon)},$$

since the ball $B(x,r)$ can intersects at most two cylinders from $G_\varepsilon(m)$. This proves $\underline{d}_\mu(x) \geq \frac{h_\mu(x)-\varepsilon}{\lambda(x)+\varepsilon}$.

In the ergodic case the last identity follows from Theorem 11. □

Remark 16. *The existence of the pointwise dimension has been prove by Young* [58] *in the case of C^2 surface diffeomorphisms with nonzero entropy. Then Ledrappier and Young* [42], *and finally Barreira, Pesin and Schmeling* [5] *extended the result in arbitrary dimensions for $C^{1+\alpha}$ diffeomorphism, for measures without zero Lyapunov exponents.*

2.3. **Equilibrium states or Gibbs measures.** An important class of invariant measures in the ergodic theory of smooth dynamical systems is equilibrium states. This notion comes from thermodynamics, via the symbolic dynamics and is central in the thermodynamic formalism of dynamical systems. In our setting they are also Gibbs measures: the behavior at small scale of these measures is precisely controlled by a function, the potential. In particular, natural measures (e.g. absolutely continuous or physical measure) are equilibrium states. The second interest of these measures lies in the fact that they posses strong statistical properties.

2.3.1. *Gibbs measures.* Let $\varphi\colon X \to \mathbb{R}$ be a Hölder continuous function. An invariant measure μ is called a Gibbs measure for the potential φ if there exists a constant $P_T(\varphi) \in \mathbb{R}$, called the pressure, such that for some $\kappa_\varphi \geq 1$, for any x and any n we have

$$\frac{1}{\kappa_\varphi} \leq \frac{\mu(\mathcal{J}_n(x))}{\exp\left(S_n\varphi(x) - nP_T(\varphi)\right)} \leq \kappa_\varphi. \tag{3}$$

The case of Markov measures is recovered if one takes the potential $\varphi(x) = \log P_{x_0,x_1}$ where P is the stochastic transition matrix. The potential $\varphi = -\log|T'|$ gives the absolutely continuous invariant measure. Note that this measure is Markov essentially in the case of piecewise affine maps.

Proposition 17. *For a Gibbs measure μ, the statement of the Shannon-Mc Millan-Breiman theorem with the partition $\mathcal{J}$ follows immediately from Birkhoff ergodic theorem and*

$$P_T(\varphi) = h_\mu(T) + \int \varphi d\mu.$$

Proof. Indeed, we have $-\frac{1}{n}\log\mu(\mathcal{J}_n(x)) \sim -\frac{1}{n}S_n\varphi(x) + P_T(\varphi)$ which converges a.e. by birkhoff ergodic theorem. The dominated convergence theorem proves the indentity. □

Remark 18. *A measure which attains the supremum*

$$\sup_\nu h_\nu(T) + \int \varphi d\nu$$

among all T-invariant measures ν, is called an equilibrium measure. According to the variational principle*, the supremum is indeed the pressure $P_T(\varphi)$.*

It turns out that for Markov expanding maps of the interval this supremum is attained at an unique measure μ_φ, which is also the (unique) gibbs measure. For a general account on equilibrium states, and a proof of these results, we refer to [10, 40]*.*

Our Gibbs measure has the following mixing property: there exist some constants $c > 0$ and $\theta \in (0,1)$ such that for any cylinder set A of rank n and any measurable set B we have

$$|\mu_\varphi(A \cap T^{-\ell}B) - \mu_\varphi(A)\mu_\varphi(B)| \leq c\theta^{\ell-n}\mu(A)\mu(B).$$

This is called the ψ-mixing property (with exponential rate). Observe that in particular such a measure μ is mixing, hence ergodic.

The mixing property can also be stated in a different and weaker way, which will be sufficient for the sequel. If f is a Lipschitz function and g an integrable function we have

$$\left|\int fg\circ T^\ell d\mu_\varphi - \int fd\mu_\varphi \int gd\mu_\varphi\right| \leq c\theta^\ell \|f\|_{Lip}\|g\|_{L^\infty}. \tag{4}$$

For the sake of completeness we provide in Section 2.3.2 below a proof of the existence of Gibbs measures and the computation of the rate of decay of correlation (4).

Remark 19. *Such a result on decay of correlations holds in a large variety of settings, where one has some expanding behavior* [43]. *This includes some dynamical systems with singularities, or without Markov partition, in any dimensions. For some nonuniformly expanding systems, the decay rate is polynomial.*

In the case of invertible maps (e.g. hyperbolic diffeomorphisms), the second function g has to be regular also [10, 58].

2.3.2. *Ruelle-Perron-Frobenius theorem.* We closely follow the presentation of the monograph [10] (See also [46]). Suppose that the potential $\varphi\colon X \to \mathbb{R}$ is α-Hölder. For convenience we will work on the subshift of finite type (Σ_A, σ) defined in Section 2.1, instead of working directly on the interval map (X, T). Let $\delta = \beta^{-\alpha}$. We endow Σ_A with the metric $d(\omega, \omega') = \delta^n$ if n is the largest integer such that $C_n(\omega) = C_n(\omega')$. Note that this makes the potential on the symbolic space $\varphi \circ \pi$ Lipschitz. To simplify notations we still denote it by φ. Let ℓ be its Lispchitz constant. For all $a \in \mathcal{A}$, if $a\omega$ and $a\omega'$ exist then $d(a\omega, a\omega') \le \delta d(\omega, \omega')$ (σ is locally expanding, its inverse contracting).

We assume that for some integer N, A^N has only positive entries. The Ruelle-Perron-Frobenius is defined by

$$L_\varphi(f)(\omega) = \sum_{a \in \mathcal{A}, A_{a\omega_0}=1} \mathrm{e}^{\varphi(a\omega)} f(a\omega).$$

L_φ acts on continuous functions. It is a bounded operator on the space of continuous function, as well as on the space of Lipschitz functions. We can iterate it

$$L_\varphi^n f(\omega) = \sum_{\sigma^n \omega' = \omega} \mathrm{e}^{S_n \varphi(\omega')} f(\omega').$$

Lemma 20. *There exists a probability measure ν and a constant $\lambda > 0$ such that $L_\varphi^* \nu = \lambda \nu$.*

Proof. The dual L_φ^* acts on probability measures. The map defined on the convex compact set $\mathcal{M}(\Sigma_A)$ of probability measures by $m \mapsto T(m) = \frac{L_\varphi^* m}{L_\varphi^* m(1)}$ has a fixed point by Schauder-Tychonoff theorem. Putting $\lambda = L_\varphi^* \nu(1)$, this fixed point ν satisfies,

$$\int_{\Sigma_A} L_\varphi f d\nu = \int_{\Sigma_A} d(L_\varphi^*)\nu = \lambda \int_{\Sigma_A} f d\nu.$$

□

Without loss of generality **we assume that** $\lambda = 1$, changing if necessary φ by $\varphi - \log \lambda$. Let $b > 0$ such that $b \ge \ell + b$ and

$$\mathcal{C}_b = \{f \ge 0\colon f(\omega) \le f(\omega')\mathrm{e}^{bd(\omega,\omega')} \text{ if } \omega_0 = \omega_0', \text{ and } \nu(f) = 1\}.$$

Lemma 21. *There exist $c \in (0, 1)$ such that if $f \in \mathcal{C}_b$ then $c \le L_\varphi^N f$ and $f \le c^{-1}$.*

Proof. Let ω and ω'' in Σ. There exists ω' such that $\omega_0 = \omega_0'$ and $\sigma^N \omega' = \omega''$. Thus

$$L_\varphi^N f(\omega'') \ge \mathrm{e}^{S_N \varphi(\omega')} f(\omega') \ge (\inf \mathrm{e}^{S_N \varphi})\mathrm{e}^{-b} f(\omega).$$

This shows that $\inf L_\varphi^N f \geq \frac{\mathrm{e}^b}{\inf \mathrm{e}^{S_N\varphi}} \sup f$. The conclusion follows from the remark that $\nu(f) = 1$ and $\nu(L_\varphi^N f) = 1$. $\square$

Notice that if $f \in \mathcal{C}_b$ then $\log f$, hence f, are Lipschitz. Moreover the Lipschitz norm

$$\|f\| := \|f\|_\infty + |f|_{\mathrm{Lip}} \leq M,$$

for some constant $M = c^{-1}\max(3, b+1)$.

Lemma 22. *There exists $h \in \mathcal{C}_b$ such that $L_\varphi h = h$ and $h > 0$.*

Proof. The set $\mathcal{C}_b$ is relatively compact in the set of continuous functions by Ascoli-Arzela . Moreover, it is clearly closed, thus compact, and convex. In addition, whenever $f \in C_b$ and $\omega_0 = \omega_0'$ we have

$$\begin{aligned}
L_\varphi f(\omega) &= \sum_a \mathrm{e}^{\varphi(a\omega)} f(a\omega) \\
&\leq \sum_a \mathrm{e}^{\varphi(a\omega)-\varphi(a\omega')} \mathrm{e}^{\varphi(a\omega')} f(a\omega') \mathrm{e}^{bd(a\omega,a\omega')} \\
&\leq \mathrm{e}^{\delta[\ell+b]d(\omega,\omega')} L_\varphi f(\omega'),
\end{aligned}$$

which shows that $L_\varphi(\mathcal{C}_b) \subset \mathcal{C}_b$. Schauder-Tychonoff theorem applies again and shows the existence of a fixed point $h \in \mathcal{C}_b$, which satisfies $L_\varphi h = h$. By Lemma 21 we have $h = L_\varphi^N h \geq c > 0$. $\square$

Note that the measure $\mu = h\nu$ is invariant. Without loss of generality **we assume that** $h = 1$, changing if necessary φ by $\varphi + \log h - \log h \circ \sigma$, and ν by $h\nu$.

Theorem 23. *The measure μ is a Gibbs measure for the potential φ.*

Proof. Let $\omega \in \Sigma_A$ and n an integer. Let $f = 1_{C^n(\omega)}$ be the indicator function of the n-cylinder about ω. We have

$$\mu(C^n(\omega)) = \mu(L_\varphi^n f) \leq \sup_{C^n(\omega)} \mathrm{e}^{S_n\varphi} \leq \kappa \mathrm{e}^{S_n\varphi(\omega)}$$

for some constant κ only depending on φ.

The argument in Lemma 21 gives $L_\varphi^n f \geq \inf_{C^n(\omega)} \mathrm{e}^{S_n\varphi} 1_{[\omega_0]}$, and $L^{n+N} f \geq (\inf \mathrm{e}^{S_N\varphi})(\inf \mathrm{e}^{S_n\varphi})$. The previous computation yields

$$\mu(C^n(\omega)) = \nu(L_\varphi^{n+N} f) \geq \frac{1}{\kappa} \mathrm{e}^{S_n\varphi(\omega)},$$

changing the constant κ if necessary. $\square$

Despite its extreme simplicity, the following lemma is the core of the estimate on decay of correlation. The interpretation is that after each iteration by T^N, at least η-percent of the remaining density is chopped out and follows the invariant measure. The exponential convergence will follow immediately.

Lemma 24. *There exists $\eta \in (0,1)$ such that for any $f \in \mathcal{C}_b$, $L_\varphi^N f = \eta + (1-\eta)f'$ with $f' \in \mathcal{C}_b$.*

Proof. Let $\eta = c^2 < 1$. Let $f \in \mathcal{C}_b$. Put $g = L_\varphi^N f$. Write $g = \eta 1 + (g - \eta)$. Since $g - \eta \geq c - \eta c^{-1} \geq 0$ and both $g \in C_b$ and $1 \in \mathcal{C}_b$, we have $g - \eta h \in \mathbb{R}\mathcal{C}_b$. $\square$

Lemma 25. *There exist constants $C > 0$ and $\theta \in (0,1)$ such that*

$$\|L_\varphi^n f - 1\| \leq C\theta^n$$

for all $f \in \mathcal{C}_b$ and $n \geq 0$, where $\|\cdot\|$ stands for the Lipschitz norm.

Proof. By applying Lemma 24 successively, we obtain that for all $p \geq 1$,

$$L_\varphi^{pN} f = (1 - (1-\eta)^p) + (1-\eta)^p f_p$$

for some $f_p \in \mathcal{C}_b$. Write $n = pN + r$. We have

$$\|L_\varphi^n f - 1\| \leq \||L_\varphi^r\||(1-\eta)^p \|1 + f_p\|.$$

Putting $\theta = (1-\eta)^{\frac{1}{N}}$ and $C = \sup_{r<N} \||L_\varphi^r\||2M/(1-\eta)$ gives the conclusion. □

Theorem 26. *The measure μ is mixing and the decay of correlation for Lipschitz f and integrable g is*

$$\left| \int fg \circ \sigma^n d\mu - \int f d\mu \int g d\mu \right| \leq c\|f\|\|g\|_\infty \theta^n.$$

Proof. Let f be a Lipschitz function. Taking $u = (1+b^{-1})\|f\|$ ensures that $f + u \in \mathbb{R}\mathcal{C}_b$. By Lemma 25 this gives the result. □

From now on we fix a Gibbs measure μ of a Hölder potential. We will prove quantitative recurrence results for the dynamical system (X, T, μ).

3. Recurrence rate

By the topological version of Poincaré recurrence theorem, i.e. Theorem 2, a.e. point x return arbitrarily close after iteration by T. A very natural question is the behavior when $r \to 0$ of the first return time

$$\tau_r(x) = \tau_{B(x,r)}(x) = \min\{n \geq 1 \colon d(T^n x, x) < r\},$$

that is the first time that the orbit of x is back in the r-neighborhood of the point x. This is quantified by the following notion.

Definition 27. *We define the lower and upper recurrence rate of a point x by the limits*

$$\underline{R}(x) := \liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r}, \quad \overline{R}(x) := \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r}.$$

When the limit exists we denote it by $R(x)$.

Theorem 28. *The recurrence rate is a.e. equal to the dimension of the measure: $R(x) = \dim_H \mu$ for μ-a.e. x.*

Recurrence rates were introduced and studied in [55] in the case of interval maps, and [6, 7] in the case of axiom A diffeomorphisms and some class of repellers. The corresponding results for hitting or waiting times were then considered by Galatolo [25, 26]; see also [27, 28] for subsequent developments.

Remark 29. *We emphasize that some assumption on the system is necessary, since there exist examples of dynamical systems where the conclusion of Theorem 28 is false (e.g. rotations with special diophantine type* [6]*).*

3.1. **The rapid mixing method.** In this section we prove Theorem 28 using the method developped in [52].

Remark 30. *The theorem is indeed valid in a more general situation than Markov expanding maps of the interval. The core assumption is the decay of correlation for Lipschitz functions* (4) *with a superpolynomial rate. Without assuming the existence of the pointwise dimension one still gets the identities* $\underline{R} = \underline{d}_\mu$ *and* $\overline{R} = \overline{d}_\mu$ μ*-a.e.*

Let $\delta = \dim_H \mu$. The following lemma is a direct consequence of Theorem 15 about the existence of the pointwise dimension and Egorov theorem.

Lemma 31. *Good set for the pointwise dimension: For any* $\varepsilon > 0$ *there exists* $r_0 > 0$ *such that the measurable set* $K \subset X$ *defined by*

$$K_\varepsilon = \{x \in X : \forall r < r_0, r^{\delta+\varepsilon} \le \mu(B(x,r)) \le r^{\delta-\varepsilon}\}$$

has measure at least $1 - \varepsilon$.

Lemma 32. *For* μ*-a.e.* $x \in K_\varepsilon$*, we have* $\liminf_{r\to0} \frac{\log \tau_r(x)}{-\log r} \ge \delta - 4\varepsilon$.

Proof. Fix $\varepsilon > 0$ and let K_ε be as in Lemma 31. Take $\alpha = \frac{1}{\delta-4\varepsilon}$, set $r_n = \frac{1}{n^\alpha}$ and define

$$L_n = \{x \in K_\varepsilon : d(T^n x, x) < r_n\}.$$

For a ball $B = B(x,r)$ and a constant $a > 0$ we denote by aB the ball $B(x,ar)$. Let $(B_i)_i$ be a collection of balls of radius r_n centered at points in K_ε that covers L_n and such that the collection of balls $(\frac12 B_i)_i$ is disjoint. We have

$$\mu(L_n) = \mu(\cup_i B_i \cap L_n) \le \sum_i \mu(B_i \cap L_n) \le \sum_i \mu(B_i \cap T^{-n} 2B_i).$$

For each i define the function $\phi_i(x) = \max(0, 1 - r_n^{-1} d(x, 2B_i))$. We remark that ϕ_i is r_n^{-1}-Lipschitz, and $1_{2B_i} \le \phi_i \le 1_{3B_i}$. We have

$$\mu(B_i \cap T^{-n} 2B_i) \le \int \phi_i \phi_i \circ T^n d\mu \le \left(\int \phi_i d\mu\right)^2 + c\theta^n \|\phi_i\|_{Lip} \le \mu(3B_i)^2 + c\theta^n r_n^{-1}.$$

using the decay of correlations formula (4). Since the balls are centered on K_ε we have $\mu(3B_i) \le (3r_n)^{\delta-\varepsilon}$ and $(\frac12 r_n)^{\delta+\varepsilon} \le \mu(\frac12 B_i)$. This last inequality and the fact that the balls are disjoint imply that their number is bounded by $(\frac12 r_n)^{-\delta-\varepsilon}$. Therefore,

$$\sum_i \mu(B_i \cap T^{-n} 2B_i) \le (\frac{1}{2} r_n)^{-\delta-\varepsilon} \left[(3r_n)^{2\delta-2\varepsilon} + c\theta^n r_n^{-1}\right].$$

Thus we have $\sum_n \mu(L_n) < +\infty$. By Borel-Cantelli lemma, for μ-a.e. x there exists $n_0(x)$ such that for any $n > n_0(x)$, $x \notin L_n$ thus $d(x, T^n x) \ge r_n$. Therefore, for any r and n such that $r_n \le r < \min\{d(x, T^j x) : j = 1, \dots, n_0(x)\}$ we have $\tau_r(x) > n$. Hence, since the set of periodic points has zero measure,

$$\liminf_{r\to0} \frac{\log \tau_r(x)}{-\log r} \ge \lim_{n\to\infty} \frac{\log n}{\log r_n} = \frac{1}{\alpha} = \delta - 4\varepsilon.$$

for μ-a.e. $x \in K_\varepsilon$. □

Lemma 33. *For* μ*-a.e.* $x \in K_\varepsilon$*, we have* $\limsup_{r\to0} \frac{\log \tau_r(x)}{-\log r} \le \delta + 2\varepsilon$.

Proof. Define
$$M_r = \{x \in K_\varepsilon \colon \tau_{2r}(x) \geq r^{-\delta-2\varepsilon}\}.$$
Let B_i be a family of balls of radius r centered at points of K_ε that covers M_r and such that the balls $\frac{1}{2}B_i$ are disjoints.

We have
$$\mu(M_r) = \mu(\cup_i B_i \cap M_r) \leq \sum_i \mu(B_i \cap M_r)$$
But observe that by the triangle inequality
$$\mu(B_i \cap M_r) \leq \mu(B_i \cap \{\tau_{B_i} \geq r^{-\delta-2\varepsilon}\}).$$
By Kač's lemma and Markov inequality this is bounded by
$$r^{\delta+2\varepsilon} \int_{B_i} \tau_{B_i} d\mu = r^{\delta+2\varepsilon}.$$
Since the number of balls B_i is bounded by $(\frac{1}{2}r)^{-\delta-\varepsilon}$, we end up with
$$\mu(M_r) \leq 2^{\delta+\varepsilon} r^\varepsilon.$$
Therefore the sequence $r_n = e^{-n}$ satisfies
$$\sum_n \mu(M_{r_n}) < +\infty.$$
By Borel Cantelli lemma, for μ-a.e. $x \in K_\varepsilon$ there exists $n_1(x)$ such that for any $n > n_1(x)$, $x \notin M_{r_n}$. Hence $\tau_{2r_n}(x) < r_n^{-\delta-2\varepsilon}$, therefore
$$\limsup_n \frac{\log \tau_{2r_n}}{-\log 2r_n} \leq \delta + 2\varepsilon.$$
The conclusion follows since $\frac{\log r_n}{\log r_{n+1}}$ converges to 1. □

Theorem 28 follows from Lemmas 31, 32 and 33.

3.2. **Repetition times, minimal distance.** In the symbolic setting, a result comparable to Theorem 28 on recurrence rate exists. Let ξ be a finite or countable measurable partition of X. We define the first repetition time of the first n-symbols by
$$R_n(x, \xi) = \min\{k \geq 1 \colon \xi_n(x) = \xi_n(T^k x)\}.$$

Theorem 34 (Ornstein-Weiss)**.** *Let (X, T, μ) be any ergodic measure preserving dynamical system. Let ξ a finite measurable partition of X. Then for μ-a.e. x we have*
$$\lim_{n\to\infty} \frac{1}{n} \log R_n(x, \xi) = h_\mu(T, \xi).$$

The initial statement was indeed for the non-overlapping return time R_n^{no} (we impose that $k \geq n$ in the definition of R_n) but Quas observed in [50] that they are a.e. eventually equal when the entropy is positive. There it is also shown that the result holds for countable partitions. Although our interest is more on smooth dynamical systems than in symbolic systems, we provide a proof of the theorem in Section 3.2.1 below.

Remark 35. *Theorem 34 can be applied with the Markov partition $\mathcal{J}$ and a Gibbs measure μ. Using the approximation argument of balls by cylinders present for example in Lemma 51, we recover the statement of Theorem 28. However this strategy, which was succesfull in the one-dimensional case* [55] *does not survive in higher dimensional systems, while the method presented in the previous section does not depend on the dimension.*

The analogy with recurrence rates is also made precise if one takes the pseudo-distance $d(x,y) = e^{-n}$ whenever n is the largest integer such that $\xi_k(x) = \xi_k(y)$ for any $k < n$. In this case one has $R_n(x,\xi) = \tau_{e^{-n}}(x)$, while the Hausdorff dimension of the ergodic measure μ is equal to the entropy $h_\mu(T,\xi)$.

Remark 36. *At the same time and independently, Boshernitzan in* [9] *established a quantitative version of the topological version of Poincaré recurrence theorem, that is Theorem 2. The statement of this elegant result is the following:*

If the α-dimensional Hausdorff measure is σ-finite on X then

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) < \infty \quad \text{for } \mu\text{-a.e. } x. \tag{5}$$

Recurrence rates are concerned with the time necessary to achieve a certain distance, while this result consider the distance as a function of time. For sure these results are correlated, and in fact it is an exercise to see that the statement (5) *implies that $\underline{R} \le \alpha$ almost surely. The reciprocal is a little bit weaker and read as follows: $\underline{R} < \alpha$ implies the statement* (5). *We do not reproduce here the proof of Boshernitzan's theorem; although much more general, it is close in spirit to our Lemma 33, where the core argument is lying in Kač's Lemma.*

3.2.1. *Repetition time and entropy.* The proof of Theorem 34 is based on Shannon-Mc Millan-Breiman theorem and some combinatorial arguments that we extract in the two lemmas below.

To simplify the notations we work directly on a space $\Sigma = \{1,\dots,p\}^{\mathbb{N}}$ for some integer p, endowed with the shift map σ and an ergodic invariant measure μ with entropy h_μ.

We call *interval* a set of consecutive integers, denote them by $[|m,n|] = \{k \in \mathbb{N} \colon m \le k \le n\}$ and denote the *singleton* $[|m,m|]$ by $[|m|]$. Given $\omega = (\omega_i)_{i\ge 0} \in \Sigma$ and $m \le n$ we denote the word $w_m \omega_{m+1} \cdots \omega_n$ by $\omega_{[|m,n|]}$. The n-repetition time reads $R_n(\omega) = \min\{k \ge 1 \colon \omega_{k+[|0,n-1|]} = \omega_{[|0,n-1|]}\}$.

We shall prove now that the exponential growth of R_n and R^{no} is governed by the entropy. Indeed, these two quantities are asymptotically equal when the entropy is positive.

Lemma 37 ([50])**.** *If then entropy $h_\mu > 0$ then $R_n(\omega) = R_n^{no}(\omega)$ eventually a.e.*

Proof. Observe that $R_n(\omega) \ne R_n^{no}(\omega)$ iff $R_n(\omega) < n$. Suppose this is the case, and let $k < n$ such that $R_n(\omega) = k$, where $\omega_{[|0,n-1|]} = \omega_{k+[|0,n-1|]}$. Hence $\omega_{[|0,k-1|]} = \omega_{[|k,2k-1|]}$.

Let $\varepsilon \in (0, h_\mu/3)$ and consider, for some integer N, the set

$$\Gamma = \Gamma(N) := \left\{ \omega \in \Sigma \colon \forall k \ge N, \left| \frac{1}{k} \log \mu(\omega_{[|0,k-1|]}) + h_\mu \right| < \varepsilon \right\}. \tag{6}$$

We can estimate the measure of the set

$$\Gamma_k := \{\omega \in \Gamma \colon \omega_{[|0,k-1|]} = \omega_{[|k,2k-1|]}\}.$$

Indeed, if we denote by ss the concatenation of a finite sequence s of length $|s| = k$, we have

$$\mu(\Gamma_k) = \sum_{|s|=k} \mu(\omega \in \Gamma_k \colon \omega_{[|0,k-1|]} = s) \leq \sum_{|s|=k} \mu(\Gamma \cap ss).$$

Remark that if $|s| = k$ and $\Gamma \cap ss \neq \emptyset$ then $\mu(ss) \leq \mathrm{e}^{-2k(h_\mu - \varepsilon)}$ and $s \cap \Gamma \neq \emptyset$, which imply that $\mu(s) \geq \mathrm{e}^{-k(h_\mu + \varepsilon)}$. Hence there can be at most $\mathrm{e}^{k(h_\mu+\varepsilon)}$ such s. Thus $\mu(\Gamma_k) \leq \mathrm{e}^{-k(h_\mu - 3\varepsilon)}$. Consequently, $\sum_k \mu(\Gamma_k) < \infty$. By Borel-Cantelli lemma, for μ-a.e. $\omega \in \Gamma$, there exists k_ω such that for all $k > k_\omega$ we have $\omega \notin \Gamma_k$. If in addition ω is not periodic then $R_n(\omega) \to \infty$ as $n \to \infty$, hence for n sufficiently large $R_n(\omega) > k_\omega$, which implies that $R_n(\omega) = R_n^{no}(\omega)$. The conclusion follows then from Shannon-McMillan-Breiman theorem, since $\mu(\cup_N \Gamma(N)) = 1$, and the measure μ is aperiodic (does not give any mass to the set of periodic points). $\square$

Given an integer L we call *pattern* a partition $\mathcal{S}$ of the interval $[|0, L-1|]$ by disjoints sub-intervals, $[|0, L-1|] = \cup_{S \in \mathcal{S}} S$. If $S = [|m, n|]$ is an element of a pattern we denote its *length* by $|S| := m - n + 1$.

For integers $1 < M < N < L$ and reals $b > 0$ and $\varepsilon \in (0, 1)$ we say that a sequence $\omega = \omega_{[|0,L-1|]} \in \{1, \ldots, p\}^L$ *follows* the pattern $\mathcal{S}$ if

- each $S \in \mathcal{S}$ is either
  - a singleton,
  - or an interval of length $|S| \in [|M, N|]$ such that for some $t \in [|M, \mathrm{e}^{bN}|]$ $\omega_{t+S} = \omega_S$, in this case we say that S is a long interval
- and the interval $[|0, L-1|]$ is almost filled by long intervals:

$$\sum_{\text{long } S \in \mathcal{S}} |S| \geq (1 - \varepsilon) L.$$

Lemma 38. *For any $\delta > 0$ there exists $\varepsilon > 0$ such that for any $M < 1/\varepsilon$, N and L, the number of admissible patterns is bounded by $\mathrm{e}^{\delta L}$.*

Proof. A number j of singletons can be in $\binom{L}{j}$ different positions in $[|0, L-1|]$. Moreover, we have $0 \leq j \leq \varepsilon L$. Hence there are at most $\sum_{j \leq \varepsilon L} \binom{L}{j}$ choices for the position of the singletons.

Each long interval S has a length at least equal to M, therefore there is at most $L/M \leq \varepsilon L$ long intervals. Once the configuration of singletons fixed, the position and the length of the long intervals is determined by the position of they left extremity, thus there are at most $\binom{L}{k}$ choices for them when there are k long intervals. Hence there is at most $\sum_{k \leq \varepsilon L} \binom{L}{k}$ choices for the configuration of the long intervals.

We then use the simple estimation

$$\sum_{j \leq \varepsilon L} \binom{L}{j} \leq \varepsilon^{-\varepsilon L} \sum_{j \leq \varepsilon L} \binom{L}{j} \varepsilon^j \leq \left(\frac{1+\varepsilon}{\varepsilon^\varepsilon} \right)^L.$$

For any $\delta > 0$, if ε is sufficiently small the number of different addmissible patterns is bounded by $\mathrm{e}^{\delta L}$. $\square$

Lemma 39. *There is at most $\mathrm{p}^{\varepsilon L} \mathrm{e}^{bL}$ admissible sequences of length L following the same pattern.*

Proof. Fix a pattern $\mathcal{S}$. We first fill the singletons. They are at most εL, which gives at most $p^{\varepsilon L}$ possibilities.

Once the configuration of singleton is fixed, we fill the long intervals, from right to left. For the first S, there exists a time $t \le \mathrm{e}^{b|S|}$ such that $\omega_S = \omega_{t+S}$. Since ω_{t+S} is on the right, it is already determined. Hence ω_S is one of the ω_{j+S}, $j \in \{N, \dots, \lfloor \mathrm{e}^{b|S|} \rfloor\}$. This leaves at most $\mathrm{e}^{b|S|}$ different choices for ω_S. We proceed similarly for the second interval, and so on and so forth. Finally, there is at most

$$\prod_{S \in \mathcal{S}} \mathrm{e}^{b|S|} \le \mathrm{e}^{bL}$$

different ways of filling the long intervals. □

Proof of Theorem 34. Let $\underline{R}(\omega) = \liminf_{n\to\infty} \dfrac{\log R_n^{no}(\omega)}{n}$ and $\overline{R}(\omega) = \limsup_{n\to\infty} \dfrac{\log R_n(\omega)}{n}$.

First we claim that $\overline{R}(\omega) \le h_\mu$ pour μ-a.e. ω. Indeed, let $\varepsilon > 0$ and $h > h_\mu + \varepsilon$. For $n \ge N$ we have

$$\begin{aligned}\mu(\Gamma(N) \cap \{R_n \ge \mathrm{e}^{nh}\}) &\le \mathrm{e}^{-nh} \int_{\Gamma(N)} R_n d\mu \\ &\le \mathrm{e}^{-nh} \sum_{|C|=n} \int_{C\cap\Gamma(N)} \tau_C d\mu \le \mathrm{e}^{-nh} e^{n(h_\mu+\varepsilon)}\end{aligned}$$

by Kač's lemma. This upper bound is summable in n, hence by Borel-Cantelli lemma we have $R_n < \mathrm{e}^{nh}$ eventually a.e. Hence $\overline{R} < h_\mu$ μ-a.e.

Assume that $h_\mu > 0$, otherwise the proof is finished. By Lemma 37, it suffices to prove that $\underline{R}(\omega) \ge h_\mu$ for μ-a.e. ω.

Since $R_n^{no}(\sigma\omega) \le R_{n+1}^{no}(\omega)$, we have $\underline{R}(\sigma\omega) \le \underline{R}(\omega)$ for all ω. This implies that $\underline{R} \circ \sigma = \underline{R}$ μ-a.e., hence $\underline{R}$ is equal to a constant b_0 a.e. Suppose for a contradiction that $b_0 < h_\mu$, and fix $b < h \in (b_0, h_\mu)$.

Let $\delta \in (0, h-b)$, take $\varepsilon > 0$ given by Lemma 38 such that $p^\varepsilon \mathrm{e}^{b+\delta} < \mathrm{e}^h$, and fix $M < 1/\varepsilon$. Let

$$A_N = \{\omega \in \Sigma \colon \exists n \in [|M, N|], \frac{\log R_n^{no}(\omega)}{n} < b\}.$$

If N is sufficiently large then $\mu(A_N) > 1 - \varepsilon/2$. Let

$$B_L = \{\omega \in \Sigma \colon \frac{1}{L}\sum_{k=0}^{L} 1_{A_N}(\sigma^k\omega) > 1 - \varepsilon/2\}.$$

If L is sufficiently large, it follows from Birkhoff ergodic theorem that $\mu(B_L) > 0.1$ and furthermore $p^{(\mathrm{e}^{bN})} < \varepsilon L/2$.

We now count the number of cylinders of length L which may contain an $\omega \in B_L$. Let $\omega \in B_L$. We see a pattern $\mathcal{S}$ in $\omega_{[|0,L-1|]}$ in that way: If $\omega \notin A_N$ then the first element of $\mathcal{S}$ is the singleton $[|0|]$, otherwise we take $[|0, n-1|]$ where n is between M and N, and $R_n^{no}(\omega) < \mathrm{e}^{bn}$. If the pattern is already constructed up to the position $k - 1 \le L - \mathrm{e}^{bN}$, then the next element will be the singleton $[|k|]$ if $\sigma^k\omega \notin A_N$, otherwise the interval $[|k, k+n-1|]$ where n is between M and N, and $R_n^{no}(\sigma^k\omega) < \mathrm{e}^{bn}$. The remaining part of the pattern is made by singletons $[|k|]$, with $k \in [|L - \mathrm{e}^{bN}, L-1|]$. There are at most $\mathrm{e}^{bN} + \varepsilon/2 < \varepsilon L$ singletons, hence the sequence $\omega_{[|0,L-1|]}$ follows the pattern $\mathcal{S}$.

By Lemma 39 there are at most $p^{\varepsilon L}\mathrm{e}^{bL}$ sequences following the pattern $\mathcal{S}$. In addition, By Lemma 38 there are at most $\mathrm{e}^{\delta L}$ patterns, hence the number of sequences is bounded by

$$p^{\varepsilon L}\mathrm{e}^{\delta L}\mathrm{e}^{bL} < \mathrm{e}^{hL}.$$

The contradiction comes from the fact that $h < h_\mu$, and $\mu(B_L) > 0.1$ for some L arbitrarily large. □

4. Fluctuation of the return time

The literature on this subject is vast and still growing rapidly in different directions. Again we will focus here on a particular aspect. We invite the reader to the reviews on this subject by Z. Coelho [15] and Abadi and Galves [1]. They are certainly an excellent starting point for a broader and also for an historical exposition on the field.

4.1. **Exponential law.**

The exponential law and the Poisson distributions are often called law of rare events. Indeed the time before the first occurrence of an event in a i.i.d. process has a geometric law; in the limit of rare events (i.e. the probability of the event is small), geometric laws are well approximated by exponential laws.

Theorem 40. *For μ-a.e. x_0, the random variable $\mu(B(x_0,r))\tau_{B(x_0,r)}(\cdot)$ converges in distribution, under the laws of μ or $\mu_{B(x_0,r)}$, to an exponential with parameter one.*

We note that most of the works on this subject are considering cylinder sets of a symbolic dynamic. Few works considering return time to balls or natural sets [16, 17, 35, 20, 19] are emerging.

If x_0 is a periodic point of period p, a large proportion of points in $B(x_0,r)$ should be back in the ball after p iterations (this depends on the measure also). Therefore the statement for return times should be false for periodic points. An extra Dirac mass at the origin of the distribution should appear in the limiting law, if it exists. Indeed, an exponential approximation for the hitting time is often valid, but with a different normalization [29].

The first approach to establish a version of the theorem was to discard points with short recurrence, e.g. the exponential law is proved for cylinders which do not recur before half of their length. And prove that this concerns almost all cylinders.

The novel approach presented here is to consider Lebesgue density points for the property that recurrence rate and dimension coincide. The advantage is that it allows to give a very short and simple proof, not based on the symbolic dynamics. This is essential when looking at return time to balls, especially in higher dimensional systems.

The next lemma exploits the basic idea that the geometric distribution appears when there is a loss of memory.

Lemma 41. *Let A be a measurable set with $\mu(A) > 0$. If*

$$\delta(A) := \sup_k |\mu(\tau_A > k) - \mu_A(\tau_A > k)|\,.$$

Then for any integer n we have

$$|\mu(\tau_A > n) - (1-\mu(A))^n| \leq \delta(A).$$

Proof. For any integer $k \geq 0$, the same argument as in (1) gives

$$\begin{aligned}\mu(\tau_A > k+1) &= \mu(\tau_A > k) - \mu(A)\mu_A(\tau_A > k)\\ &= (1-\mu(A))\mu(\tau_A > k) + \mu(A)(\mu(\tau_A > k) - \mu_A(\tau_A > k)).\end{aligned}$$

by invariance of the measure. Thus

$$|\mu(\tau_A > k+1) - (1-\mu(A))\mu(\tau_A > k)| \leq \mu(A)\delta(A).$$

Therefore, by an immediate recurrence, for any integer n

$$|\mu(\tau_A > n) - (1-\mu(A))^n| \leq \sum_{k=0}^{n-1}(1-\mu(A))^k\delta(A)\mu(A) \leq \frac{1}{\mu(A)}\delta(A)\mu(A) \leq \delta(A).$$

□

The point is now to estimate the distance $\delta(B(x_0,r))$ between the distribution of return and entrance times in $B(x_0,r)$. Clearly this has to do with the mixing property. However, for short returns the mixing may not be strong enough, therefore one has to take care of them in a special manner. For short returns, this is done by a direct application of the recurrence rate result Theorem 28.

Lemma 42. *For any $d \in (0, \dim_H \mu)$, we have*

$$\mu_{B(x_0,r)}(\tau_{B(x_0,r)} \leq r^{-d}) \to 0 \quad \text{as } r \to 0 \tag{7}$$

for μ-a.e. $x_0 \in X$. We call such x_0 a non-sticky point.

Proof. Let

$$L = \{x \in X \colon \forall r < r_0, \tau_{2r}(x) > r^{-d}\}.$$

Let $x_0 \in L$ be a Lebesgue density point of L, that is $\mu_{B(x_0,r)}(L) \to 1$ as $r \to 0$.

Let $r < r_0$. If $x \in B(x_0,r)$ and $\tau_{B(x_0,r)}(x) \leq r^{-d}$ we have $\tau_{2r}(x) \leq r^{-d}$ as well, hence $x \in L^c$. Therefore

$$\mu_{B(x_0,r)}(\tau_{B(x_0,r)} \leq r^{-d}) \leq \mu_{B(x_0,r)}(L^c).$$

The conclusion follows by taking $r_0 \to 0$, which by Theorem 28 ensures that $\mu(L) \to 1$, and the Lebesgue density theorem which says that density points form a set of full measure. □

Lemma 43. *For any $d \in (0, \dim_H \mu)$, one has for μ-a.e. $x_0 \in X$*

$$\mu(\tau_{B(x_0,r)} \leq r^{-d}) \to 0 \quad \text{as } r \to 0.$$

Proof. This is a direct consequence of the inequality

$$\mu(\tau_A \leq n) \leq n\mu(A)$$

valid for any measurable set A and the existence of the pointwise dimension. □

Inevitably, one will need a geometric measure theoretic hypothesis of the form:
Hypothesis A. x_0 is such that there exists $a > 0$ and $b \geq 0$ such that

$$\mu(B(x_0,r) \setminus B(x_0, r-\rho)) \leq r^{-b}\rho^a \tag{8}$$

for any $r > 0$ sufficiently small.

Lemma 44. *In our setting the hypothesis A is satisfied for any points x_0.*

Proof. The Gibbs property and uniform expansion implies that there exists two constants $a, b > 0$ such that for any integer k

$$\mu(\mathcal{J}_k(x)) \leq e^{-ak} \quad \text{and} \quad e^{-bk} \leq \operatorname{diam}(\mathcal{J}_k(x)).$$

Therefore any interval of length e^{-bk} has non empty intersection with at most two cylinders, hence its measure is bounded by $2e^{-ak}$. Hence there exists $c, d > 0$ such that any interval I has a measure

$$\mu(I) \leq c \operatorname{diam}(I)^d. \tag{9}$$

□

For large times, and around such a point x_0, one can use the mixing property to estimate $\delta(B(x_0, r))$.

Lemma 45. *For μ-a.e. x_0 we have $\delta(B(x_0, r)) \to 0$ as $r \to 0$.*

Proof. Let $d \in (0, \dim_H \mu)$ and let x_0 be a non-sticky point. Write for simplicity $A = B(x_0, r)$ and $E_n = \{\tau_A \geq n\}$. Let $A' = B(x_0, r - \rho)$, $g \leq n$ be an integer and define the function $\phi(x) = \max(0, 1 - \rho^{-1} d(x, A'))$. We remark that ϕ is ρ^{-1}-Lipschitz and that $1_{A'} \leq \phi \leq 1_A$. We make several approximations:

$$|\mu(A \cap E_n) - \mu(A \cap T^{-g} E_{n-g})| \leq \mu(A \cap \{\tau_A \leq g\}) \tag{10}$$

$$|\mu(A \cap T^{-g} E_{n-g}) - \int \phi 1_{E_{n-g}} \circ T^g d\mu| \leq \mu(A \setminus A') \tag{11}$$

$$|\int \phi 1_{E_{n-g}} \circ T^g d\mu - \mu(E_{n-g}) \int \phi d\mu| \leq c\theta^g \rho^{-1} \tag{12}$$

$$|\int \phi d\mu - \mu(A)| \leq \mu(A \setminus A') \tag{13}$$

$$|\mu(T^{-g} E_{n-g}) - \mu(E_n)| \leq \mu(\tau_A \leq g). \tag{14}$$

Putting together all these estimates gives

$$\begin{aligned} |\mu_A(E_n) - \mu(E_n)| &= \frac{1}{\mu(A)} |\mu(A \cap E_n) - \mu(A)\mu(E_n)| \\ &\leq \mu_A(\tau_A \leq g) + \frac{2\mu(A \setminus A')}{\mu(A)} + \frac{c}{\mu(A)} \theta^g \rho^{-1} + \mu(\tau_A \leq g). \end{aligned}$$

Observe that this upper bound holds even for $n \leq g$, so that it is also an upper bound for $\delta(B(x_0, r))$. Now we choose $g = \lfloor r^{-d} \rfloor$ and $\rho = \theta^{g/2}$. The first term goes to zero by Lemma 42, the last one by Lemma 43. The measure $\mu(A \setminus A')$ is bounded by $r^{-b} \theta^{ag/2}$ by (8) thanks to Lemma 44. This proves $\delta(B(x_0, r)) \to 0$. □

Proof of the theorem. Fix $t > 0$. We still write $A = B(x_0, r)$. Taking $n = \lfloor t/\mu(A) \rfloor$, since $\mu(\mu(A)\tau_A > t) = \mu(\tau_A > n)$ we get by Lemma 41 that

$$|\mu(\mu(A)\tau_A > t) - e^{-t}| \leq \delta(A) + |(1 - \mu(A))^n - e^{-t}|.$$

By Lemma 45 it suffices to show that the last term goes to zero when $r \to 0$. It is bounded by

$$|(1 - \frac{t}{n})^n - e^{-t}| + |(1 - \mu(A))^n - (1 - \frac{t}{n})^n|.$$

It is well known that the first term goes to zero as $n \to \infty$, and the second term is bounded by $n|\mu(A) - \frac{t}{n}| \leq \frac{t}{n}$. This shows that the hitting time, rescaled by the measure of the ball, converges in distribution to an exponential. The statement for the return time follows since the two distributions differ by $\delta(B(x_0, r))$, which goes to zero as $r \to 0$. □

Remark 46. *It was shown by Lacroix* [41] *that if instead of a ball $B(x, r)$ one allows any type of neighborhoods, then any possible limiting distributions can appear; see also* [21]*. On the other side, if a limiting distribution exists for the return time, then it also exists for the hitting time, and the two are related by an integral relation* [31]*. The only fixed point of this relation is, not by chance, the exponential distribution.*

For successive return times, one expect a Poisson limit distribution [17, 34, 30, 49, 54]*, or more generally a compound Poisson.*

5. Smallest return time in balls

We now investigate the first return time of a set $A \subset X$:

$$\tau(A) := \min\{n \geq 1 \colon A \cap T^{-n}A \neq \emptyset\} = \inf_{x \in A} \tau_A(x).$$

This quantity arised in two different contexts. First, it is the basic ingredient in the definition of the dimension for Poincaré recurrence introduced by Afraimovich [2]. The second motivation was the proof of exponential law [18, 35]. As already said before, for cylinders with a short periodic orbit the distribution of return times is not exponential. It is also related to the speed of approximation to the exponential law [51].

5.1. **Rate of recurrence for cylinders under positive entropy.**

We show now that in a symbolic system with positive entropy, the first return time of a cylinder is at least of the order of its size. This result was established in [55]. The proof presented here is from [3].

Theorem 47. *Let ξ be a finite measurable partition with strictly positive entropy $h_\mu(T, \xi)$. Then the lower rate of Poincaré recurrences for cylinders is almost surely larger than one, i.e. for μ-a.e. $x \in X$ one has*

$$\liminf_{n \to \infty} \frac{\tau(\xi_n(x))}{n} \geq 1.$$

Proof. We keep the notations from Section 3.2.1 and write the proof in the symbolic representation. Fix $\varepsilon \in (0, h_\mu/3)$. We choose N so large that $\Gamma = \Gamma(N)$ (see (6)) has a measure at least $1 - \varepsilon$. We can choose c so large that for any $\omega \in \Gamma$ and any positive integer n

$$c^{-1} e^{[-nh_\mu - n\varepsilon]} \leq \mu(C_n(\omega)) \leq c e^{[-nh_\mu + n\varepsilon]}. \tag{15}$$

Let $\delta = 1 - \frac{3}{h_\mu}\varepsilon$ and set

$$\mathcal{A}_n := \{\omega \in \Gamma \colon \tau(C_n(\omega)) \leq \delta n\}.$$

Obviously $\mathcal{A}_n = \cup_{k=1}^{\lfloor \delta n \rfloor} P_n(k)$ where

$$P_n(k) := \{\omega \in \Gamma : \tau(C_n(\omega)) = k\}.$$

We shall prove that $\sum_n \mu(\mathcal{A}_n) < \infty$. Let n be a positive integer and $0 \leq k \leq n$. If the return time of the cylinder $C = [w_0 \cdots w_{n-1}]$ is equal to k, i.e. $\tau(C) = k$, then it can be readily checked that $\omega_{j+k} = \omega_j$, for all $0 \leq j \leq n - k - 1$. This means that any block made with k consecutive symbols completely determines the cylinder C. Let

$$\mathcal{Z} = \{C_k(\omega) : \omega \in P_n(k)\}.$$

Because of the structure of cylinders under consideration, for any cylinder $Z \in \mathcal{Z}$ there is a unique cylinder C_Z of length n such that $C_Z \subset Z$ and one has $Z \cap P_n(k) \subset C_Z$. This implies

$$\mu(P_n(k)) = \sum_{Z \in \mathcal{Z}} \mu(Z \cap P_n(k)) \leq \sum_{Z \in \mathcal{Z}} \mu(C_Z).$$

But for each $Z \in \mathcal{Z}$ we have $Z \cap \Gamma \neq \emptyset$ and $C_Z \cap \Gamma \neq \emptyset$, thus there exists $\omega \in \Gamma$ such that $Z = C_k(\omega)$ and $C_Z = C_n(\omega)$. Using (15) we get

$$\mu(C_n(\omega)) \leq c \exp[-nh_\mu + n\varepsilon] \quad \text{and} \quad 1 \leq c\mu(C_k(\omega)) \exp[kh_\mu + k\varepsilon].$$

Multiplying these inequalities we get

$$\mu(C_Z) \leq c^2 \exp[-nh_\mu + n\varepsilon] \exp[kh_\mu + k\varepsilon]\mu(Z).$$

Summing up on $Z \in \mathcal{Z}$ we get (recall that $k \leq n$)

$$\mu(P_n(k)) \leq c^2 \exp[-(n-k)h_\mu + 2n\varepsilon].$$

This implies that

$$\begin{aligned} \mu(\mathcal{A}_n) &= \sum_{k=1}^{\lfloor \delta n \rfloor} \mu(P_n(k)) \\ &\leq c^2 \frac{e^{h_\mu}}{e^{h_\mu} - 1} \exp\left[-n(h_\mu - \delta h_\mu - 2\varepsilon)\right]. \end{aligned}$$

Since $h_\mu - \delta h_\mu - 2\varepsilon = h_\mu - (1 - \frac{3}{h_\mu}\varepsilon)h_\mu - 2\varepsilon = \varepsilon > 0$, we get that

$$\sum_{n \geq 1} \mu(\mathcal{A}_n) < +\infty.$$

In view of Borel-Cantelli Lemma, we finally get that for μ-almost every $\omega \in \Gamma$ $\tau(C_n(\omega)) \geq (1 - \frac{3}{h_\mu}\varepsilon)n$, except for finitely many integers n. Since in addition $\mu(\Gamma) > 1 - \varepsilon$, the arbitrariness of ε implies the desired result. □

In the special case of a Markov partition the other inequality is easy:

Proposition 48. *For the Markov partition $\mathcal{J}$ we have for any $x \in X$*

$$\limsup_{n \to \infty} \frac{\tau(\mathcal{J}_n(x))}{n} \leq 1.$$

Proof. By the Markov property, any cylinder $\mathcal{J}_n(x)$ contains a periodic point of period at most $n + m_0$. Therefore, $\tau(\mathcal{J}_n(x)) \leq n + m_0$. □

5.2. **Local rate of return for balls.** These symbolic recurrence rate can be translated to estimate return time of balls. That is to estimate $\tau(B(x, r))$.

Definition 49. *We call a point $x \in X$ super-regular with respect to a partition ξ if its orbit does not approach exponentially fast the boundary of the partition:* $\lim \frac{1}{n} \log d(T^n x, \partial\xi) = 0$.

Lemma 50. *Let μ be any Gibbs measure of an Hölder potential. Then almost every points are super-regular with respect to the Markov partition $\mathcal{J}$.*

Proof. The boundary of the partition is composed by a finite number p of points therefore by equation (9) in the proof of Lemma 44 we get

$$\mu(x\colon d(x,\partial\xi)<\varepsilon)\le pc\varepsilon^d$$

for any $\varepsilon>0$. By invariance of the measure this implies that for any $\nu>0$

$$\sum_n \mu(x\colon d(T^nx,\partial\xi)<\mathrm{e}^{-\nu n})<\infty.$$

The conclusion follows then by Borel Cantelli lemma. □

Lemma 51. *If x is super-regular with respect to the partition $\mathcal{J}$ and $\lambda>\bar\lambda(x):=\limsup\frac{1}{n}\log|(T^n)'(x)|$, we have $B(x,\mathrm{e}^{-\lambda n})\subset\mathcal{J}_n(x)$ for any n sufficiently large.*

Proof. Let $\nu>0$ be such that $\bar\lambda(x)+\nu<\lambda$. By super-regularity there exists $c>0$ such that for any integer k,

$$d(T^kx,\partial\xi)\ge c\mathrm{e}^{-\nu k}. \tag{16}$$

Let n_0 be such that for any $n>n_0$,

$$|(T^n)'(x)|\le\frac{c}{D}\mathrm{e}^{(\lambda-\nu)n}. \tag{17}$$

We claim that for any $n>n_0$, and any $k\le n$,$B(x,\mathrm{e}^{-\lambda n})\subset\mathcal{J}_k(x)$. Indeed this is true for $k=0$ since $\mathrm{e}^{-\lambda n}\le c$. Moreover, if this holds for some integer $k<n$ then we get

$$T^k(B(x,\mathrm{e}^{-\lambda n}))\subset B(T^kx,D|(T^k)'(x)|\mathrm{e}^{-\lambda n})\subset\mathcal{J}_0(T^kx),$$

by (16) and (17). Therefore the ball $B(x,\mathrm{e}^{-\lambda n})$ is contained in $\mathcal{J}_{k+1}(x)$. This proves the claim by recurrence. □

Theorem 52. *Let μ be a Gibbs measure of a Hölder potential. Then for μ-a.e. x we have*

$$\lim_{r\to0}\frac{\tau(B(x,r))}{|\log r|}=\frac{1}{\lambda_\mu}.$$

Proof. Let x be a point which has a Lyapunov exponent $\lambda(x)$ equal to λ_μ, which is super-regular with respect to the Markov partition and such that the lower recurrence rate for cylinders $\mathcal{J}_n(x)$ is at least equal to one. This concerns a.e. points by Lemma 50 and Theorem 47. By Lemma 51, for any $\lambda>\lambda_\mu$ we have

$$\liminf_{n\to\infty}\frac{\tau(B(x,\mathrm{e}^{-\lambda n}))}{|\log\mathrm{e}^{-\lambda n}|}\ge\liminf_{n\to\infty}\frac{1}{\lambda}\frac{\tau(\mathcal{J}_n(x))}{n}.$$

This proves the lower bound.

By Proposition 9 we have $\mathrm{diam}(\mathcal{J}_n(x))\le c_1|(T^n)'(x)|^{-1}$. Taking $n=n(r)$ the smallest integer such that the upper bound is less than r, we get that $\mathcal{J}_n(x)\subset B(x,r)$. The conclusion follows now by Proposition 48. □

Remark 53. *The upper bound $\limsup_{r\to0}\frac{\tau(B(x,r))}{|\log r-|}\le\frac{1}{\lambda_\mu}$ still holds in higher dimension for (non-conformal) expanding maps, under some Markov assumption. The lower bound in the first part of the proof may be generalized to higher dimensional dynamical systems* [56]*, under a weak regularity condition (of the type in* [39] *which ensures the existence of Lyapunov charts). In that case one has to replace λ_μ by the largest Lyapunov exponent $\Lambda_\mu:=\lim\frac{1}{n}\int\log\|d_xT^n\|d\mu$.*

Unfortunately these two inequalities only give a range of possible values for the local rate of return for balls. There are examples [56] *where the bounds are attained, and also where there are not sharp. This suggests that the existence and the computation of the local rate of return in the non-conformal case is still far away.*

5.3. **Dimension for Poincaré recurrence.** These rate of return for balls are the base ingredient of the definition of the dimension for Poincaré recurrence, or Afraimovich-Pesin dimension [2, 47]. Define for $A \subset X$, $q \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ the quantity

$$M(A, q, \alpha) = \lim_{\varepsilon \to 0} \inf_{\{B_i\}} \sum_i \mathrm{e}^{-q\tau(B_i)} (\operatorname{diam} B_i)^\alpha$$

where the infimum is taken among all countable covers of A by balls B_i. Let $\alpha(A, q)$ denote the transition point of $M(A, q, \alpha)$ from $+\infty$ to zero. The spectrum $\alpha(\cdot, \cdot)$ is a generalization of the Hausdorff dimension and has been introduced and computed for some geometric constructions and Markov maps of the interval [4, 23]. The behavior of $\tau(B(x, r))/|\log r|$ is closely related to a corresponding pointwise dimension [14, 3]. This spectrum has also been computed for surface diffeomorphisms [56] and for a general class of interval maps [36].

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