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MULTIFRACTAL ANALYSIS OF HYPERBOLIC FLOWS

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Abstract. We establish the multifractal analysis of hyperbolic flows and of suspension flows over subshifts of finite type. A non-trivial consequence of our results is that for every Hölder continuous function non-cohomologous to a constant, the set of points without Birkhoff average has full topological entropy.

1. Introduction

Much attention has been given by physicists and applied mathematicians to the study of chaotic behavior. Several techniques were put forward as a mean to deal with the enormous amount of data provided by the associated time series. In particular, there has been a growing interest in the study of multifractal spectra, such as the dimension spectrum for pointwise dimensions. These spectra conveniently encode information about the "multifractal" structure of complicated invariant sets. The rigorous mathematical theory of multifractal analysis has been quite developed during the last decade. We refer the reader to the book [7] for the description of results, and for a list of references.

We briefly describe here the main elements of multifractal analysis. Let $T\colon X \to X$ be a continuous map of a compact metric space, and $g\colon X \to \mathbb{R}$ a continuous function. For each $\alpha \in \mathbb{R}$, let

$$K_\alpha = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} g(T^i x) = \alpha \right\}.$$

We also consider the set

$$K = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} g(T^i x) \text{ does not exist} \right\}.$$

Clearly

$$X = K \cup \bigcup_{\alpha\in\mathbb{R}} K_\alpha. \tag{1}$$

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This union is formed by pairwise disjoint T-invariant sets, and is called a *multifractal decomposition* of X.

For each $\alpha \in \mathbb{R}$ such that $K_\alpha \neq \varnothing$, set

$$\mathcal{D}(\alpha) = \dim_H K_\alpha,$$

where $\dim_H Z$ denotes the Hausdorff dimension of Z. Given $Z \subset X$ and $\alpha > 0$, recall that $\dim_H Z = \inf\{\alpha : m(Z, \alpha) = 0\}$, where

$$m(Z, \alpha) = \lim_{\delta \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha,$$

and the infimum is taken over all finite or countable covers $\mathcal{U}$ of Z by sets of diameter at most δ. The function $\mathcal{D}$ is called *dimension spectrum for the Birkhoff averages of* g, and is one of the main elements of multifractal analysis.

By Birkhoff's ergodic theorem, if μ is a T-invariant finite ergodic measure on X, and $\alpha = \int_X g \, d\mu/\mu(X)$, then $\mu(K_\alpha) = \mu(X)$. That is, there exists a set K_α in the multifractal decomposition with full μ-measure. Of course that this does not mean that the other sets in the multifractal decomposition are empty. In fact, for several classes of hyperbolic dynamical systems it has been proved that:

1. if $K_\alpha \neq \varnothing$, then K_α is a proper dense set;
2. the set $\{\alpha \in \mathbb{R} : K_\alpha \neq \varnothing\}$ is an interval (in particular it contains an uncountable number of points);
3. the function $\mathcal{D}$ is real analytic and strictly convex;
4. the irregular set K is everywhere dense and has full Hausdorff dimension, that is, $\dim_H K = \dim_H X$.

This implies that the multifractal decomposition in (1) is composed of an uncountable number of T-invariant sets, all being everywhere dense, and all having positive Hausdorff dimension. Thus, multifractal analysis reveals a very rich "multifractal" structure for hyperbolic dynamical systems. In particular, this analysis has been effected when g is a Hölder continuous function, and T is either a subshift of finite type, an expanding map, or an axiom A diffeomorphism. We refer to [7] for details and a list of references.

One of the main objectives of our paper is to establish a version of the multifractal analysis for a class of hyperbolic flows and suspension flows over subshifts of finite type. In the multifractal analysis of a flow $\Phi = \{\varphi_t\}_t$ on X, the sets K_α and K are replaced respectively by

$$K_\alpha = \left\{ x \in X : \lim_{t \to \infty} \frac{1}{t} \int_0^t g(\varphi_\tau x) \, d\tau = \alpha \right\}$$

and

$$K = \left\{ x \in X : \lim_{t \to \infty} \frac{1}{t} \int_0^t g(\varphi_\tau x) \, d\tau \text{ does not exist} \right\}.$$

Recall that a set $A \subset X$ is Φ-invariant if $\varphi_t A = A$ for every $t \in \mathbb{R}$. Each of the sets K_α and K are Φ-invariant.

For several classes of hyperbolic flows we establish Properties 1, 2, 3, and 4 above. For example, we can give a complete description when Φ is the geodesic flow on a compact surface with negative curvature.

Recall that a Borel finite measure μ on X is Φ-invariant if $\mu(\varphi_t A) = \mu(A)$ for every measurable set $A \subset X$ and every $t \in \mathbb{R}$. If μ is ergodic, i.e., if Φ-invariant measurable set has either zero or full μ-measure. By Birkhoff's ergodic theorem, if $g\colon X \to \mathbb{R}$ is a μ-integrable measurable function then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(\varphi_\tau x)\, d\tau = \frac{1}{\mu(X)}\int_X g\, d\mu \tag{2}$$

for μ-almost every $x \in X$. Therefore, it is a rare event from the point of view of measure theory that the limit in (2) does not exist for a given point $x \in X$. Property 4 above shows that, surprisingly, for suspension flows over subshifts of finite type, and a generic Hölder continuous function g, the set of points where the limit in (2) does not exist is everywhere dense and has *full* topological entropy. In particular, from the point of view of topology it is a rather common event that the limit in (2) does not exist for a given point $x \in X$. Our results are counterparts of the corresponding results for diffeomorphisms on hyperbolic sets developed by Barreira and Schmeling in [3].

The main theme of our proofs is to use Markov systems and the associated symbolic dynamics developed by Bowen [4] and Ratner [10] to reduce the setup for flows to the setup for maps, and then apply the results that are already available in the case of maps. This is done through a study of suspension flows over subshifts of finite type associated to Markov systems, and a careful analysis of the relation between cohomology for flows and cohomology for the maps associated to Markov systems.

After the completion of this draft, we learned that Pesin and Sadovskaya [8] recently obtained results related to ours. They use a different approach, involving the construction of Moran covers associated to Markov systems.

2. Hyperbolic flows

2.1. **Preliminaries.** Let $\Phi = \{\varphi_t\}_t$ be a C^1 flow of the smooth compact manifold M. A Φ-invariant set $\Lambda \subset M$ is called *hyperbolic* for Φ if there exists a continuous splitting $T_\Lambda M = E^s \oplus E^u \oplus E^0$, and constants $c > 0$ and $\lambda \in (0,1)$ such that for each $x \in \Lambda$ the following properties hold:

1. $\frac{d}{dt}(\varphi_t x)|_{t=0}$ generates $E^0(x)$;
2. $d_x\varphi_t E^s(x) = E^s(\varphi_t x)$ and $d_x\varphi_t E^u(x) = E^u(\varphi_t x)$ for each $t \in \mathbb{R}$;
3. $\|d_x\varphi_t v\| \le c\lambda^t\|v\|$ for every $v \in E^s(x)$ and $t > 0$;
4. $\|d_x\varphi_{-t} v\| \le c\lambda^t\|v\|$ for every $v \in E^u(x)$ and $t > 0$.

For example, geodesic flows on compact Riemannian manifolds with negative sectional curvature have the whole manifold as a hyperbolic set. Furthermore, time changes and small C^1 perturbations of flows with a hyperbolic set also possess a hyperbolic set.

A closed Φ-invariant set $\Lambda \subset M$ is called a *basic set* of Φ if Λ is hyperbolic, locally maximal, topologically transitive, and the periodic orbits of Φ are dense in Λ.

2.2. **Irregular sets.** For each continuous function $g\colon \Lambda \to \mathbb{R}$ we define the *irregular set for the Birkhoff averages of* g (with respect to $\Phi = \{\varphi_t\}_t$) by

$$\mathcal{B}(g) = \left\{ x \in \Lambda : \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi_\tau x)\, d\tau \text{ does not exist} \right\}.$$

One can easily verify that $\mathcal{B}(g)$ is Φ-invariant. By Birkhoff's ergodic theorem, the set $\mathcal{B}(g)$ has zero measure with respect to any Φ-invariant finite measure.

We say that $g\colon \Lambda \to \mathbb{R}$ is Φ-*cohomologous* to a function $h\colon \Lambda \to \mathbb{R}$ on Λ if there exists a bounded measurable function $q\colon \Lambda \to \mathbb{R}$ such that

$$g(x) - h(x) = \lim_{t\to 0} \frac{q(\varphi_t x) - q(x)}{t} \tag{3}$$

for every $x \in \Lambda$. If $g\colon \Lambda \to \mathbb{R}$ is Φ-cohomologous to a constant $c \in \mathbb{R}$ on Λ, then

$$\begin{aligned}
\left| \frac{1}{t} \int_0^t g(\varphi_\tau x)\, d\tau - c \right| &= \frac{1}{t} \lim_{s\to 0} \frac{1}{s} \left| \int_s^{s+t} q(\varphi_\tau x)\, d\tau - \int_0^t q(\varphi_\tau x)\, d\tau \right| \\
&= \frac{1}{t} \lim_{s\to 0} \frac{1}{s} \left| \int_t^{s+t} q(\varphi_\tau x)\, d\tau - \int_0^s q(\varphi_\tau x)\, d\tau \right| \\
&\le \frac{2 \sup|q|}{t}
\end{aligned} \tag{4}$$

for every $x \in \Lambda$ and $t > 0$, and hence, $\mathcal{B}(g) = \varnothing$.

We now present the main result of this section. It shows that for hyperbolic flows, if $g\colon \Lambda \to \mathbb{R}$ is not Φ-cohomologous to a constant, then *the set* $\mathcal{B}(g)$ *is non-empty, is everywhere dense, and has full topological entropy.* See Section 4.1 for the definition of topological entropy $h(\Phi|Z)$ on an arbitrary set Z (not necessarily compact nor invariant).

Theorem 1. *Let Λ be a compact basic set of a topologically mixing $C^{1+\varepsilon}$ flow Φ, for some $\varepsilon > 0$, and let $g\colon \Lambda \to \mathbb{R}$ be a Hölder continuous function. Then the following properties are equivalent:*

1. *g is not Φ-cohomologous to a constant on Λ;*
2. *$\mathcal{B}(g)$ is a non-empty proper dense set with*

$$h(\Phi|\mathcal{B}(g)) = h(\Phi|\Lambda). \tag{5}$$

In [3], Barreira and Schmeling studied irregular sets with respect to diffeomorphisms on hyperbolic sets. Theorem 1 is a counterpart of their results in the case of flows, and follows from the more general statements formulated below.

We now show that "most" Hölder continuous functions are not Φ-cohomologous to a constant. Let $C^\alpha(\Lambda)$ be the space of Hölder continuous functions on Λ with Hölder exponent α. For a function $\varphi \in C^\alpha(\Lambda)$ we define its norm by

$$\|\varphi\|_\alpha = \sup\{|\varphi(x)|\colon x \in \Lambda\} + \sup\left\{\frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\alpha} : x,\, y \in \Lambda \text{ and } x \neq y\right\},$$

where d denotes the distance on M.

Theorem 2. *Let Λ be a compact basic set of a topologically transitive C^1 flow Φ. Then, for each $\alpha \in (0,1)$, the family of functions in $C^\alpha(\Lambda)$ which are not Φ-cohomologous to a constant is open and dense in $C^\alpha(\Lambda)$.*

Theorems 1 and 2 immediately imply the following statement, whose formulation has the advantage of not using the notion of cohomology.

Theorem 3. *Let Λ be a compact basic set of a topologically mixing $C^{1+\varepsilon}$ flow Φ, for some $\varepsilon > 0$. Given $\alpha > 0$, for an open and dense family of functions $g \in C^\alpha(\Lambda)$, the set $\mathcal{B}(g)$ is a non-empty proper dense set with $h(\Phi|\mathcal{B}(g)) = h(\Phi|\Lambda)$.*

2.3. **Multifractal analysis.** Let $g\colon \Lambda \to \mathbb{R}$ be a continuous function. For each $\alpha \in \mathbb{R}$, consider the set

$$K_\alpha = \left\{x \in \Lambda : \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi_\tau x)\, d\tau = \alpha\right\}.$$

One can easily verify that K_α is Φ-invariant. By (4), if g is Φ-cohomologous to a constant $c \in \mathbb{R}$ on Λ, then $K_c = \Lambda$.

Given $\alpha \in \mathbb{R}$, set

$$\mathcal{E}(\alpha) = h(\Phi|K_\alpha).$$

The function $\mathcal{E}$ is called the *entropy spectrum for the Birkhoff averages of g*.

For every real number q, let ν_q be the equilibrium measure of qg, and write

$$T(q) = P_\Phi(qg),$$

where $P_\Phi(qg)$ is the topological pressure of qg with respect to Φ. It is well known that T is a real analytic function. We denote by $h_\nu(\Phi|\Lambda)$ the entropy of $\Phi|\Lambda$ with respect to the Φ-invariant measure ν. See Section 4.1 for the definition.

We now present a multifractal analysis of the spectrum $\mathcal{E}$ on basic sets.

Theorem 4. *Let Λ be a compact basic set of a topologically mixing $C^{1+\varepsilon}$ flow Φ, for some $\varepsilon > 0$, and let $g\colon \Lambda \to \mathbb{R}$ be a Hölder continuous function with $P_\Phi(g) = 0$. Then the following properties hold:*

1. *the domain of $\mathcal{E}$ is a closed interval in $[0,\infty)$, which coincides with the range of the function $\alpha = -T'$, and if $q \in \mathbb{R}$ then*

$$\mathcal{E}(\alpha(q)) = T(q) + q\alpha(q) = h_{\nu_q}(\Phi|\Lambda);$$

2. *if g is not Φ-cohomologous to a constant on Λ, then $\mathcal{E}$ and T are real analytic strictly convex functions.*

See Section 4.2 for a more detailed description of the spectrum $\mathcal{E}$.

2.4. **Markov systems.** Let Λ be a compact basic set of the C^1 flow $\Phi = \{\varphi_t\}_t$, and let

$$V^s_\varepsilon(x) = \{y \in B(x,\varepsilon) : d(\varphi_t y, \varphi_t x) \to 0 \text{ as } t \to +\infty\}$$

and

$$V^u_\varepsilon(x) = \{y \in B(x,\varepsilon) : d(\varphi_t y, \varphi_t x) \to 0 \text{ as } t \to -\infty\}$$

be the local stable and unstable manifolds of size ε at the point $x \in \Lambda$. For each sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if x, $y \in \Lambda$ are at a distance $d(x,y) \le \delta$ then there is a unique time $t = t(x,y) \in [-\varepsilon,\varepsilon]$ for which the set $[x,y] \stackrel{\text{def}}{=} V^s_\varepsilon(\varphi_t x) \cap V^u_\varepsilon(y)$ consists of a single point, and $[x,y] \in \Lambda$.

Let $D \subset M$ be an open disk of dimension $\dim M - 1$ which is transversal to the flow Φ, and let $x \in D$. There exists a diffeomorphism from $D \times (-\varepsilon,\varepsilon)$ onto an open neighborhood $U(x)$ of x. The projection map $\pi_D \colon U(x) \to D$ defined by $\pi_D(\varphi_t y) = y$ is differentiable. A closed set $R \subset \Lambda \cap D$ is called a *rectangle* if $R = \overline{\operatorname{int} R}$ (where the interior is computed with respect to the topology of $\Lambda \cap D$), and $\pi_D[x,y] \in R$ whenever $x, y \in R$.

Consider a collection of rectangles $R_1, \ldots, R_k \subset \Lambda$ (each contained in some disk transversal to the flow) with $R_i \cap R_j = \partial R_i \cap \partial R_j$ for $i \ne j$ such that there exists $\varepsilon > 0$ with:

1. $\Lambda = \bigcup_{t\in[0,\varepsilon]} \varphi_t(\bigcup_{i=1}^k R_i)$;
2. for each $i \ne j$ either $(\varphi_t R_i) \cap R_j = \varnothing$ for all $t \in [0,\varepsilon]$ or $(\varphi_t R_j) \cap R_i = \varnothing$ for all $t \in [0,\varepsilon]$.

We define the *transfer function* $\tau \colon \Lambda \to [0,\infty)$ by

$$\tau(x) = \min\left\{ t > 0 : \varphi_t x \in \bigcup_{i=1}^k R_i \right\}.$$

Let $T \colon \Lambda \to \bigcup_{i=1}^k R_i$ be the *transfer map* given by $Tx = \varphi_{\tau(x)} x$. We note that the restriction of T to $\bigcup_{i=1}^k R_i$ is invertible.

We say that the rectangles R_1, …, R_k form a *Markov system* for Φ on Λ if

$$T(\operatorname{int}(V^s_\varepsilon(x) \cap R_i)) \subset \operatorname{int}(V^s_\varepsilon(Tx) \cap R_j)$$

and

$$T^{-1}(\operatorname{int}(V^u_\varepsilon(Tx) \cap R_j)) \subset \operatorname{int}(V^u_\varepsilon(x) \cap R_i)$$

whenever $x \in \operatorname{int} TR_i \cap \operatorname{int} R_j$. Any basic set Λ of a C^1 flow possesses Markov systems of arbitrary small diameter (see [4, 10]). Furthermore, the map τ is Hölder continuous on each domain of continuity, and

$$0 < \inf_{x\in\Lambda} \tau \le \sup_{x\in\Lambda} \tau < \infty. \tag{6}$$

Given a Markov system R_1, …, R_k for Φ on the basic set Λ we define a $k \times k$ matrix A with entries $a_{ij} = 1$ if $\operatorname{int} TR_i \cap \operatorname{int} R_j \ne \varnothing$, and $a_{ij} = 0$

otherwise. Consider the set $X \subset \{1, \ldots, k\}^{\mathbb{Z}}$ defined by

$$X = \{(\cdots i_{-1} i_0 i_1 \cdots) : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{Z}\},$$

and the shift map $\sigma \colon X \to X$ given by $\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots)$, where $j_n = i_{n+1}$ for every $n \in \mathbb{Z}$. The map $\sigma|X$ is called a (two-sided) *subshift of finite type* with *transfer matrix* A. We fix $\beta > 1$ and equip X with the distance d_X defined by

$$d_X((\cdots i_{-1} i_0 i_1 \cdots), (\cdots j_{-1} j_0 j_1 \cdots)) = \sum_{n=-\infty}^{\infty} \beta^{-|n|} |i_n - j_n|. \tag{7}$$

We define a *coding map* $\pi \colon X \to \bigcup_{i=1}^{k} R_i$ for the basic set by

$$\pi(\cdots i_0 \cdots) = \bigcap_{j \in \mathbb{Z}} \overline{T^{-j} \operatorname{int} R_{i_j}}.$$

One can easily check that $\pi \circ \sigma = T \circ \pi$. As observed in [4], it is always possible to choose the constant β in such a way that the function $\tau \circ \pi \colon X \to [0, \infty)$ is Lipschitz.

Markov systems will be used in the proof of Theorem 1.

2.5. **Cohomology for flows and maps.** We now discuss the cohomology assumption in Theorem 1. We show how to use a Markov system to reduce this assumption to a cohomology assumption using the associated transfer map instead of the original flow. This relation is crucial to our approach.

Given a continuous function $g \colon \Lambda \to \mathbb{R}$ and a Markov system for the flow $\Phi = \{\varphi_t\}_t$ on the basic set Λ with transfer function $\tau \colon \Lambda \to [0, \infty)$, we define a new function $I_g \colon \Lambda \to \mathbb{R}$ by

$$I_g(x) = \int_0^{\tau(x)} g(\varphi_s x)\, ds.$$

In particular, if $c \in \mathbb{R}$, then $I_c = c\tau$.

We say that a function $G \colon \Lambda \to \mathbb{R}$ is *T-cohomologous* to a function $H \colon \Lambda \to \mathbb{R}$ on Λ if there exists a bounded measurable function $q \colon \Lambda \to \mathbb{R}$ such that

$$G - H = q \circ T - q \text{ on } \Lambda.$$

Theorem 5. *Let Λ be a basic set of the C^1 flow Φ, $g \colon \Lambda \to \mathbb{R}$ and $h \colon \Lambda \to \mathbb{R}$ continuous functions, and τ the transfer function of some Markov system for Φ on Λ. Then the following properties are equivalent:*

1. *g is Φ-cohomologous to h on Λ and* (3) *holds for every $x \in \Lambda$;*
2. *I_g is T-cohomologous to I_h on Λ with*

$$I_g(x) - I_h(x) = q(Tx) - q(x) \textit{ for every } x \in \Lambda.$$

Theorem 5 allow us to translate the results obtained in [3] in the setting of subshifts of finite type and hyperbolic sets to the setting of hyperbolic flows.

Theorem 5 implies that a function g is Φ-cohomologous to a constant $c \in \mathbb{R}$ if and only if I_g is T-cohomologous to $c\tau$. In particular, the cohomology assumption in Theorem 1 can be replaced by one in terms of the transfer map T (associated to some Markov system). Therefore, it would be of interest to also describe the convergence and the non-convergence of the Birkhoff averages of the flow Φ in terms of T. This is effected in the following statement.

Proposition 6. *Let Λ be a basic set of the C^1 flow $\Phi = \{\varphi_t\}_t$, $g\colon \Lambda \to \mathbb{R}$ a continuous function, and τ the transfer function of some Markov system for Φ on Λ. Then the following properties hold:*

1. *if $g\colon \Lambda \to \mathbb{R}$ is Hölder continuous, then I_g is Hölder continuous on each domain of continuity of τ;*
2. *if $x \in \Lambda$, then*
$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t g(\varphi_s x)\,ds = \liminf_{m\to\infty} \frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m \tau(T^i x)}$$
and
$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t g(\varphi_s x)\,ds = \limsup_{m\to\infty} \frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m \tau(T^i x)};$$
3. $$\mathcal{B}(g) = \left\{ x \in \Lambda : \lim_{m\to\infty} \frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m \tau(T^i x)} \text{ does not exist} \right\}. \tag{8}$$

The identity (8) tells us that any irregular set for a hyperbolic flow can be described in terms of the map T. However, contrarily to the maps considered in [3], T is not invertible nor hyperbolic.

3. Suspension flows

3.1. **Preliminaries.** Let $T\colon X \to X$ be a homeomorphism of the compact metric space X, and $\tau\colon X \to (0,\infty)$ a Lipschitz function. Consider the space
$$Y = \{(x,s) \in X \times \mathbb{R} : 0 \le s \le \tau(x)\}, \tag{9}$$
with the points $(x,\tau(x))$ and $(Tx,0)$ identified for each $x \in X$. One can introduce in a natural way a topology on Y which makes Y a compact topological space. This topology is induced by a distance introduced by Bowen and Walters in [5] (see Appendix A for details). The metric structure shall first be used in Section 3.2.

The *suspension flow over T with height function τ* is the flow $\Psi = \{\psi_t\}_t$ on Y where $\psi_t\colon Y \to Y$ is defined by
$$\psi_t(x,s) = (x, s+t). \tag{10}$$

We extend τ to a function $\tau\colon Y \to \mathbb{R}$ by
$$\tau(y) = \min\{t > 0 : \psi_t y \in X \times \{0\}\},$$

and extend T to a map $T\colon Y \to X \times \{0\}$ by

$$T(y) = \psi_{\tau(y)} y.$$

Since there is no danger of confusion we continue to use the symbols τ and T for the extensions. Given a continuous function $g\colon Y \to \mathbb{R}$ we define a function $I_g\colon Y \to \mathbb{R}$ by

$$I_g(y) = \int_0^{\tau(y)} g(\psi_s y)\, ds. \tag{11}$$

Theorem 7. *If $\Psi = \{\psi_t\}_t$ is a suspension flow on Y over $T\colon X \to X$, and $g\colon Y \to \mathbb{R}$ and $h\colon Y \to \mathbb{R}$ are continuous functions, then the following properties are equivalent:*

1. *g is Ψ-cohomologous to h on Y with*
$$g(y) - h(y) = \lim_{t \to 0} \frac{q(\psi_t y) - q(y)}{t} \text{ for every } y \in Y;$$
2. *I_g is T-cohomologous to I_h on Y with*
$$I_g(y) - I_h(y) = q(Ty) - q(y) \text{ for every } y \in Y;$$
3. *$I_g|X \times \{0\}$ is T-cohomologous to $I_h|X \times \{0\}$ on $X \times \{0\}$ with*
$$I_g(y) - I_h(y) = q(Ty) - q(y) \text{ for every } y \in X \times \{0\}.$$

By Theorem 7 (see Properties 2 and 3), each cohomology class in the base space X induces a cohomology class in the whole space Y, and all cohomology classes in Y appear in this way.

We also obtain a version of Proposition 6 for suspension flows.

Proposition 8. *Let $\Psi = \{\psi_t\}_t$ be a suspension flow on Y over $T\colon X \to X$ with height function τ, and $g\colon Y \to \mathbb{R}$ a continuous function. If $x \in X$ and $s \in [0, \tau(x)]$, then*

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t g(\psi_\tau(x,s))\, d\tau = \liminf_{m \to \infty} \frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m \tau(T^i x)} \tag{12}$$

and

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t g(\psi_\tau(x,s))\, d\tau = \limsup_{m \to \infty} \frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m \tau(T^i x)}. \tag{13}$$

Note that for a fixed $x \in X$ the limits in (12) and (13) are independent of s.

One can also consider the case when $T\colon X \to X$ is continuous but not necessarily a homeomorphism. More precisely, let T be a local homeomorphism in an open neighborhood of each point of the compact metric space X, $\tau\colon X \to (0, \infty)$ a Lipschitz function, and Y as in (9). Note that even if X is a topological manifold and τ is a constant function, then Y may not be a topological manifold. The *suspension semi-flow over T with height function τ* is the semi-flow $\Psi = \{\psi_t\}_t$ on Y where $\psi_t\colon Y \to Y$ is defined by (10). The statements in Theorem 7 and Proposition 8 also hold for suspension semi-flows.

3.2. **Suspension flows over subshifts of finite type.** Let now $\Psi = \{\psi_t\}_t$ be a suspension flow on Y over $T\colon X \to X$. The space Y is equipped with the Bowen–Walters distance (see Appendix A for the definition). Let $g\colon Y \to \mathbb{R}$ be a continuous function. For each $\alpha \in \mathbb{R}$, set

$$\mathcal{E}(\alpha) = h(\Psi|K_\alpha),$$

where

$$K_\alpha = \left\{x \in Y : \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\psi_\tau x)\, d\tau = \alpha\right\}.$$

The topological entropy is computed with respect to (the topology induced by) the Bowen–Walters distance on Y. The function $\mathcal{E}$ is called the *entropy spectrum for the Birkhoff averages of* g. For every real number q, let ν_q be the equilibrium measure of qg, and write

$$T(q) = P_\Psi(qg).$$

The following is a version of Theorem 4 for suspension flows over subshifts of finite type.

Theorem 9. *Let Ψ be a suspension flow on Y over a topologically mixing two-sided subshift of finite type, and $g\colon Y \to \mathbb{R}$ a Hölder continuous function with $P_\Psi(g) = 0$. Then the following properties hold:*

1. *the domain of $\mathcal{E}$ is a closed interval in $[0,\infty)$, which coincides with the range of the function $\alpha = -T'$, and if $q \in \mathbb{R}$ then*
$$\mathcal{E}(\alpha(q)) = T(q) + q\alpha(q) = h_{\nu_q}(\Psi);$$
2. *if g is not Ψ-cohomologous to a constant on Y, then $\mathcal{E}$ and T are real analytic strictly convex functions.*

Given a continuous function $g\colon Y \to \mathbb{R}$ we consider the irregular set

$$\mathcal{B}(g) = \left\{y \in Y : \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\psi_\tau y)\, d\tau \text{ does not exist}\right\}.$$

Set

$$\mathcal{C} = \left\{x \in X : \lim_{m\to\infty} \frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m \tau(T^i x)} \text{ does not exist}\right\}.$$

For a suspension flow Ψ and a continuous function g on Y, it follows from Proposition 8 that

$$\mathcal{B}(g) = \{(x,s) \in Y : x \in \mathcal{C} \text{ and } s \in [0,\tau(x)]\}.$$

We now formulate a version of Theorem 1 for suspension flows over subshifts of finite type.

Theorem 10. *Let Ψ be a suspension flow on Y over a topologically mixing two-sided subshift of finite type, and $g\colon Y \to \mathbb{R}$ a Hölder continuous function. Then the following properties are equivalent:*

1. *g is not Ψ-cohomologous to a constant on Y;*
2. *$\mathcal{B}(g)$ is a non-empty proper dense set with*
$$h(\Psi|\mathcal{B}(g)) = h(\Psi). \tag{14}$$

Abramov's entropy formula shows that

$$h(\Psi) = \sup_{\mu} \frac{h_\mu(T)}{\int_X \tau \, d\mu} = \frac{h_\nu(T)}{\int_X \tau \, d\nu},$$

where the supremum is taken over all T-invariant probability measures. Here, $h_\mu(T)$ is the entropy of T with respect to μ, and ν is the equilibrium measure of $-h(\Psi)\tau$.

One can also consider one-sided subshifts of finite type $T\colon X \to X$. It is easy to verify that in this case T is a local homeomorphism in an open neighborhood of each point. The statements in Theorem 10 hold for suspension semi-flows over one-sided subshifts of finite type. See also Section 5.1 below.

Given a basic set of a hyperbolic flow, each Markov system has naturally associated a suspension flow over a two-sided subshift of finite type. In fact these are the primary examples of suspensions flows. We now describe this construction. If Λ is a basic set of the C^1 flow $\Phi = \{\varphi_t\}_t$, then given a Markov system there is an associated transfer function $\tau\colon \Lambda \to \mathbb{R}$ (which is Hölder continuous on each domain of continuity), and an associated two-sided subshift of finite type $\sigma\colon X \to X$ with coding map $\pi\colon X \to \Lambda$ (see Section 2.4). Therefore, to each Markov system one can naturally associate the suspension flow $\Psi = \{\psi_t\}_t$ on Y over σ with Lipschitz height function $\tau \circ \pi$ (see Section 2.4). We extend π to a finite-to-one surjection $\pi\colon Y \to \Lambda$ by $\pi(x,s) = (\varphi_s \circ \pi)(x)$ for every $(x,s) \in Y$. Then

$$\pi \circ \psi_t = \varphi_t \circ \pi. \tag{15}$$

Observe that the function $g \circ \pi\colon Y \to \mathbb{R}$ is Hölder continuous whenever $g\colon \Lambda \to \mathbb{R}$ is Hölder continuous. Using (15) one can show that

$$\mathcal{B}(g) = \pi(\mathcal{B}(g \circ \pi)).$$

This can be used to establish the identity in (5) from the identity in (14).

4. Multifractal analysis and irregular sets

4.1. **A new Carathéodory dimension for flows.** We introduce here a new Carathéodory dimension characteristic for flows. It is a generalization of the topological entropy, and is a flow version of a Carathéodory dimension characteristic introduced in [3] in the case of maps.

Let $\Psi = \{\psi_t\}_t$ be a continuous flow of the compact metric space (Y,d). Given $x \in Y$, $t > 0$, and $\varepsilon > 0$, we write

$$B(x,t,\varepsilon) = \{y \in Y : d(\psi_\tau y, \psi_\tau x) < \varepsilon \text{ whenever } 0 \le \tau \le t\}$$

Let $u\colon Y \to \mathbb{R}$ be a strictly positive continuous function. We write

$$U(x,t,\varepsilon) = \sup \left\{ \int_0^t u(\psi_\tau y)\, d\tau : y \in B(x,t,\varepsilon) \right\}$$

if $B(x,t,\varepsilon) \ne \varnothing$, and $U(x,t,\varepsilon) = -\infty$ otherwise.

For each set $Z \subset Y$ and each $\alpha \in \mathbb{R}$, we define

$$M(Z, \alpha, u, \varepsilon) = \lim_{T\to\infty} \inf_{\Gamma} \sum_{(x,t)\in\Gamma} \exp(-\alpha U(x, t, \varepsilon)),$$

where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i, t_i)\}_i$ such that $(x_i, t_i) \in Y \times [T, \infty)$ for each i, and $\bigcup_i B(x_i, t_i, \varepsilon) \supset Z$. We define the number

$$\dim_{u,\varepsilon} Z = \inf\{\alpha \colon M(Z, \alpha, u, \varepsilon) = 0\}.$$

The limit

$$\dim_u Z \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \dim_{u,\varepsilon} Z$$

exists, and is called the *u-dimension* of Z (with respect to Ψ).

If u is the constant function equal to 1, then $\dim_u Z$ is called *topological entropy of Ψ on Z*, and is denoted by $h(\Psi|Z)$. If Z is compact and Ψ-invariant, then we recover the well-known notion of topological entropy

$$h(\Psi|Z) = \lim_{\varepsilon\to 0} \liminf_{t\to\infty} \frac{\log N_Z(t, \varepsilon)}{t} = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} \frac{\log N_Z(t, \varepsilon)}{t},$$

where $N_Z(t, \varepsilon)$ is the least number of sets $B(x, t, \varepsilon)$ needed to cover Z.

For every Borel probability measure ν on Y, let

$$\dim_{u,\varepsilon} \nu = \inf\{\dim_{u,\varepsilon} Z \colon \nu(Z) = 1\}.$$

The limit

$$\dim_u \nu \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \dim_{u,\varepsilon} \nu$$

exists, and is called the *u-dimension* of ν. If $u = 1$, then $\dim_u \mu$ is called the *entropy of Ψ with respect to ν*, and is denoted by $h_\nu(\Psi)$. We also define the *lower* and *upper u-pointwise dimensions* of ν at the point $x \in Y$ by

$$\underline{d}_{\nu,u}(x) = \lim_{\varepsilon\to 0} \liminf_{t\to\infty} -\frac{\log \nu(B(x, t, \varepsilon))}{U(x, t, \varepsilon)}$$

and

$$\overline{d}_{\nu,u}(x) = \lim_{\varepsilon\to 0} \limsup_{t\to\infty} -\frac{\log \nu(B(x, t, \varepsilon))}{U(x, t, \varepsilon)}.$$

4.2. **Suspension flows over subshifts of finite type.** Let $\Psi = \{\psi_t\}_t$ be a suspension flow on Y over a homeomorphism $T \colon X \to X$ of the compact metric space X, and μ a T-invariant Borel probability measure in X. It is well known that μ induces a Ψ-invariant probability measure ν in Y such that

$$\int_Y g \, d\nu = \int_X \int_0^{\tau(x)} g(x, s) \, ds d\mu(x) \Big/ \int_X \tau \, d\mu \tag{16}$$

for every continuous function $g \colon Y \to \mathbb{R}$, and that any Ψ-invariant measure ν in Y is of this form for some T-invariant Borel probability measure μ in X. We remark that the identity in (16) is equivalent to

$$\int_Y g \, d\nu = \int_X I_g \, d\mu \Big/ \int_X \tau \, d\mu, \tag{17}$$

where the function I_g is defined by (11).

We now consider the space Y equipped with the Bowen–Walters distance (see Appendix A). For every real number α, set

$$K_\alpha = \{y \in Y : \underline{d}_{\nu,u}(y) = \overline{d}_{\nu,u}(y) = \alpha\}.$$

Whenever $K_\alpha \neq \varnothing$ and $y \in K_\alpha$, the common value α of $\underline{d}_{\nu,u}(y)$ and $\overline{d}_{\nu,u}(y)$ is denoted by $d_{\nu,u}(y)$, and is called *u-pointwise dimension of ν at y*. We set

$$\mathcal{D}_u(\alpha) = \dim_u K_\alpha.$$

The function $\mathcal{D}_u$ is called the *u-dimension spectrum for u-pointwise dimensions* (with respect to the measure ν).

We now consider the special case when T is a subshift of finite type.

Proposition 11. *Let $\Psi = \{\psi_t\}_t$ be a suspension flow on Y over a topologically mixing two-sided subshift of finite type, ν is an equilibrium measure for Ψ with Hölder continuous potential, and $u\colon Y \to \mathbb{R}$ is a Hölder continuous positive function. If $y \in Y$ and $\varepsilon > 0$, then*

$$\underline{d}_{\nu,u}(y) = \liminf_{t\to\infty} -\frac{\log \nu(B(y,t,\varepsilon))}{\int_0^t u(\psi_\tau y)\, d\tau}$$

and

$$\overline{d}_{\nu,u}(y) = \limsup_{t\to\infty} -\frac{\log \nu(B(y,t,\varepsilon))}{\int_0^t u(\psi_\tau y)\, d\tau}.$$

Notice that the limits in the proposition are independent of ε.

Let $g\colon Y \to \mathbb{R}$ be a continuous function. By (17) and Abramov's entropy formula, we obtain

$$h_\nu(\Psi) + \int_Y g\, d\nu = \frac{h_\mu(T) + \int_X I_g\, d\mu}{\int_X \tau\, d\mu}, \tag{18}$$

whenever μ is a T-invariant probability measure in X, and ν is the Ψ-invariant probability measure induced by μ in Y. Since $\tau > 0$, we conclude from (18) that $P_\Psi(g) = 0$ if and only if $P_T(I_g) = 0$, where $P_T(I_g)$ is the topological pressure of I_g with respect to T. Therefore, when $P_\Psi(g) = 0$ the measure ν is an equilibrium measure of g (with respect to Ψ) if and only if μ is an equilibrium measure of $I_g|X$ (with respect to T).

For every real number q, we define the function $g_q\colon Y \to \mathbb{R}$ by

$$g_q = -T_u(q)u + qg,$$

where the number $T_u(q)$ is chosen so that $P_\Psi(g_q) = 0$. The above discussion shows that $T_u(q)$ is equivalently specified by the equation $P_T(I_{g_q}) = 0$, where $P_T(I_{g_q})$ is the topological pressure of I_g with respect to T. We denote by ν_q and m_u, respectively, the equilibrium measures of g_q and $-\dim_u X \cdot u$ with respect to Ψ.

The following is a complete multifractal analysis of the spectrum $\mathcal{D}_u$ for suspension flows over subshifts of finite type.

Theorem 12. *Let Ψ be a suspension flow on Y over a topologically mixing two-sided subshift of finite type, $u\colon Y \to \mathbb{R}$ a Hölder continuous positive function, and ν an equilibrium measure for Ψ with Hölder continuous potential $g\colon Y \to \mathbb{R}$ such that $P_\Psi(g) = 0$. Then the following properties hold:*

1. *for ν-almost every $y \in Y$,*
$$d_{\nu,u}(y) = \frac{h_\nu(\Psi)}{\int_Y u \, d\nu};$$
2. *T_u is real analytic, and satisfies $T_u'(q) \le 0$ and $T_u''(q) \ge 0$ for every $q \in \mathbb{R}$, with $T_u(0) = \dim_u Y$ and $T_u(1) = 0$;*
3. *the domain of $\mathcal{D}_u$ is a closed interval in $[0, \infty)$, which coincides with the range of the function $\alpha_u = -T_u'$, and if $q \in \mathbb{R}$ then*
$$\mathcal{D}_u(\alpha_u(q)) = T_u(q) + q\alpha_u(q);$$
4. *for every $q \in \mathbb{R}$, $\nu_q(K_{\alpha_u(q)}) = 1$, and*
$$d_{\nu_q,u}(x) = T_u(q) + q\alpha_u(q)$$
for ν_q-almost all $x \in K_{\alpha_u(q)}$; moreover,
$$\overline{d}_{\nu_q,u}(x) \le T_u(q) + q\alpha_u(q)$$
for every $x \in K_{\alpha_u(q)}$, and $\mathcal{D}_u(\alpha_u(q)) = \dim_u \nu_q$ for every $q \in \mathbb{R}$;
5. *if $\nu \neq m_u$, then $\mathcal{D}_u$ and T_u are real analytic strictly convex functions.*

Theorem 12 is a flow version of Theorem 6.6 in [3], which in turn follows from work of Pesin and Weiss [9], and Schmeling [11].

Setting $u = 1$ in Theorem 12 we obtain a complete multifractal analysis of the spectrum
$$\mathcal{E}(\alpha) = h(\Psi|\{y \in Y : h_\nu(y) = \alpha\}),$$
where
$$h_\nu(y) = \lim_{t\to\infty} -\frac{\log \nu(B(y,t,\varepsilon))}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t g(\psi_\tau y)\, d\tau. \tag{19}$$
The function $\mathcal{E}$ is called *entropy spectrum for local entropies* (with respect to the measure ν), and coincides with the entropy spectrum for the Birkhoff averages of g. In the case of axiom A diffeomorphisms this spectrum was studied in [1].

We note that the statements in Proposition 11 and Theorem 12 also hold for suspension semi-flows over one-sided subshifts of finite type.

4.3. **Irregular sets.** In this section we establish a version of the results in Section 3 for u-dimension. Consider again a continuous flow $\Psi = \{\psi_t\}_t$ on Y. Given continuous functions $g_1, \ldots, g_k \colon Y \to \mathbb{R}$ and $u\colon Y \to \mathbb{R}$, with u positive, we define the *irregular set* $\mathcal{F}(g_1, \ldots, g_k; u)$ by
$$\left\{ y \in Y : \lim_{t\to\infty} \frac{\int_0^t g_j(\psi_s y)\, ds}{\int_0^t u(\psi_s y)\, ds} \text{ does not exist for } j = 1, \ldots, k \right\}. \tag{20}$$

One can show that

$$\mathcal{F}(g_1, \dots, g_k; u) = \{(x, s) : x \in \mathcal{C}(g_1, \dots, g_k; u) \text{ and } s \in [0, \tau(x)]\}, \tag{21}$$

where $\mathcal{C}(g_1, \dots, g_k; u)$ is the set

$$\left\{ x \in X : \lim_{m\to\infty} \frac{\sum_{i=0}^{m} I_{g_j}(T^i x)}{\sum_{i=0}^{m} I_u(T^i x)} \text{ does not exist for } j = 1, \dots, k \right\}. \tag{22}$$

The proof is a modification of the proof of Proposition 6.

Theorem 13. *Let Ψ be a suspension flow on Y over a topologically mixing two-sided subshift of finite type, and $g_1, \dots, g_k, u\colon Y \to \mathbb{R}$ Hölder continuous functions, with u positive. Then the following properties are equivalent:*

1. *the function g_j is not Ψ-cohomologous to a multiple of u on Y for each $j = 1, \dots, k$;*
2. $\dim_u \mathcal{F}(g_1, \dots, g_k; u) = \dim_u Y$.

Setting $u = 1$, we have

$$\mathcal{F}(g_1, \dots, g_k; 1) = \bigcap_{j=1}^{k} \mathcal{B}(g_j).$$

Hence, under the hypotheses of Theorem 13, if the function g_j is not Ψ-cohomologous to a constant on Y for each $j = 1, \dots, k$, then

$$h(\Psi| \bigcap_{j=1}^{k} \mathcal{B}(g_j)) = h(\Psi).$$

One can also consider suspension semi-flows over one-sided subshifts of finite type, and obtain a corresponding version of Theorem 13. An application of this is given in the following section.

5. Suspensions over hyperbolic dynamical systems

5.1. **Suspension semi-flows over expanding maps.** Let $T\colon M \to M$ be a C^1 map of a smooth compact manifold M, and $\Lambda \subset M$ a T-invariant set such that T is expanding on Λ. This means that there exist constants $c > 0$ and $\beta > 1$ such that $\|d_x T^n v\| \geq c\beta^n \|v\|$ for all $x \in \Lambda$, $v \in T_x M$, and $n \in \mathbb{N}$. We say that Λ is a *repeller* of T. It is well known that repellers admit Markov partitions of arbitrarily small diameter. Each Markov partition has associated a one-sided subshift of finite type $\sigma\colon X \to X$, and a coding map $\pi\colon X \to \Lambda$ for the repeller, which is Hölder continuous, onto, finite-to-one, and satisfies $T \circ \pi = \pi \circ \sigma$.

Consider a Markov partition for Λ, and the associated coding map $\pi\colon X \to \Lambda$. Let Ψ be the associated suspension semi-flow on Y over the one-sided subshift of finite type $\sigma\colon X \to X$, with Y equipped with the Bowen–Walters distance. We define a function $u\colon X \to \mathbb{R}$ by

$$u(x) = \log\|d_{\pi x} T\|. \tag{23}$$

One says that T is *conformal* on Λ if d_xT is a multiple of an isometry for each $x \in \Lambda$. One can show that if T is conformal on Λ, then

$$\dim_H Z = 1 + \dim_u \pi^{-1} Z$$

for every Ψ-invariant set $Z \subset \Lambda$. This follows from work of Schmeling [12].

Let ν be a Ψ-invariant probability measure on Y. For every real number α, set

$$K_\alpha = \left\{ y \in Y : \lim_{r\to 0} \frac{\log \nu(B(y,r))}{\log r} = \alpha \right\},$$

where $B(y,r) \subset Y$ is the Bowen–Walters ball with radius r centered at $y \in Y$. The function

$$\mathcal{D}(\alpha) = \dim_H K_\alpha$$

is called the *dimension spectrum for pointwise dimensions* (with respect to the measure ν). Let μ be the measure in X associated to ν as in Section 4.2. By Proposition 17 in Appendix A, for each $y = (x,s) \in Y$ there exists $c \geq 1$ such that if r is sufficiently small, then

$$B_X(x, r/c) \times (s - r/c, s + r/c) \subset B(y,r) \subset B_X(x, cr) \times (s - cr, s + cr).$$

Therefore,

$$K_\alpha = \left\{ (x,s) \in Y : \lim_{r\to 0} \frac{\log \mu(B_X(x,r))}{\log r} = \alpha - 1 \right\}. \tag{24}$$

Since each set K_α is Ψ-invariant, if u is as in (23), then

$$\mathcal{D}(\alpha) = 1 + \mathcal{D}_u(\alpha - 1).$$

Proceeding in a similar way to that in Section 4.2 one can now effect a multifractal analysis of the spectrum $\mathcal{D}$. We use the same notation as in Section 4.2. The following is an immediate consequence of Theorem 12 and the above discussion, together with the appropriate versions of Propositions 17 and 19 in Appendix A for locally invertible maps.

Theorem 14. *For a repeller Λ of a topologically mixing C^1 map which is conformal on Λ, let Ψ be the suspension semi-flow on Y over the one-sided subshift of finite type associated to some Markov partition of Λ, and ν an equilibrium measure for Ψ with Hölder continuous potential $g\colon Y \to \mathbb{R}$ such that $P_\Psi(g) = 0$. Then the following properties hold:*

1. *for ν-almost every $y \in Y$,*
$$\lim_{r\to 0} \frac{\log \nu(B(y,r))}{\log r} = 1 + \frac{h_\mu(T)}{\int_X (\log\|dT\| \circ \pi)\, d\mu};$$
2. *if $T = T_u + 1$, $\alpha = -T'$, and $q \in \mathbb{R}$, then $\mathcal{D}(\alpha(q)) = T(q) + q\alpha(q)$;*
3. *for every $q \in \mathbb{R}$, $\nu_q(K_{\alpha(q)}) = 1$, and*
$$\lim_{r\to 0} \frac{\log \nu_q(B(y,r))}{\log r} = T(q) + q\alpha(q)$$

for ν_q-almost all $x \in K_{\alpha(q)}$; moreover,

$$\limsup_{r\to 0} \frac{\log \nu_q(B(y,r))}{\log r} \le T(q) + q\alpha(q)$$

for every $x \in K_{\alpha(q)}$, and $\mathcal{D}(\alpha(q)) = \dim_H \nu_q$ for every $q \in \mathbb{R}$;

4. *if $\nu \ne m_u$, then $\mathcal{D}$ and T are real analytic strictly convex functions.*

The following statement follows easily from a version of Theorem 13 for suspension semi-flows over one-sided subshifts of finite type.

Theorem 15. *Under the hypothesis of Theorem 14, if $\nu \ne m_u$ then*

$$\dim_H \left\{ y \in Y : \lim_{r\to 0} \frac{\log \nu(B(y,r))}{\log r} \text{ does not exist} \right\} = \dim_H Y.$$

5.2. **Suspension flows over axiom A diffeomorphisms.** Let Λ be a basic set of a C^1 flow Φ. Given a Markov system, we consider the associated two-sided subshift of finite type $\sigma\colon X \to X$, and coding map $\pi\colon X \to \Lambda$ (see Sections 2.4 and 3.2).

Let $\beta_s\colon X \to \mathbb{R}$ and $\beta_u\colon X \to \mathbb{R}$ be Hölder continuous positive functions. For each cylinder set

$$C_{i_{-n}\cdots i_m} = \{(\cdots j_0 \cdots) : j_k = i_k \text{ for } -n \le k \le m\},$$

write

$$\beta_s(C_{i_{-n}\cdots i_m}) = \sup\left\{ \prod_{k=0}^{m} \beta_s(\sigma^k x) : x \in C_{i_{-n}\cdots i_m} \right\}$$

and

$$\beta_u(C_{i_{-n}\cdots i_m}) = \sup\left\{ \prod_{k=0}^{n} \beta_u(\sigma^{-k} x) : x \in C_{i_{-n}\cdots i_m} \right\}.$$

Given $\alpha \in \mathbb{R}$, consider the function

$$M(Z,\alpha) = \lim_{\ell\to 0} \inf_{\Gamma} \sum_{C\in\Gamma} \exp(-\alpha\beta_s(C) - \alpha\beta_u(C)),$$

where the infimum is taken over all covers Γ of Z by cylinders $C_{i_{-n}\cdots i_m}$ with $m > \ell$ and $n > \ell$. We define the (β_s, β_u)-*dimension* of Z by

$$\dim_{\beta_s,\beta_u} Z = \inf\{\alpha : M(Z,\alpha) = 0\}.$$

Let again Λ be a basic set of a C^1 flow $\Phi = \{\varphi_t\}_t$. We say that the flow Φ is *conformal* on Λ if the maps

$$d_x\varphi_t | E^s(x)\colon E^s(x) \to E^s(\varphi_t x) \quad \text{and} \quad d_x\varphi_t | E^u(x)\colon E^u(x) \to E^u(\varphi_t x)$$

are multiples of isometries for each $x \in \Lambda$ and $t \in \mathbb{R}$. We give two examples of (β_s, β_u)-dimension:

1. Let Λ be a basic set of a C^1 flow Φ such that Φ is conformal on Λ. Let T be the transfer map associated to some Markov system for Φ on

Λ, and $\pi\colon Y \to \Lambda$ the associated coding map. Consider the functions $\beta_s\colon X \to \mathbb{R}$ and $\beta_u\colon X \to \mathbb{R}$ defined by

$$\beta_s(x) = \log\|d_{\pi x}\varphi_{\tau(\pi x)}|E^s(\pi x)\| \tag{25}$$

and

$$\beta_u(x) = -\log\|d_{\pi x}\varphi_{\tau(\pi x)}|E^u(\pi x)\|. \tag{26}$$

Note that

$$\prod_{k=0}^{n-1}\beta_s(\sigma^k x) = \log\|d_{\pi x}\varphi_{\tau_n(\pi x)}|E^s(\pi x)\|,$$

$$\prod_{k=0}^{n-1}\beta_u(\sigma^{-k} x) = \log\|d_{\pi x}\varphi_{-\tau_n(\pi x)}|E^u(\pi x)\|,$$

where

$$\tau_n(\pi x) = \sum_{k=0}^{n-1}\tau(\pi(\sigma^k x)).$$

Then

$$\dim_H Z = 1 + \dim_{\beta_s,\beta_u}\pi^{-1}Z \tag{27}$$

for every Ψ-invariant set $Z \subset \Lambda$.

2. Let Λ be a basic set of a C^1 axiom A diffeomorphism f such that $d_x f|E^s(x)$ and $d_x f|E^u(x)$ are multiples of isometries for each $x \in \Lambda$. Consider a Markov partition for Λ, and the associated coding map $\pi\colon X \to \Lambda$. Define functions $\beta_s\colon X \to \mathbb{R}$ and $\beta_u\colon X \to \mathbb{R}$ by

$$\beta_s(x) = \log\|d_{\pi x}f|E^s(\pi x)\| \quad \text{and} \quad \beta_u(x) = -\log\|d_{\pi x}f|E^u(\pi x)\|.$$

Then

$$\dim_H Z = \dim_{\beta_s,\beta_u}\pi^{-1}Z \tag{28}$$

for every set $Z \subset \Lambda$.

The identities in (27) and (28) follow from work of Schmeling [12]. In what follows we shall only consider the first situation. A straightforward modification applies to the second one.

We briefly present another description of the (β_s, β_u)-dimension. When X is equipped with the distance in (7), the map π is in general only Hölder continuous. We will introduce a new distance $\widehat{d}_X$ in X (inducing the same topology as d_X) such that for a certain class of flows (the flows which are conformal on Λ; see the definition below in this section) the map $\pi\colon (X, \widehat{d}_X) \to \pi(X)$ is locally Lipschitz with Lipschitz inverse, and thus it preserves the Hausdorff dimension. We define a new distance $\widehat{d}_X$ in X by

$$\widehat{d}_X((\cdots i_0\cdots),(\cdots j_0\cdots)) = |i_0 - j_0| + \beta_s(C_{i_{-n_u}\cdots i_{n_s}}) + \beta_u(C_{i_{-n_u}\cdots i_{n_s}}),$$

where

$$n_s = \max\{n \in \mathbb{N} : i_k = j_k \text{ for all } k \le n\}$$

and

$$n_u = \max\{n \in \mathbb{N} : i_k = j_k \text{ for all } k \ge -n\}.$$

Since

$$\operatorname{diam}_{\widehat{d}_X} C = \beta_s(C) + \beta_u(C)$$

for every cylinder C, the (β_s, β_u)-dimension coincides with the Hausdorff dimension with respect to $\widehat{d}_X$. The distance $\widehat{d}_X$ induces a new Bowen–Walters distance in Y. One can easily verify that this distance induces the same topology in Y as the original Bowen–Walters distance obtained from d_X.

Let Λ be a basic set of a C^1 flow Φ, and ν be a Φ-invariant probability measure on Λ. For every $\alpha \in \mathbb{R}$, let

$$K_\alpha = \left\{ y \in \Lambda : \lim_{r \to 0} \frac{\log \nu(B(y,r))}{\log r} = \alpha \right\},$$

With the help of a Markov system, one can show that the set K_α satisfies an identity similar to that in (24). It follows from work of Barreira, Pesin, and Schmeling [2] that $\nu(\bigcup_{\alpha \in \mathbb{R}} K_\alpha) = 1$.

Consider now the *dimension spectrum for pointwise dimensions* (with respect to the measure ν) defined by

$$\mathcal{D}(\alpha) = \dim_H K_\alpha.$$

In a similar way to that in Section 5.1, if Φ is conformal on Λ, then

$$\mathcal{D}(\alpha) = 1 + \dim_{\beta_s, \beta_u}(X \cap \pi^{-1} K_{\alpha - 1}),$$

with β_s and β_u as in (25) and (26).

Given a continuous function $g \colon \Lambda \to \mathbb{R}$, let $t_s(q)$ and $t_u(q)$ be the unique numbers such that

$$P_T(-t_s(q)\beta_s + qg \circ \pi) = P_T(-t_u(q)\beta_u + qg \circ \pi) = 0$$

We write

$$T(q) = 1 + t_s(q) + t_u(q).$$

One can also formulate a version of Theorem 14 for basic sets.

Theorem 16. *Let Λ be a compact basic set of a topologically mixing $C^{1+\varepsilon}$ flow Φ, for some $\varepsilon > 0$, such that Φ is conformal on Λ, and ν an equilibrium measure for Φ with Hölder continuous potential $g \colon \Lambda \to \mathbb{R}$ such that $P_\Phi(g) = 0$. Then the following properties hold:*

1. *for ν-almost every $y \in \Lambda$,*

$$\lim_{r \to 0} \frac{\log \nu(B(y,r))}{\log r} = 1 - \frac{h_\mu(T)}{\int_X \beta_s \, d\mu} - \frac{h_\mu(T)}{\int_X \beta_u \, d\mu};$$

2. *if $\alpha = -T'$ then $\mathcal{D}(\alpha(q)) = T(q) + q\alpha(q)$ for every $q \in \mathbb{R}$;*
3. *if μ is not a measure of maximal dimension on X, then $\mathcal{D}$ and T are real analytic strictly convex functions.*

The proof can be obtained from that of the corresponding result for axiom A diffeomorphisms due to Simpelaere [13].

6. Proofs

6.1. **Proofs of the results in Section 3.**

Proof of Theorem 7. Assume that g is Ψ-cohomologous to h on Y. If $x \in Y$ then

$$\begin{aligned}
I_g(x) - I_h(x) &= \int_0^{\tau(x)} \lim_{t\to 0} \frac{q(\psi_t\psi_s x) - q(\psi_s x)}{t}\, ds \\
&= \lim_{t\to 0} \frac{1}{t}\left(\int_t^{\tau(x)+t} q(\psi_s x)\, ds - \int_0^{\tau(x)} q(\psi_s x)\, ds\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(\int_0^{\tau(x)+t} q(\psi_s x)\, ds - \int_0^{\tau(x)} q(\psi_s x)\, ds\right) \\
&\quad - \lim_{t\to 0} \frac{1}{t}\int_0^t q(\psi_s x)\, ds \\
&= q(\psi_{\tau(x)} x) - q(x) \\
&= q(Tx) - q(x).
\end{aligned}$$

Therefore, I_g is T-cohomologous to I_h on Y.

Assume now that I_g is T-cohomologous to I_h on Y. If $x \in Y$ then $\tau(\psi_t x) = \tau(x) - t$ for every sufficiently small $t > 0$ (depending on x). Thus, $T(\psi_t x) = Tx$, and

$$I_g(\psi_t x) - I_h(\psi_t x) = q(Tx) - q(\psi_t x).$$

Since

$$\lim_{t\to 0^+} \frac{I_g(\psi_t x) - I_g(x)}{t} = \lim_{t\to 0} -\frac{1}{t}\int_0^t g(\psi_s x)\, ds = -g(x),$$

we obtain

$$\begin{aligned}
g(x) - h(x) &= \lim_{t\to 0^+}\left(-\frac{I_g(\psi_t x) - I_h(\psi_t x)}{t} + \frac{I_g(x) - I_h(x)}{t}\right) \\
&= \lim_{t\to 0^+}\left(-\frac{q(Tx) - q(\psi_t x)}{t} + \frac{q(Tx) - q(x)}{t}\right) \qquad (29)\\
&= \lim_{t\to 0^+} \frac{q(\psi_t x) - q(x)}{t}.
\end{aligned}$$

We also have

$$\tau(\psi_{-t} x) = \begin{cases} \tau(x) + t & \text{if } x \notin X \times \{0\} \\ t & \text{if } x \in X \times \{0\} \end{cases}$$

for every sufficiently small $t > 0$ (depending on x). When $x \notin X \times \{0\}$ we have $T(\psi_{-t} x) = Tx$ and one can proceed in a similar fashion to the one above to show that

$$g(x) - h(x) = \lim_{t\to 0^-} \frac{q(\psi_t x) - q(x)}{t}. \qquad (30)$$

When $x \in X \times \{0\}$ we have $T(\psi_{-t}x) = x$, and

$$I_g(\psi_{-t}x) - I_h(\psi_{-t}x) = q(x) - q(\psi_{-t}x).$$

Since

$$\lim_{t\to 0^+} \frac{I_g(\psi_{-t}x)}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_{-t}^{0} g(\psi_s x)\,ds = g(x),$$

we obtain

$$g(x) - h(x) = \lim_{t\to 0^-} \frac{I_g(\psi_t x) - I_h(\psi_t x)}{-t} = \lim_{t\to 0^-} \frac{q(x) - q(\psi_t x)}{-t}. \tag{31}$$

By (29), (30), and (31), if $x \in Y$ then

$$g(x) - h(x) = \lim_{t\to 0} \frac{q(\psi_t x) - q(x)}{t}$$

Therefore, g is Ψ-cohomologous to h on Y.

It remains to prove that Property 3 implies Property 2. Assume that Property 3 holds with the function $q\colon X \times \{0\} \to \mathbb{R}$. We can extend q to a function $q\colon Y \to \mathbb{R}$ by

$$q(\psi_t y) = q(y) - \int_0^t [g(\psi_s y) - h(\psi_s y)]\,ds$$

for every $y = (x, 0)$ and $t \in [0, \tau(x))$. For every $t \in [0, \tau(x))$ we have $T\psi_t y = Ty$ and by (11) we obtain

$$\begin{aligned} q(T\psi_t y) - q(\psi_t y) &= q(Ty) - q(\psi_t) \\ &= \int_t^{\tau(y)} [g(\psi_s y) - h(\psi_s y)]\,ds \\ &= I_g(\psi_t y) - I_h(\psi_t y). \end{aligned}$$

This completes the proof of the theorem. $\square$

Proof of Proposition 8. The proof is a straightforward modification of the corresponding arguments in the proof of Proposition 6 (see Section 6.2 below). $\square$

Proof of Theorem 9. By (19), the desired statements follow immediately from Theorem 12 by setting $u = 1$. $\square$

Proof of Theorem 10. This follows immediately from Theorem 13 by setting $k = 1$, $g_1 = g$, and $u = 1$. $\square$

6.2. **Proofs of the results in Section 2.**

Proof of Theorem 1. If g is Φ-cohomologous to a constant, then $\mathcal{B}(g) = \varnothing$.

Assume now that g is not Φ-cohomologous to a constant. Consider a Markov system, and the associated suspension flow $\Psi = \{\psi_t\}_t$ and coding map $\pi\colon Y \to \Lambda$ satisfying (15). The map π can be used to transfer the results from the symbolic dynamics to the dynamics on the manifold.

By (15), we obtain $\pi(\mathcal{B}(g\circ\pi))\subset\mathcal{B}(g)$. A priori one cannot discard that there exists a point $x\in X$ such that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t(g\circ\pi)(\psi_\tau x)\,d\tau<\limsup_{t\to\infty}\frac{1}{t}\int_0^t(g\circ\pi)(\psi_\tau x)\,d\tau \tag{32}$$

and

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t g(\psi_\tau(\pi x))\,d\tau=\limsup_{t\to\infty}\frac{1}{t}\int_0^t g(\psi_\tau(\pi x))\,d\tau. \tag{33}$$

With slight changes to the proof of Theorem 7.4 in [3] (see also the proof of Theorem 21.1 in [7], and in particular that of Lemmas 2 and 3 inside Theorem 21.1) one can prove the following.

Lemma 1. *We have*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t(g\circ\pi)(\psi_\tau x)\,d\tau=\limsup_{t\to\infty}\frac{1}{t}\int_0^t(g\circ\pi)(\psi_\tau x)\,d\tau=\alpha$$

if and only if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t g(\psi_\tau(\pi x))\,d\tau=\limsup_{t\to\infty}\frac{1}{t}\int_0^t g(\psi_\tau(\pi x))\,d\tau=\alpha.$$

The lemma shows that (32) and (33) cannot hold simultaneously, and hence, $\mathcal{B}(g)\subset\pi(\mathcal{B}(g\circ\pi))$. Therefore,

$$\mathcal{B}(g)=\pi(\mathcal{B}(g\circ\pi)). \tag{34}$$

We now proceed as in [3]. Let $R\subset\Lambda$ be the "boundary" of the Markov system, i.e., the set of points $y\in\Lambda$ such that $\varphi_t x$ is in the boundary of some element of the Markov system for some $t\in\mathbb{R}$. Note that R is Φ-invariant, and that $\pi\colon\pi^{-1}(\Lambda\setminus R)\to\Lambda\setminus R$ is a homeomorphism. Furthermore, since there exist cylinders $C\subset X$ such that $\pi(C)$ is disjoint from R, we have

$$h(\Psi|\pi^{-1}R)<h(\Psi)\quad\text{and}\quad h(\Phi|R)<h(\Phi|\Lambda).$$

By (34), we conclude that

$$h(\Phi|\mathcal{B}(g))=h(\Psi|\mathcal{B}(g\circ\pi)).$$

By Theorem 10, we obtain

$$h(\Phi|\Lambda)=h(\Psi)=h(\Psi|\mathcal{B}(g\circ\pi))=h(\Phi|\mathcal{B}(g)).$$

This completes the proof of the theorem. □

Proof of Theorem 2. Let

$$G\stackrel{\text{def}}{=}\{g\in C^\alpha(\Lambda):g\text{ is not }\Phi\text{-cohomologous to a constant}\},$$

and $g\in G$. By Livschitz's theorem for flows (see, for example, Theorem 19.2.4 in [6]), there exist two points $x_i=\varphi_{T_i}x_i$ for $i=0,\,1$ such that

$$\delta=\left|\frac{1}{T_0}\int_0^{T_0}g(\varphi_\tau x_0)\,d\tau-\frac{1}{T_1}\int_0^{T_1}g(\varphi_\tau x_1)\,d\tau\right|\neq0.$$

For any $f \in C^\alpha(\Lambda)$ such that $\|f-g\|_\alpha < \delta/2$ we have

$$\left|\frac{1}{T_i}\int_0^{T_i}(f-g)(\varphi_\tau x_i)\,d\tau\right| \le \sup\{|f(x)-g(x)| : x \in \Lambda\} \le \|f-g\|_\alpha < \frac{\delta}{2},$$

for $i = 0$, 1, and hence,

$$\frac{1}{T_0}\int_0^{T_0} f(\varphi_\tau x_0)\,d\tau \neq \frac{1}{T_1}\int_0^{T_1} f(\varphi_\tau x_1)\,d\tau.$$

This implies that f is not Φ-cohomologous to a constant. Hence, G is open.

Let Γ_0 and Γ_1 be two distinct periodic orbits, and choose a function $h \in C^\alpha(\Lambda)$ such that $h|_{\Gamma_i} = i$ for $i = 0$, 1. Let $g \notin G$. For any $\varepsilon > 0$, the function $g_\varepsilon = g + \varepsilon h \in C^\alpha(\Lambda)$ is not Φ-cohomologous to a constant, because averages on Γ_0 and Γ_1 differ by ε. Moreover, $\|g_\varepsilon - g\|_\alpha \le \varepsilon\|h\|_\alpha$, and hence the function g can be arbitrarily well approximated by functions in G. Therefore, G is dense in $C^\alpha(\Lambda)$. □

Proof of Theorem 4. Consider a Markov system, and the associated suspension flow $\Psi = \{\psi_t\}_t$ and coding map $\pi\colon Y \to \Lambda$ satisfying (15). By Lemma 1 (see the proof of Theorem 1), we have $\mathcal{E}(\alpha) = \mathcal{D}_u(\alpha)$ for every α, with $u = 1$, and $\mathcal{D}_u$ as in Section 4.2. Therefore, the desired statements follow immediately from Theorem 12 by setting $u = 1$. □

Proof of Theorem 5. The proof is a straightforward modification of the corresponding arguments in the proof of Theorem 7 (see Section 6.1 above). □

Proof of Proposition 6. Given $m \in \mathbb{N}$, define a function $\tau_m\colon \Lambda \to \mathbb{R}$ by

$$\tau_m(x) = \sum_{i=0}^{m-1} \tau(T^i x). \tag{35}$$

If $x \in \Lambda$ and $m \in \mathbb{N}$ then

$$\begin{aligned}\int_0^{\tau_m(x)} g(\varphi_s x)\,ds &= \sum_{i=0}^{m-1}\int_{\tau_i(x)}^{\tau_{i+1}(x)} g(\varphi_s x)\,ds\\ &= \sum_{i=0}^{m-1}\int_0^{\tau(T^i x)} g(\varphi_s T^i x)\,ds\\ &= \sum_{i=0}^{m-1} I_g(T^i x).\end{aligned} \tag{36}$$

Given $t > 0$ there exists a unique $m \in \mathbb{N}$ such that $\tau_m(x) \le t < \tau_{m+1}(x)$. One can write $t = \tau_m(x) + \kappa$ for some $\kappa \in (\inf\tau, \sup\tau)$ and thus

$$\frac{1}{t}\int_0^t g(\varphi_s x)\,ds = \frac{\int_0^{\tau_m(x)} g(\varphi_s x)\,ds + \int_0^{\tau_m(x)+\kappa} g(\varphi_s x)\,ds}{\tau_m(x)+\kappa}$$

and

$$\begin{aligned}
&\left|\frac{1}{t}\int_0^t g(\varphi_s x)\,ds - \frac{1}{\tau_m(x)}\int_0^{\tau_m(x)} g(\varphi_s x)\,ds\right| \\
&\le \left|\frac{1}{\tau_m(x)+\kappa} - \frac{1}{\tau_m(x)}\right| \int_0^{\tau_m(x)} |g(\varphi_s x)|\,ds + \frac{\kappa \sup|g|}{\tau_m(x)+\kappa} \\
&\le \frac{\kappa}{(\tau_m(x)+\kappa)\tau_m(x)} \cdot \tau_m(x) \sup|g| + \frac{\kappa \sup|g|}{\tau_m(x)+\kappa} \\
&\le \frac{2\sup\tau \sup|g|}{\tau_m(x)}.
\end{aligned}$$

By (6), if $t \to \infty$, then $m \to \infty$ and $\tau_m(x) \to \infty$. Hence, by (36),

$$\left|\frac{1}{t}\int_0^t g(\varphi_s x)\,ds - \frac{1}{\tau_m(x)}\sum_{i=0}^{m-1} I_g(T^i x)\right| \to 0 \text{ as } t \to \infty.$$

This immediately implies Statements 2 and 3.

Assume now that g is Hölder continuous. If x and y lie in the same domain of continuity of τ, then

$$\begin{aligned}
|I_g(x) - I_g(y)| &= \left|\int_{\tau(y)}^{\tau(x)} g(\varphi_s x)\,ds + \int_0^{\tau(y)} [g(\varphi_s x) - g(\varphi_s y)]\,ds\right| \\
&\le \sup|g| \cdot |\tau(x) - \tau(y)| \\
&\quad + \sup\tau \cdot \sup_{s\in(0,\tau(y))} |g(\varphi_s x) - g(\varphi_s y)| \\
&\le c d(x,y)^\alpha + c \sup_{s\,\in\,(0,\sup\tau),\ z\,\in\,M} \|d_z\varphi_s\|^\alpha d(x,y)^\alpha,
\end{aligned}$$

for some positive constants c and α. This shows that I_g is Hölder continuous on each domain of continuity of τ. ☐

6.3. **Proofs of the results in Section 4.**

Proof of Proposition 11. For each $m \in \mathbb{N}$, let $\tau_m\colon X \to \mathbb{R}$ be the function defined by (35). Given $x \in X$, let $m = m(x,t) \in \mathbb{N}$ be the unique integer satisfying $\tau_{m-1}(x) \le t < \tau_m(x)$. By Proposition 19 in Appendix A there exists a constant $c \ge 1$ such that if $y = (x,s) \in Y$, $t > 0$, and $\varepsilon > 0$ is sufficiently small, then

$$B_X(x,m,\varepsilon) \times \left(s - \frac{\varepsilon}{c}, s + \frac{\varepsilon}{c}\right) \subset B(y,t,\varepsilon) \subset B_X(x,m-1,\varepsilon) \times (s - c\varepsilon, s + c\varepsilon), \tag{37}$$

where

$$B_X(x,m,\varepsilon) = \{x' \in X : d_X(T^k x', T^k x) < \varepsilon \text{ for } k = 0, \ldots, m\}. \tag{38}$$

By Proposition 18 in Appendix A the function I_g is Hölder continuous on X. Since μ is an equilibrium measure of I_g it has the Gibbs property. Therefore, the limit

$$\liminf_{t\to\infty} -\frac{\log \nu(B(y,t,\varepsilon))}{U(y,t,\varepsilon)} = \liminf_{t\to\infty} -\frac{\log \mu(B_X(x,m,\varepsilon))}{U(y,t,\varepsilon)} \tag{39}$$

is independent of ε. Let

$$\delta(\varepsilon) = \sup\{|u(y_1) - u(y_2)| : d_Y(y_1,y_2) < \varepsilon\}$$

and observe that

$$1 \le \frac{U(y,t,\varepsilon)}{\int_0^t u(\psi_\tau y)\,d\tau} \le \frac{\int_0^t [u(\psi_\tau y) + \delta(\varepsilon)]\,d\tau}{\int_0^t u(\psi_\tau y)\,d\tau} \le 1 + \frac{\delta(\varepsilon)}{t \inf u}. \tag{40}$$

By (39) and (40), we conclude that

$$\underline{d}_{\nu,u}(y) = \liminf_{t\to\infty} -\frac{\log \nu(B(y,t,\varepsilon))}{U(y,t,\varepsilon)} = \liminf_{t\to\infty} -\frac{\log \nu(B(y,t,\varepsilon))}{\int_0^t u(\psi_\tau y)\,d\tau}.$$

A similar argument applies to $\overline{d}_{\nu,u}(y)$. ☐

Proof of Theorem 12. We shall reduce our setup to the case of maps.

Lemma 2. *If $y = (x,s) \in Y$, then*

$$\underline{d}_{\nu,u}(y) = \liminf_{m\to\infty} -\frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m I_u(T^i x)}$$

and

$$\overline{d}_{\nu,u}(y) = \limsup_{m\to\infty} -\frac{\sum_{i=0}^m I_g(T^i x)}{\sum_{i=0}^m I_u(T^i x)}.$$

Proof of the lemma. Let $\tau_m \colon Y \to \mathbb{R}$ be the function defined by (35). Given $t > 0$, let $m \in \mathbb{N}$ be the unique integer such that $\tau_m(x) \le t < \tau_{m+1}(x)$, and write $t = \tau_m(x) + \kappa$ with $\kappa \in (\inf \tau, \sup \tau)$. Proceeding as in the proof of Proposition 6 we obtain

$$\left|\frac{1}{t}\int_0^t u(\psi_\tau y)\,d\tau - \frac{1}{\tau_m(y)}\sum_{i=0}^{m-1} I_u(T^i y)\right| \to 0 \text{ as } t \to \infty. \tag{41}$$

Let $B_X(x,m,\varepsilon)$ be as in (38). By (37),

$$\left|\frac{-\log \nu(B(y,t,\varepsilon))}{t} + \frac{\log \mu(B_X(x,m,\varepsilon))}{\tau_m(x)}\right| \to 0 \text{ as } t \to \infty. \tag{42}$$

Note that $T^i(x,s) = T^i(x,0)$ for every $i \in \mathbb{N}$, and hence,

$$\sum_{i=0}^{m-1} I_u(T^i y) = \sum_{i=0}^{m-1} I_u(T^i x).$$

Write

$$A = \frac{-\log \nu(B(y,t,\varepsilon))}{\int_0^t u(\psi_\tau y)\,d\tau} + \frac{\log \mu(B_X(x,m,\varepsilon))}{\sum_{i=0}^{m-1} I_u(T^i x)}.$$

Since $0 < \inf u \le \sup u < \infty$, by (41) and (42) we obtain

$$A = \left(\frac{-\log \mu(B_X(x,m,\varepsilon))}{\tau_m(x)} + o(t) \right) \frac{t}{\int_0^t u(\psi_\tau y)\, d\tau} + \frac{\log \mu(B_X(x,m,\varepsilon))}{\tau_m(x)} \left(\frac{t}{\int_0^t u(\psi_\tau y)\, d\tau} + o(t) \right),$$

and hence,

$$|A| \le \left(\frac{1}{\inf u} + \frac{h_\mu(T)}{\inf \tau} \right) o(t).$$

This completes the proof of the lemma. □

Given $Z \subset X$ and $\beta \in \mathbb{R}$, set

$$N_\beta(Z) = \lim_{\ell\to\infty} \inf_\Gamma \sum_{C\in\Gamma} \exp\left(-\beta \sup \left\{ \sum_{i=0}^{m(C)-1} I_u(T^i x) : x \in C \right\} \right), \tag{43}$$

with the infimum taken over all covers Γ of Z by cylinders $C_{i_{-n}\cdots i_m}$ such that $m(C_{i_{-n}\cdots i_m}) = m \ge \ell$.

Lemma 3. *If $Z \subset X$ is T-invariant, then*

$$\dim_u\{(x,s) \in Y : x \in Z \text{ and } s \in [0,\tau(x)]\} = \inf\{\beta : N_\beta(Z) = 0\}.$$

Proof of the lemma. We use the same notation as in the proof of Lemma 2. The inequality

$$\left| \int_0^t u(\psi_\tau x)\, d\tau - \sum_{i=0}^{m-1} I_u(T^i x) \right| \le \kappa \sup u$$

implies the desired statement. □

By Lemmas 2 and 3 we have

$$K_\alpha = \{(x,s) \in Y : x \in Z_\alpha \text{ and } s \in [0,\tau(x)]\},$$

where

$$Z_\alpha = \left\{ x \in X : \lim_{m\to\infty} -\frac{\sum_{i=0}^{m-1} I_g(T^i x)}{\sum_{i=0}^{m-1} I_u(T^i x)} = \alpha \right\},$$

and

$$\mathcal{D}_u(\alpha) = \inf\{\beta : N_\beta(Z_\alpha) = 0\}.$$

Lemma 4. *We have*

$$h_\mu(T) \Big/ \int_X I_u \, d\mu = h_\nu(\Psi) \Big/ \int_Y u \, d\nu.$$

Proof of the lemma. By (11),

$$\begin{aligned}\int_X I_u \, d\mu \Big/ \int_X \tau \, d\mu &= \int_X \int_0^{\tau(x)} u(\psi_s x) \, ds d\mu(x) \\ &= \int_X \int_0^{\tau(x)} u((x,s)) \, d\nu(x,s) \\ &= \int_Y u \, d\nu.\end{aligned}$$

Abramov's entropy formula shows that

$$h_\mu(T) \Big/ \int_X I_u \, d\mu = h_\nu(\Psi) \int_X \tau \, d\mu \Big/ \int_X I_u \, d\mu = h_\nu(\Psi) \Big/ \int_Y u \, d\nu.$$

This establishes the desired identity. □

By Lemmas 2 and 4, we obtain $d_{\nu,u}(y) = h_\nu(\Psi)/\int_Y u \, d\nu$ for ν-almost every $y \in Y$. We can now apply Theorem 6.6 in [3] to obtain the remaining properties in the theorem. □

Proof of Theorem 13. Proceeding as in the proof of Theorem 12, one can reduce our setup to the case of maps. More precisely, Lemma 2 establishes the identity (21), with $\mathcal{F}(g_1, \ldots, g_k; u)$ and $\mathcal{C}(g_1, \ldots, g_k; u)$ as in (20) and (22). Furthermore, by Lemma 3 we have

$$\dim_u Y = \inf\{\beta : N_\beta(X) = 0\} \tag{44}$$

and

$$\dim_u \mathcal{F}(g_1, \ldots, g_k; u) = \inf\{\beta : N_\beta(\mathcal{C}(g_1, \ldots, g_k; u)) = 0\}, \tag{45}$$

with $N_\beta(Z)$ as in (43). Note that the set $\mathcal{C}(g_1, \ldots, g_k; u)$ is defined entirely in terms of the map T, and the functions I_u and I_{g_j} for each j. By Theorem 7, the function g_j is Ψ-cohomologous to a multiple of u on Y if and only if I_{g_j} is T-cohomologous to a multiple of I_u on X, and hence, if and only if I_{g_j} is T-cohomologous to $I_{\alpha_j u} = \alpha_j I_u$ on X, where α_j is the unique number such that $P_T(I_{g_j}) = P_T(\alpha_j I_u)$. Therefore we have the setup of Theorem 7.1 in [3], which implies that

$$\inf\{\beta : N_\beta(\mathcal{C}(g_1, \ldots, g_k; u)) = 0\} = \inf\{\beta : N_\beta(X) = 0\}.$$

The desired result follows from (44) and (45). □

Appendix A. Bowen–Walters distance for suspension flows

We recall here a distance introduced by Bowen and Walters in [5] for suspension flows with an arbitrary height function. We also establish several properties which are needed in the proofs of the statements in Sections 2–5. We would like to thank Valentin Afraimovich and Jean-René Chazottes for bringing the paper [5] to our attention.

As in Section 3.1, let $T \colon X \to X$ be a homeomorphism of the compact metric space (X, d_X), and $\tau \colon X \to (0, \infty)$ a Lipschitz function. Without loss of generality one can assume that the diameter $\operatorname{diam} X$ of X is at most 1.

If this is not the case then since X is compact one can simply consider the new distance $d_X/\operatorname{diam} X$ on X.

We also consider the space Y in (9) with the points $(x,\tau(x))$ and $(Tx,0)$ identified for each $x\in X$. The suspension flow over T with height function τ is the flow $\Psi=\{\psi_t\}_t$ on Y with $\psi_t\colon Y\to Y$ defined as in (10).

We first assume that $\tau=1$ on X, and introduce the Bowen–Walters distance d_1 on the corresponding space Y. We shall first consider horizontal and vertical segments. Given x, $y\in X$ and $t\in[0,1]$ we define the length of the *horizontal segment* $[(x,t),(y,t)]$ by

$$\rho_h((x,t),(y,t))=(1-t)d_X(x,y)+td_X(Tx,Ty). \tag{46}$$

Note that

$$\rho_h((x,0),(y,0))=d_X(x,y) \quad \text{and} \quad \rho_h((x,1),(y,1))=d_X(Tx,Ty).$$

Furthermore, given (x,t), $(y,s)\in Y$ on the same orbit we define the length of the *vertical segment* $[(x,t),(y,s)]$ by

$$\rho_v((x,t),(y,s))=\inf\{|r|:\psi_r(x,t)=(y,s) \text{ and } r\in\mathbb{R}\}. \tag{47}$$

Finally, given two points (x,t), $(y,s)\in Y$ the distance $d_1((x,t),(y,s))$ is defined as the infimum of the lengths of paths between (x,t) and (y,s) composed by a finite number of horizontal and vertical segments.

More precisely, for each $n\in\mathbb{N}$ we consider all finite chains $z_0=(x,t)$, z_2, …, z_{n-1}, $z_n=(y,s)$ of points in Y such that for each i either z_i and z_{i+1} are on the same segment $X\times\{t\}$ for some $t\in[0,1]$ (in which case $[z_i,z_{i+1}]$ is called a *horizontal segment*), or z_i and z_{i+1} are on the same orbit of the flow (in which case $[z_i,z_{i+1}]$ is called a *vertical segment*). The lengths of horizontal and vertical segments are defined respectively in (46) and (47). We remark that when $[z_i,z_{i+1}]$ is simultaneously a horizontal and a vertical segment, since by hypothesis the space X has diameter at most 1, the length of $[z_i,z_{i+1}]$ is computed thinking of it as a horizontal segment. The length of the chain from z_0 to z_n is finally defined as the sum of the lengths of the segments $[z_i,z_{i+1}]$ for $i=0$, …, $n-1$.

We now consider the case of an arbitrary Lipschitz height function $\tau\colon X\to(0,\infty)$, and introduce the Bowen–Walters distance d_Y on Y. Given two points (x,t), $(y,s)\in Y$, we set

$$d_Y((x,t),(y,s))=d_1((x,t/\tau(x)),(y,s/\tau(s))),$$

where d_1 is the Bowen–Walters distance when τ is the constant 1. Note that a horizontal segment is now of the form $w=[(x,t\tau(x)),(y,t\tau(y))]$, and that its length is

$$\ell_h(w)=(1-t)d_X(x,y)+td_X(Tx,Ty).$$

The length of a vertical segment $w=[(x,t),(x,s)]$ now becomes

$$\ell_v(w)=|t-s|/\tau(x),$$

provided that t and s are sufficiently close (or otherwise when x is not a fixed point of T).

We shall from now on assume that T is an invertible Lipschitz map with Lipschitz inverse. We consider a number $L \geq \max\{1/\min\tau, \sup\tau, 1\}$ which is simultaneously a Lipschitz constant for T, T^{-1}, and τ.

Given $(x,t),(y,s) \in Y$ we define

$$d_\pi((x,t),(y,s)) = \min\left\{\begin{array}{l} d_X(x,y)+|t-s|, \\ d_X(Tx,y)+\tau(x)-t+s, \\ d_X(x,Ty)+\tau(y)-s+t \end{array}\right\}. \tag{48}$$

Note that d_π need not be a metric. Nevertheless, the following statement relates d_π with the Bowen–Walters distance d_Y.

Proposition 17. *There exists a constant $c>1$ such that for each p, $q \in Y$ the following property holds:*

$$c^{-1}d_\pi(p,q) \leq d_Y(p,q) \leq c d_\pi(p,q). \tag{49}$$

Proof. Let (x,t), $(y,s) \in Y$. We easily obtain

$$L^{-1}|t-s| - L^2 d_X(x,y) \leq \left|\frac{t}{\tau(x)} - \frac{s}{\tau(y)}\right| \leq L|t-s| + L^2 d_X(x,y). \tag{50}$$

We now consider the chain formed by the points (x,t), $(y,t\tau(y)/\tau(x))$, and (y,s), which is composed of a horizontal and a vertical segment. We obtain

$$\begin{aligned} &d_Y((x,t),(y,s)) \\ &\quad\leq \ell_h((x,t),(y,t\tau(y)/\tau(x))) + \ell_v((y,t\tau(y)/\tau(x)),(y,s)) \\ &\quad\leq \left(1-\frac{t}{\tau(x)}\right)d_X(x,y) + \frac{t}{\tau(x)}d_X(Tx,Ty) + \left|\frac{t}{\tau(x)} - \frac{s}{\tau(y)}\right| \\ &\quad\leq L d_X(x,y) + L|t-s| + L^2 d_X(x,y), \end{aligned} \tag{51}$$

using (50). Therefore

$$d_Y((x,t),(y,s)) \leq c[d_X(x,y)+|t-s|] \tag{52}$$

whenever $c \geq L+L^2$. Considering the chain formed by the points (x,t), $(x,\tau(x)) = (Tx,0)$, $(y,0)$, and (y,s) we obtain

$$\begin{aligned} d_Y((x,t),(y,s)) &\leq \frac{\tau(x)-t}{\tau(x)} + d_X(Tx,y) + \frac{s}{\tau(y)} \\ &\leq L[d_X(Tx,y)+\tau(x)-t+s]. \end{aligned} \tag{53}$$

By (52), (53), and the symmetry of d_Y we conclude that

$$d_Y((x,t),(y,s)) \leq c d_\pi((x,t),(y,s))$$

whenever $c \geq L+L^2$.

Consider now a chain $z_0,\ldots,z_n$ between (x,t) and (y,s), and denote its length by $\ell(z_0,\ldots,z_n)$. Assume further that the chain does not intersect the roof of Y. Let H and V denote the set of indices in the chain corresponding respectively to horizontal and vertical segments, and write

$$\ell_H = \sum_{i\in H}\ell_h(z_i,z_{i+1}) \quad \text{and} \quad \ell_V = \sum_{i\in V}\ell_v(z_i,z_{i+1}).$$

Let us denote $z_i = (x_i, r_i) \in Y$. Since the chain does not cross the roof, for the horizontal length we have

$$\begin{aligned} \ell_H &= \sum_{i\in H}(1-r_i)d_X(x_i,x_{i+1}) + r_i d_X(Tx_i,Tx_{i+1}) \\ &\geq L^{-1}\sum_{i\in H}(1-r_i)d_X(x_i,x_{i+1}) + r_i d_X(x_i,x_{i+1}) \geq L^{-1}d_X(x,y). \end{aligned} \tag{54}$$

For the vertical length, using (50) we obtain

$$\ell_V \geq |t/\tau(x) - s/\tau(y)| \geq L^{-1}|t-s| - L^2 d_X(x,y). \tag{55}$$

It follows from (54) and (55) that

$$2L^4\ell(z_1,\ldots,z_n) \geq (L^4+L)\ell_H + L\ell_V \geq d_X(x,y) + |t-s|. \tag{56}$$

It is easy to see that for any chain of length ℓ there exists another chain with the same endpoints and of length at most $L\ell$, such that at most one segment intersects the roof of Y. Notice that if a chain crosses the roof of Y at least two times in the same direction then its length is at least 2, which is always larger than the length of the chain used to establish (51). Hence $Ld_Y((x,t),(y,s))$ is bounded from below by the infimum of the length of all chains between (x,t) and (y,s) which intersect the roof at most once. Let then $z_0,\ldots,z_n$ be a chain intersecting the roof of Y exactly once. Without loss of generality one can assume that there exists $1 \leq j \leq n$ such that $r_j = \tau(x_j)$ where $z_j = (x_j, r_j)$, and that $[z_{j-1}, z_j]$ is a vertical segment. If z_j is after z_{j-1} along the orbit then by (56) we obtain

$$2L^4[\ell(z_0\ldots,z_j) + \ell(z_j\ldots,z_n)] \geq d_X(x,x_j) + \tau(x) - t + d_X(Tx_j,y) + s.$$

Since

$$Ld(x,x_j) + d(Tx_j,y) \geq d(Tx,Tx_j) + d(Tx_j,y) \geq d(Tx,y)$$

we conclude that

$$2L^5\ell(z_1,\ldots,z_n) \geq d_X(Tx,y) + \tau(x) - t + s. \tag{57}$$

If z_j is before z_{j-1} along the orbit then a similar computation gives

$$2L^5\ell(z_1,\ldots,z_n) \geq d_X(x,Ty) + \tau(y) - s + t. \tag{58}$$

By (56), (57), and (58) we conclude that

$$d_\pi((x,t),(y,s)) \leq c d_Y((x,t),(y,s))$$

provided that $c \geq 2L^6$.

Setting $c = 2L^6$ we obtained the desired inequalities in (49). □

Given a continuous function $g\colon Y \to \mathbb{R}$ we define a new function $I_g\colon X \to \mathbb{R}$ by (11).

Proposition 18. *If g is a Hölder continuous function on Y, then I_g is Hölder continuous on X.*

Proof. We proceed in a similar way to that in the proof of Proposition 6. Let x, $y \in X$ and assume without loss of generality that $\tau(x) \geq \tau(y)$. We obtain

$$\begin{aligned}
|I_g(x) - I_g(y)| &= \left| \int_{\tau(y)}^{\tau(x)} g(\varphi_s x)\, ds + \int_0^{\tau(y)} [g(\varphi_s x) - g(\varphi_s y)]\, ds \right| \\
&\leq \sup|g| \cdot |\tau(x) - \tau(y)| \\
&\quad + \sup \tau \cdot \sup_{s \in (0, \tau(y))} |g(\varphi_s x) - g(\varphi_s y)| \\
&\leq \sup|g| \cdot L d_X(x,y) + b \sup_{s \in (0,\tau(y))} d_Y((x,s),(y,s))^\alpha,
\end{aligned} \tag{59}$$

for some positive constants α and b. It follows from Proposition 17 and (59) (see also (48)) that

$$\begin{aligned}
|I_g(x) - I_g(y)| &\leq \sup|g| \cdot L d_X(x,y) + b\,(c d_\pi((x,s),(y,s)))^\alpha \\
&\leq [\sup|g| \cdot L + bc^\alpha] d_X(x,y)^\alpha.
\end{aligned}$$

This shows that I_g is Hölder continuous on X. ☐

We now consider Bowen balls in X and Y, defined respectively by

$$\begin{aligned}
B_X(x,m,\varepsilon) &\stackrel{\text{def}}{=} \bigcap_{0 \leq n \leq m} T^{-n} B_X(T^n x, \varepsilon), \\
B_Y(y,\rho,\varepsilon) &\stackrel{\text{def}}{=} \bigcap_{0 \leq t \leq \rho} \psi_{-t} B_Y(\psi_t y, \varepsilon).
\end{aligned}$$

We say that T has *bounded distortion* if for each Hölder continuous function $g \colon X \to \mathbb{R}$ there exists a constant $D > 0$ such that if $x \in X$, $m \in \mathbb{N}$, $\varepsilon > 0$, and $y \in B_X(x,m,\varepsilon)$ then

$$\left| \sum_{k=0}^{m-1} g(T^k x) - \sum_{k=0}^{m-1} g(T^k y) \right| \leq D\varepsilon.$$

Recall also the definition of the function τ_m in (35).

Proposition 19. *Assume that T has bounded distortion. There exists $\kappa > 0$ such that for every $x \in X$, $0 < s < \tau(x)$, and $m \in \mathbb{N}$, if $\varepsilon > 0$ is sufficiently small then*

$$B_Y((x,s), \tau_m(x), \frac{1}{\kappa}\varepsilon) \subset B_X(x,m,\varepsilon) \times (s-\varepsilon, s+\varepsilon) \subset B_Y((x,s), \tau_m(x), \kappa\varepsilon). \tag{60}$$

Proof. Let $\varepsilon \in (0, \frac{1}{2c})$ with c as in Proposition 17. Let also $(x,t) \in Y$ with $t \in (c\varepsilon, \tau(x) - c\varepsilon)$, and $(y,t) \in B_Y((x,s), \tau_m(x), \varepsilon)$.

If $m = 0$ then by Proposition 17 we have $d_\pi((x,t),(y,s)) \leq c\varepsilon$. Since

$$\tau(x) - t + s \geq \tau(x) - t \geq c\varepsilon \quad \text{and} \quad \tau(y) - s + t \geq t \geq c\varepsilon$$

we must have

$$d_X(x,y) + |t - s| = d_\pi((x,t),(y,s)) \leq c\varepsilon,$$

which implies that $d_X(x,y) \le c\varepsilon$ and $|t-s| \le c\varepsilon$. This establishes the first inclusion when $m=0$.

For any $1 \le n \le m$ set $t_n = \tau_n(x) - t$ and $s_n = \tau_n(y) - s$. It is easy to see that $\psi_{t_n}(x,t) = (T^n x, 0)$ and $\psi_{s_n}(y,s) = (T^n y, 0)$. By Proposition 17 we obtain

$$\begin{aligned} d_X(T^n x, T^n y) &\le c d_Y(\psi_{t_n}(x,t), \psi_{s_n}(y,s)) \\ &\le c d_Y(\psi_{t_n}(x,t), \psi_{t_n}(y,s)) + c d_Y(\psi_{t_n}(y,s), \psi_{s_n}(y,s)) \\ &\le c\varepsilon + c|t_n - s_n|. \end{aligned} \tag{61}$$

Furthermore, by (48) we have

$$d_\pi(\psi_{t_n}(x,t), \psi_{t_n}(y,s)) \le c\varepsilon.$$

Thus there exists $y_n \in X$ and $r_n \in (t_n - c\varepsilon, t_n + c\varepsilon)$ such that $\psi_{r_n}(y,s) = (y_n, 0)$. Moreover the sequence r_n is strictly increasing, since $t_{n+1} - t_n > 2c\varepsilon$. Hence $s_n \le r_n \le t_n + c\varepsilon$. By symmetry we obtain $t_n \le s_n + c\varepsilon$, and hence $|t_n - s_n| \le c\varepsilon$. By (61) we conclude that

$$d_X(T^n x, T^n y) \le c(1+c)\varepsilon.$$

This establishes the first inclusion in (60) provided that $\kappa \ge c(1+c)$.

Let now $y \in B_X(x,m,\varepsilon)$ and $s \in (t-\varepsilon, t+\varepsilon)$. Take $r \in (0, \tau_m(x))$ and choose n such that $\tau_n(x) \le r + t < \tau_{n+1}(x)$. Write $r' = r + t - \tau_n(x) \ge 0$. By Proposition 17, the bounded distortion property, and (48) we obtain

$$\begin{aligned} d_Y(\psi_r(x,t), \psi_r(y,s)) &\le d_Y((T^n x, r'), (T^n y, r')) + d_Y((T^n y, r'), \psi_r(y,s)) \\ &\le c d_\pi((T^n x, r'), (T^n y, r')) + c d_\pi((T^n y, r'), \psi_r(y,s)) \\ &\le c d_X(T^n x, T^n y) + c|r' + \tau_n(y) - r - s| \\ &\le c d_X(T^n x, T^n y) + c|t-s| + c|\tau_n(x) - \tau_n(y)| \\ &\le c(2+D)\varepsilon. \end{aligned}$$

This establishes the second inclusion in (60) provided that $\kappa \ge c(2+D)$.

Setting $\kappa = \max\{c(1+c), c(2+D)\}$ we obtain the desired inclusions. □

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