Rare events and extremal indices via spectral perturbation

Gerhard Keller

Universität Erlangen

25.3.2014

Introduction

- $T : M \to M$, $A_\epsilon \subset M$ small
- $\tau_\epsilon : M \to \mathbb{N}$, $\tau_\epsilon(x) = \min\{k \in \mathbb{N} : T^k x \in A_\epsilon\}$

Introduction

- $T : M \to M$, $A_\epsilon \subset M$ small
- $\tau_\epsilon : M \to \mathbb{N}$, $\tau_\epsilon(x) = \min\{k \in \mathbb{N} : T^k x \in A_\epsilon\}$
- m probability measure on M
- Questions:

$$m\{\tau_\epsilon \geq n\} \approx \lambda_\epsilon^n \text{ ?} \qquad (\epsilon \text{ fixed, } n \to \infty)$$

$$m\{\tau_\epsilon \geq t/\mu_0(A_\epsilon)\} \approx e^{-\theta t} \text{ ?} \qquad (t \text{ fixed, } \epsilon \to 0)$$

Introduction

- $T : M \to M$, $A_\epsilon \subset M$ small
- $\tau_\epsilon : M \to \mathbb{N}$, $\tau_\epsilon(x) = \min\{k \in \mathbb{N} : T^k x \in A_\epsilon\}$
- m probability measure on M
- Questions:

$$m\{\tau_\epsilon \geq n\} \approx \lambda_\epsilon^n \text{ ?} \qquad (\epsilon \text{ fixed, } n \to \infty)$$

$$m\{\tau_\epsilon \geq t/\mu_0(A_\epsilon)\} \approx e^{-\theta t} \text{ ?} \qquad (t \text{ fixed, } \epsilon \to 0)$$

- Background references: Better ask the specialists at this conference!
- This talk based on:
  - G. Keller, C. Liverani: *Stability of the spectrum for transfer operators*. Ann. Mat. Sc. Norm. Pisa 28 (1999), 141-152.
  - G. Keller, C. Liverani: *Rare Events, Escape Rates and Quasistationarity: Some Exact Formulae*. Journal of Stat. Phys. 135 (2009) 519-534.
  - G. Keller: *Rare events, exponential hitting times and extremal indices via spectral perturbation*. Dynamical Systems 27 (2012) 11-27.

Introduction

- $T : M \to M$, $A_\epsilon \subset M$ small
- $\tau_\epsilon : M \to \mathbb{N}$, $\tau_\epsilon(x) = \min\{k \in \mathbb{N} : T^k x \in A_\epsilon\}$
- m probability measure on M
- Questions:

$$m\{\tau_\epsilon \geq n\} \approx \lambda_\epsilon^n \; ? \qquad (\epsilon \text{ fixed}, \; n \to \infty)$$
$$m\{\tau_\epsilon \geq t/\mu_0(A_\epsilon)\} \approx e^{-\theta t} \; ? \qquad (t \text{ fixed}, \; \epsilon \to 0)$$

- Background references: Better ask the specialists at this conference!
- This talk based on:
  - G. Keller, C. Liverani: *Stability of the spectrum for transfer operators.* Ann. Mat. Sc. Norm. Pisa 28 (1999), 141-152.
  - G. Keller, C. Liverani: *Rare Events, Escape Rates and Quasistationarity: Some Exact Formulae.* Journal of Stat. Phys. 135 (2009) 519-534.
  - G. Keller: *Rare events, exponential hitting times and extremal indices via spectral perturbation.* Dynamical Systems 27 (2012) 11-27.

- f probab. density w.r.t. m: $\quad T_*(f \cdot m) =: Pf \cdot m$
- $P_\epsilon f := P(f \cdot 1_{M \setminus A_\epsilon})$ linear operator, $P_0 = P$
- $P_\epsilon^n 1 = P^n 1_{\bigcap_{k=0}^{n-1} T^{-k}(M \setminus A_\epsilon)} = P^n 1_{\{\tau_\epsilon \geqslant n\}}$
- $m\{\tau_\epsilon \geqslant n\} = \int 1_{\{\tau_\epsilon \geqslant n\}} \, dm = \int P^n 1_{\{\tau_\epsilon \geqslant n\}} \, dm = \int P_\epsilon^n 1 \, dm$
- Idea: P_ϵ has leading eigenvalue λ_ϵ, $\lambda_0 = 1$,

$$\lambda_\epsilon = 1 - (1 - \lambda_\epsilon) \sim e^{-(1-\lambda_\epsilon)}.$$

Then:

$$m\{\tau_\epsilon \geqslant n\} \sim \lambda_\epsilon^n$$

and

$$m\left\{\tau_\epsilon \geqslant \frac{t}{\mu_0(A_\epsilon)}\right\} \sim \lambda_\epsilon^{t/\mu_0(A_\epsilon)} \sim e^{-\frac{1-\lambda_\epsilon}{\mu_0(A_\epsilon)} \cdot t} \sim e^{-\theta t}$$

Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon$, $\lambda_0 = 1$, $m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1$, $Q_\epsilon \varphi_\epsilon = 0$, $m_\epsilon Q_\epsilon = 0$

Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon$, $m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$

Eigenvalue perturbation

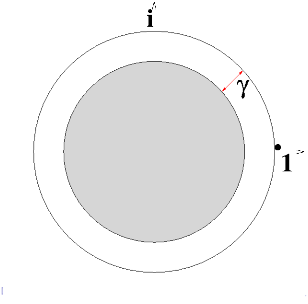
$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon,\ \lambda_0 = 1,\ m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1,\ Q_\epsilon \varphi_\epsilon = 0,\ m_\epsilon Q_\epsilon = 0$

   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon,\ m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \leq C \cdot (1-\gamma)^n$ (γ: spectral gap)

   (Uniform summability suffices.)



Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon$, $\lambda_0 = 1$, $m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1$, $Q_\epsilon \varphi_\epsilon = 0$, $m_\epsilon Q_\epsilon = 0$
   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon$, $m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \leq C \cdot (1 - \gamma)^n$ (γ: spectral gap)
4. $m(\varphi_\epsilon) = 1$, $\|\varphi_\epsilon\| \leq C$

Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon$, $\lambda_0 = 1$, $m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1$, $Q_\epsilon \varphi_\epsilon = 0$, $m_\epsilon Q_\epsilon = 0$

   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon$, $m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \leq C \cdot (1-\gamma)^n$ (γ: spectral gap)
4. $m(\varphi_\epsilon) = 1$, $\|\varphi_\epsilon\| \leq C$
5. $\eta_\epsilon := \|m(P_0 - P_\epsilon)\| \leq C \cdot |\overbrace{m(P_0 - P_\epsilon)\varphi_0}^{=:\Delta_\epsilon}| \to 0$ as $\epsilon \to 0$

Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon,\ \lambda_0 = 1,\ m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1,\ Q_\epsilon \varphi_\epsilon = 0,\ m_\epsilon Q_\epsilon = 0$

   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon,\ m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \le C \cdot (1-\gamma)^n$ (γ: spectral gap)
4. $m(\varphi_\epsilon) = 1,\ \|\varphi_\epsilon\| \le C$
5. $\eta_\epsilon := \|m(P_0 - P_\epsilon)\| \le C \cdot |\overbrace{m(P_0 - P_\epsilon)\varphi_0}^{=:\Delta_\epsilon}| \to 0$ as $\epsilon \to 0$

Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon,\ \lambda_0 = 1,\ m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1,\ Q_\epsilon \varphi_\epsilon = 0,\ m_\epsilon Q_\epsilon = 0$
   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon,\ m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \leq C \cdot (1-\gamma)^n$ (γ: spectral gap)
4. $m(\varphi_\epsilon) = 1,\ \|\varphi_\epsilon\| \leq C$
5. $\eta_\epsilon := \|m(P_0 - P_\epsilon)\| \leq C \cdot |\overbrace{m(P_0 - P_\epsilon)\varphi_0}^{=:\Delta_\epsilon}| \to 0$ as $\epsilon \to 0$

- $(P - P_\epsilon)(f) = P\big(f \cdot 1_{A_\epsilon}\big)$
- For Ass. 5 observe:

$$\Delta_\epsilon = \int P(\varphi_0 \cdot 1_{A_\epsilon})\, dm = \int_{A_\epsilon} \varphi_0\, dm = \mu_0(A_\epsilon)$$

5. $\Leftrightarrow |m(1_{A_\epsilon} f)| \leq C \cdot \|f\| \cdot |m(1_{A_\epsilon} \varphi_0)| \leq C \cdot \|f\| \cdot |\mu_0(A_\epsilon)|$ for all $f \in V$

Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon,\ \lambda_0 = 1,\ m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1,\ Q_\epsilon \varphi_\epsilon = 0,\ m_\epsilon Q_\epsilon = 0$

   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon,\ m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \leq C \cdot (1-\gamma)^n$ (γ: spectral gap)
4. $m(\varphi_\epsilon) = 1,\ \|\varphi_\epsilon\| \leq C$
5. $\eta_\epsilon := \|m(P_0 - P_\epsilon)\| \leq C \cdot |\overbrace{m(P_0 - P_\epsilon)\varphi_0}^{=:\Delta_\epsilon}| \to 0$ as $\epsilon \to 0$

Eigenvalue Perturbation Theorem [Keller/Liverani, JSP '09]

$$\frac{1-\lambda_\epsilon}{\Delta_\epsilon} = \theta \cdot (1 + o(1)) \text{ as } \epsilon \to 0.$$

Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon,\ \lambda_0 = 1,\ m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1,\ Q_\epsilon \varphi_\epsilon = 0,\ m_\epsilon Q_\epsilon = 0$
   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon,\ m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \le C \cdot (1-\gamma)^n$ (γ: spectral gap)
4. $m(\varphi_\epsilon) = 1,\ \|\varphi_\epsilon\| \le C$
5. $\eta_\epsilon := \|m(P_0 - P_\epsilon)\| \le C \cdot |\overbrace{m(P_0 - P_\epsilon)\varphi_0}^{=:\Delta_\epsilon}| \to 0$ as $\epsilon \to 0$

Eigenvalue Perturbation Theorem [Keller/Liverani, JSP '09]
Under one additional assumption defining θ,

$$\frac{1-\lambda_\epsilon}{\Delta_\epsilon} = \theta \cdot (1 + o(1)) \text{ as } \epsilon \to 0.$$

Eigenvalue perturbation

$(V, \|.\|)$ B-space, $P_\epsilon : V \to V$ linear bounded, $P = P_0$

Assumptions: $\exists \lambda_\epsilon \in \mathbb{C}, \varphi_\epsilon \in V, m_\epsilon \in V', Q_\epsilon : V \to V$:

1. $\lambda_\epsilon^{-1} P_\epsilon = \varphi_\epsilon \otimes m_\epsilon + Q_\epsilon$, $\lambda_0 = 1$, $m_0 = m$
2. $m_\epsilon(\varphi_\epsilon) = 1$, $Q_\epsilon \varphi_\epsilon = 0$, $m_\epsilon Q_\epsilon = 0$
   Hence $P_\epsilon \varphi_\epsilon = \lambda_\epsilon \varphi_\epsilon$, $m_\epsilon P_\epsilon = \lambda_\epsilon m_\epsilon$
3. $\|Q_\epsilon^n\| \le C \cdot (1-\gamma)^n$ (γ: spectral gap)
4. $m(\varphi_\epsilon) = 1$, $\|\varphi_\epsilon\| \le C$
5. $\eta_\epsilon := \|m(P_0 - P_\epsilon)\| \le C \cdot |\overbrace{m(P_0 - P_\epsilon)\varphi_0}^{=:\Delta_\epsilon}| \to 0$ as $\epsilon \to 0$

Eigenvalue Perturbation Theorem [Keller/Liverani, JSP '09]
Under one additional assumption defining θ,

$$\frac{1-\lambda_\epsilon}{\Delta_\epsilon} = \theta \cdot (1 + o(1)) \text{ as } \epsilon \to 0.$$

Basic identity:

$$\begin{aligned}\lambda_0 - \lambda_\epsilon &= \lambda_0\, m(\varphi_\epsilon) - m(\lambda_\epsilon\, \varphi_\epsilon) = (m\, P)(\varphi_\epsilon) - m(P_\epsilon \varphi_\epsilon) \\ &= m\big((P - P_\epsilon)\varphi_\epsilon\big)\end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $m_\epsilon(\varphi_0) \cdot (1 - \lambda_\epsilon)$
  $= m_\epsilon(\varphi_0) \cdot m\left((P - P_\epsilon)\varphi_\epsilon\right) = m\Big((P - P_\epsilon)\big(\overbrace{m_\epsilon(\varphi_0) \cdot \varphi_\epsilon}^{\varphi_\epsilon \otimes m_\epsilon(\varphi_0)}\big)\Big)$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $$\begin{aligned} & m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon) \\ &= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\big(\overbrace{m_\epsilon(\varphi_0)\cdot\varphi_\epsilon}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)}\big)\Big) \\ &= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big) \end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $$\begin{aligned} & m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon) \\ &= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\big(\overbrace{m_\epsilon(\varphi_0)\cdot\varphi_\epsilon}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)}\big)\Big) \\ &= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big) \\ &= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)} \end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $$\begin{aligned} & m_\epsilon(\varphi_0) \cdot (1-\lambda_\epsilon) \\ &= m_\epsilon(\varphi_0) \cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\overbrace{(m_\epsilon(\varphi_0)\cdot\varphi_\epsilon)}^{\varphi_\epsilon \otimes m_\epsilon(\varphi_0)}\Big) \\ &= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon \otimes m_\epsilon - \mathbb{I})\varphi_0\Big) \\ &= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)}_{\leqslant \eta_\epsilon \|Q_\epsilon^n\varphi_0\| = \mathcal{O}((1-\gamma)^n \Delta_\epsilon)} \end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $$\begin{aligned} & m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon) \\ &= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\big(\overbrace{m_\epsilon(\varphi_0)\cdot\varphi_\epsilon}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)}\big)\Big) \\ &= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big) \\ &= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)}_{\leqslant \eta_\epsilon\,\|Q_\epsilon^n\varphi_0\| = \mathcal{O}((1-\gamma)^n\Delta_\epsilon)} \\ &= \Delta_\epsilon + \sum_{k=0}^{n-1} m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(\lambda_\epsilon^{-1}P_\epsilon - \mathbb{I})\varphi_0\right) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon) \end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $$\begin{aligned} & m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon) \\ &= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\big(\overbrace{m_\epsilon(\varphi_0)\cdot\varphi_\epsilon}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)}\big)\Big) \\ &= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big) \\ &= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)}_{\leqslant\eta_\epsilon\|Q_\epsilon^n\varphi_0\|=\mathcal{O}((1-\gamma)^n\Delta_\epsilon)} \\ &= \Delta_\epsilon + \sum_{k=0}^{n-1} m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(\lambda_\epsilon^{-1}P_\epsilon - \mathbb{I})\varphi_0\right) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon) \end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $$\begin{aligned} & m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon) \\ &= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\overbrace{(m_\epsilon(\varphi_0)\cdot\varphi_\epsilon)}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)}\Big) \\ &= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big) \\ &= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)}_{\leqslant\eta_\epsilon\,\|Q_\epsilon^n\varphi_0\| = \mathcal{O}((1-\gamma)^n\Delta_\epsilon)} \\ &= \Delta_\epsilon + \sum_{k=0}^{n-1} m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(\lambda_\epsilon^{-1}P_\epsilon - P)\varphi_0\right) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon) \end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon)$

$$= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)(\overbrace{m_\epsilon(\varphi_0)\cdot\varphi_\epsilon}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)})\Big)$$

$$= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big)$$

$$= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)}_{\leqslant\eta_\epsilon\,\|Q_\epsilon^n\varphi_0\|=\mathcal{O}((1-\gamma)^n\Delta_\epsilon)}$$

$$= \Delta_\epsilon + \sum_{k=0}^{n-1} m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(\lambda_\epsilon^{-1}P_\epsilon - P)\varphi_0\right) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon)$$

$$= \Delta_\epsilon - \Delta_\epsilon\cdot\sum_{k=0}^{n-1}\frac{m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(P_\epsilon-P)\varphi_0\right)}{m\Big((P-P_\epsilon)\varphi_0\Big)}$$

$$+ (1-\lambda_\epsilon)\cdot\sum_{k=1}^{n} m\Big((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k\varphi_0\Big) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon)$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $$\begin{aligned}
&m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon)\\
&= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\overbrace{(m_\epsilon(\varphi_0)\cdot\varphi_\epsilon)}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)}\Big)\\
&= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big)\\
&= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)}_{\leqslant\eta_\epsilon\,\|Q_\epsilon^n\varphi_0\|=\mathcal{O}((1-\gamma)^n\Delta_\epsilon)}\\
&= \Delta_\epsilon + \sum_{k=0}^{n-1} m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(\lambda_\epsilon^{-1}P_\epsilon - P)\varphi_0\right) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon)\\
&= \Delta_\epsilon - \Delta_\epsilon\cdot\sum_{k=0}^{n-1}\frac{m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(P_\epsilon - P)\varphi_0\right)}{m\Big((P-P_\epsilon)\varphi_0\Big)}\\
&\quad + (1-\lambda_\epsilon)\cdot\sum_{k=1}^{n} m\Big((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k\varphi_0\Big) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon)\\
&= \Delta_\epsilon\cdot\left(1 - \sum_{k=0}^{n-1} q_{k,\epsilon} + \mathcal{O}((1-\gamma)^n)\right) + (1-\lambda_\epsilon)\cdot\mathcal{O}(n\cdot\eta_\epsilon)
\end{aligned}$$

- $m_\epsilon(\varphi_0) = 1 + \mathcal{O}(\eta_\epsilon)$,
- $m_\epsilon(\varphi_0)\cdot(1-\lambda_\epsilon)$

$$= m_\epsilon(\varphi_0)\cdot m\left((P-P_\epsilon)\varphi_\epsilon\right) = m\Big((P-P_\epsilon)\overbrace{(m_\epsilon(\varphi_0)\cdot\varphi_\epsilon)}^{\varphi_\epsilon\otimes m_\epsilon(\varphi_0)}\Big)$$

$$= m\left((P-P_\epsilon)\varphi_0\right) + m\Big((P-P_\epsilon)(\varphi_\epsilon\otimes m_\epsilon - \mathbb{I})\varphi_0\Big)$$

$$= \Delta_\epsilon + m\left((P-P_\epsilon)((\lambda_\epsilon^{-1}P_\epsilon)^n - \mathbb{I})\varphi_0\right) - \underbrace{m\Big((P-P_\epsilon)Q_\epsilon^n\varphi_0\Big)}_{\leqslant \eta_\epsilon\,\|Q_\epsilon^n\varphi_0\| = \mathcal{O}((1-\gamma)^n\Delta_\epsilon)}$$

$$= \Delta_\epsilon + \sum_{k=0}^{n-1} m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(\lambda_\epsilon^{-1}P_\epsilon - P)\varphi_0\right) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon)$$

$$= \Delta_\epsilon - \Delta_\epsilon\cdot\sum_{k=0}^{n-1}\frac{m\left((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k(P_\epsilon - P)\varphi_0\right)}{m\Big((P-P_\epsilon)\varphi_0\Big)}$$

$$+ (1-\lambda_\epsilon)\cdot\sum_{k=1}^{n} m\Big((P-P_\epsilon)(\lambda_\epsilon^{-1}P_\epsilon)^k\varphi_0\Big) + \mathcal{O}((1-\gamma)^n\Delta_\epsilon)$$

$$= \Delta_\epsilon\cdot\left(1 - \sum_{k=0}^{n-1} q_{k,\epsilon} + \mathcal{O}((1-\gamma)^n)\right) + (1-\lambda_\epsilon)\cdot\mathcal{O}(n\cdot\eta_\epsilon)$$

Rare event Perron Frobenius operators (REPFO)

- $T : M \to M$, $\tau_\epsilon \to \mathbb{N}$, $A_\epsilon \subset M$ and $P, P_\epsilon : V \to V$ as before.
- **Problem:** Make sure that P_ϵ satisfies assumptions ❶ - ❺.

Rare event Perron Frobenius operators (REPFO)

- $T : M \to M$, $\tau_\epsilon \to \mathbb{N}$, $A_\epsilon \subset M$ and $P, P_\epsilon : V \to V$ as before.
- **Problem:** Make sure that P_ϵ satisfies assumptions 1 - 5.
- **Assumptions:** $\exists \alpha \in (0,1), D > 0 \; \exists |.|_w \leq \|.\|$ on V such that
  - (A) $\sigma(P_\epsilon) \cap \{|z| > \alpha\}$ contains only isolated eigenvalues.
  - (B) $|P_\epsilon^n f|_w \leq D \cdot |f|_w$
  - (C) $\|P_\epsilon^n f\| \leq D \cdot (\alpha^n \|f\| + |f|_w)$
  - (D) $\pi_\epsilon := \sup\{|P_\epsilon f - Pf|_w : \; \|f\| \leq 1\} \to 0$ as $\epsilon \to 0$

Rare event Perron Frobenius operators (REPFO)

- $T : M \to M$, $\tau_\epsilon \to \mathbb{N}$, $A_\epsilon \subset M$ and $P, P_\epsilon : V \to V$ as before.
- **Problem:** Make sure that P_ϵ satisfies assumptions ❶ - ❺.
- **Assumptions:** $\exists \alpha \in (0,1), D > 0$ $\exists |.|_w \leq \|.\|$ on V such that
  (A) $\sigma(P_\epsilon) \cap \{|z| > \alpha\}$ contains only isolated eigenvalues.
  (B) $|P_\epsilon^n f|_w \leq D \cdot |f|_w$
  (C) $\|P_\epsilon^n f\| \leq D \cdot (\alpha^n \|f\| + |f|_w)$
  (D) $\pi_\epsilon := \sup\{|P_\epsilon f - Pf|_w : \ \|f\| \leq 1\} \to 0$ as $\epsilon \to 0$
- **Assume (REPFO):**
  - the P_ϵ satisfy (A) - (D)of the spectral perturbation theorem,
  - 1 is a simple eigenvalue of P, all other eigenvalues have modulus < 1.
  - $1_{A_\epsilon} f \in V$ for all $f \in V$,
  - $|m(1_{A_\epsilon} f)| \leq C \cdot \|f\| \cdot |\mu_0(A_\epsilon)|$ for all $f \in V$,
    where $\mu_0 = \varphi_0 \, m$ stationary measure for T.

Corollary:
(REPFO) implies: ❶ - ❺ of the eigenvalue perturbation theorem.
(Uses spectral perturbation theorem of Keller/Liverani 1999; see last slide.)

Examples of (REPFO)-settings

Known:

- Piecewise expanding interval maps [Rychlik 1983]
- Piecewise expanding maps in higher dimensions [Saussol 2000]
- Gibbs measures on subshifts of finite type [Ferguson/Pollicott 2011] (includes Markov chains over finite alphabets)

Examples of (REPFO)-settings

Known:

- Piecewise expanding interval maps [Rychlik 1983]
- Piecewise expanding maps in higher dimensions [Saussol 2000]
- Gibbs measures on subshifts of finite type [Ferguson/Pollicott 2011] (includes Markov chains over finite alphabets)

Further candidates:

- Piecewise hyperbolic maps [Demers/Liverani 2008, Baladi/Gouëzel 2009, 2010]
- Coupled map lattices of piecewise expanding interval maps [Keller/Liverani 2006, 2009]
- Collet-Eckmann maps [Keller/Nowicki 1992]
- Maps with suitable hyperbolic Young towers

Eigenvalue perturbation for REPFOs

Theorem [Keller/Liverani '09]
1) If the (REPFO) assumptions are satisfied, then, for arbitrary $N \in \mathbb{N}$,

$$\frac{1-\lambda_\epsilon}{\mu_0(A_\epsilon)} = \Big(\theta_{N,\epsilon} + \mathcal{O}((1-\gamma)^N)\Big) \cdot (1 + \mathcal{O}(N\,\eta_\epsilon))$$

where

$$\theta_{N,\epsilon} = 1 - \sum_{k=0}^{N-1} \lambda_\epsilon^{-k}\, q_{k,\epsilon}$$

$$q_{k,\epsilon} = \frac{\mu_0\left(A_\epsilon \cap T^{-1}A_\epsilon^c \cap T^{-2}A_\epsilon^c \cap \cdots \cap T^{-k}A_\epsilon^c \cap T^{-(k+1)}A_\epsilon\right)}{\mu_0(A_\epsilon)}$$

Eigenvalue perturbation for REPFOs

Theorem [Keller/Liverani '09]
1) If the (REPFO) assumptions are satisfied, then, for arbitrary $N \in \mathbb{N}$,

$$\frac{1-\lambda_\epsilon}{\mu_0(A_\epsilon)} = \left(\theta_{N,\epsilon} + \mathcal{O}((1-\gamma)^N)\right) \cdot (1 + \mathcal{O}(N\,\eta_\epsilon))$$

where

$$\theta_{N,\epsilon} = 1 - \sum_{k=0}^{N-1} \lambda_\epsilon^{-k}\, q_{k,\epsilon}$$

$$q_{k,\epsilon} = \frac{\mu_0\left(A_\epsilon \cap T^{-1}A_\epsilon^c \cap T^{-2}A_\epsilon^c \cap \dots \cap T^{-k}A_\epsilon^c \cap T^{-(k+1)}A_\epsilon\right)}{\mu_0(A_\epsilon)}$$

2) **If** $q_k := \lim_{\epsilon\to 0} q_{k,\epsilon}$ **for all** k**,** then $\theta_N := \lim_{\epsilon\to 0}\theta_{N,\epsilon} = 1 - \sum_{k=0}^{N-1} q_k$ and

$$\lim_{\epsilon\to 0}\frac{1-\lambda_\epsilon}{\mu_0(A_\epsilon)} = \left(\theta_N + \mathcal{O}((1-\gamma)^N)\right) \quad \text{for all } N.$$

Eigenvalue perturbation for REPFOs

Theorem [Keller/Liverani '09]
1) If the (REPFO) assumptions are satisfied, then, for arbitrary $N \in \mathbb{N}$,

$$\frac{1-\lambda_\epsilon}{\mu_0(A_\epsilon)} = \left(\theta_{N,\epsilon} + \mathcal{O}((1-\gamma)^N)\right) \cdot (1 + \mathcal{O}(N\,\eta_\epsilon))$$

where

$$\theta_{N,\epsilon} = 1 - \sum_{k=0}^{N-1} \lambda_\epsilon^{-k}\, q_{k,\epsilon}$$

$$q_{k,\epsilon} = \frac{\mu_0\left(A_\epsilon \cap T^{-1}A_\epsilon^c \cap T^{-2}A_\epsilon^c \cap \cdots \cap T^{-k}A_\epsilon^c \cap T^{-(k+1)}A_\epsilon\right)}{\mu_0(A_\epsilon)}$$

2) **If** $q_k := \lim_{\epsilon\to 0} q_{k,\epsilon}$ **for all** k**,** then $\theta_N := \lim_{\epsilon\to 0}\theta_{N,\epsilon} = 1 - \sum_{k=0}^{N-1} q_k$ and

$$\lim_{\epsilon\to 0} \frac{1-\lambda_\epsilon}{\mu_0(A_\epsilon)} = 1 - \sum_{k=0}^{\infty} q_k =: \theta$$

Holes shrinking to a submanifold

Example: Two coupled interval maps. $M = [0,1]^2$

$$\hat{T} : M \to M, \quad \hat{T}(x,y) = ((1-\delta)T(x) + \delta T(y), (1-\delta)T(y) + \delta T(x)) \ .$$

$$\lim_{\epsilon \to 0} \frac{1-\lambda_\epsilon}{2\epsilon} = \int_0^1 h_\delta(x,x) \left(1 - \frac{1}{(1-2\delta)|T'(x)|}\right) dx$$

Hitting times

$$\left|\Pr_{fm}\left\{\tau_\epsilon \geq \frac{t}{\mu_0(A_\epsilon)}\right\} - e^{-\xi_\epsilon t}\right| \leq C\,\eta_\epsilon\,|\log \eta_\epsilon|\,(t \vee 1)e^{-t} \quad \text{where } \xi_\epsilon \to \theta.$$

Hitting times

$$\left| \Pr_{fm}\left\{\tau_\epsilon \geq \frac{t}{\mu_0(A_\epsilon)}\right\} - e^{-\xi_\epsilon t}\right| \leq C\,\eta_\epsilon\,|\log\eta_\epsilon|\,(t\vee 1)e^{-t} \quad \text{where } \xi_\epsilon \to \theta.$$

- f probability density w.r.t. m, arbitrary $n, N \in \mathbb{N}$:

$$\begin{aligned}
\Pr_{fm}\{\tau_\epsilon \geq n\} &= \int P_\epsilon^n f \; dm \\
&= \lambda_\epsilon^n \cdot m_\epsilon(f)\cdot m(\varphi_\epsilon) + \mathcal{O}((1-\gamma)^n) \\
&= \left(\left(1 - \mu_0(A_\epsilon)\underbrace{\left(\theta_{N,\epsilon} + \mathcal{O}((1-\gamma)^N)\right)}_{=:\xi_\epsilon \text{ with } N=\mathcal{O}(\log\eta_\epsilon)}\right)\cdot(1+\mathcal{O}(N\eta_\epsilon))\right)^n (1+\mathcal{O}(\eta_\epsilon)) \\
&\qquad + \mathcal{O}((1-\gamma)^n) \\
&= \left[e^{-\mu_0(A_\epsilon)\xi_\epsilon\cdot(1+\mathcal{O}(\eta_\epsilon\log\eta_\epsilon))}\right]^n \cdot (1+\mathcal{O}(\eta_\epsilon)) \\
&= e^{-\xi_\epsilon t\cdot(1+\mathcal{O}(\eta_\epsilon\log\eta_\epsilon))}\cdot(1+\mathcal{O}(\eta_\epsilon)) \qquad n := \left\lceil \frac{t}{\mu_0(A_\epsilon)}\right\rceil
\end{aligned}$$

Exchange rates

- $T : I \to I$ continuous, p.w. expanding with inv. density $h > 0$
- ergodic decomposition $I = J_1 \cup J_2$, $J_1 \cap J_2 = \{z\}$
- stochastic perturbation with a kernel $(x, y) \mapsto \epsilon^{-1} K(\epsilon(y - x))$.
- $V = \{f \in L^1_{Leb} : \int f = 0\}$, $\varphi_0(x) = h\psi$, $m_0 = \psi m$ with $\psi = 1_{J_1} - 1_{J_2}$.
-

$$\lim_{\epsilon \to 0} \frac{1 - \lambda_\epsilon}{2\epsilon} = \frac{\alpha + \beta}{2} \cdot \mathbb{E}[|W|] + \frac{\alpha - \beta}{2} \cdot \mathbb{E}[Z]$$

$$\alpha = \frac{h(z^-)}{2m(J_1)} \text{ and } \beta = \frac{h(z^+)}{2m(J_2)}$$

Z a random variable distributed with density K,
$Z_0, Z_1, \ldots$ independent copies,

$$W := \frac{\sum_{k=0}^{\infty} \frac{Z_k}{|T'(z)|^k}}{\sum_{k=0}^{\infty} \frac{1}{|T'(z)|^k}}$$

Spectral perturbation

- $T : M \to M$, $\tau_\epsilon \to \mathbb{N}$, $A_\epsilon \subset M$ and $P, P_\epsilon : V \to V$ as before.
- **Problem:** Make sure that P_ϵ satisfies assumptions ❶ - ❺.

Assumptions: $\exists \alpha \in (0,1), D > 0\ \exists |.|_w \leq \|.\|$ on V such that

(A) $\sigma(P_\epsilon) \cap \{|z| > \alpha\}$ contains only isolated eigenvalues.

(B) $|P_\epsilon^n f|_w \leq D \cdot |f|_w$

(C) $\|P_\epsilon^n f\| \leq D \cdot (\alpha^n \|f\| + |f|_w)$

(D) $\pi_\epsilon := \underbrace{\sup\{|P_\epsilon f - Pf|_w :\ \|f\| \leq 1\}}_{=:|\!|\!| P_\epsilon - P |\!|\!|} \to 0$ as $\epsilon \to 0$

Spectral perturbation

- $T : M \to M$, $\tau_\epsilon \to \mathbb{N}$, $A_\epsilon \subset M$ and $P, P_\epsilon : V \to V$ as before.
- **Problem:** Make sure that P_ϵ satisfies assumptions 1 - 5.

Assumptions: $\exists \alpha \in (0,1), D > 0\ \exists |.|_w \leq \|.\|$ on V such that

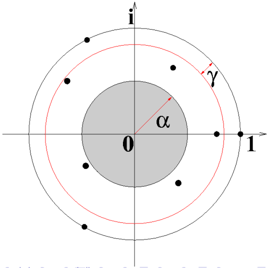
(A) $\sigma(P_\epsilon) \cap \{|z| > \alpha\}$ contains only isolated eigenvalues.

(B) $|P_\epsilon^n f|_w \leq D \cdot |f|_w$

(C) $\|P_\epsilon^n f\| \leq D \cdot (\alpha^n \|f\| + |f|_w)$

(D) $\pi_\epsilon := \underbrace{\sup\{|P_\epsilon f - Pf|_w : \ \|f\| \leq 1\}}_{=:|||P_\epsilon - P|||} \to 0$ as $\epsilon \to 0$

Remark:
(B) - (D) implies (A), if
$\{f \in V : \|f\| \leq 1\}$
is compact in $(V, |.|_w)$.



Spectral perturbation

- $T : M \to M$, $\tau_\epsilon \to \mathbb{N}$, $A_\epsilon \subset M$ and $P, P_\epsilon : V \to V$ as before.
- **Problem:** Make sure that P_ϵ satisfies assumptions 1 - 5.

Assumptions: $\exists \alpha \in (0,1), D > 0 \; \exists |.|_w \leq \|.\|$ on V such that

(A) $\sigma(P_\epsilon) \cap \{|z| > \alpha\}$ contains only isolated eigenvalues.

(B) $|P_\epsilon^n f|_w \leq D \cdot |f|_w$

(C) $\|P_\epsilon^n f\| \leq D \cdot (\alpha^n \|f\| + |f|_w)$

(D) $\pi_\epsilon := \underbrace{\sup\{|P_\epsilon f - Pf|_w : \; \|f\| \leq 1\}}_{=:|||P_\epsilon - P|||} \to 0$ as $\epsilon \to 0$

Spectral Perturbation Theorem [Keller/Liverani, Ann. Sc. N. Pisa '99]
(A) - (D)implies that "all spectral quantities of the P_ϵ in $\{|z| > \alpha\}$ are Hölder continuous in $|||.|||$-norm". In particular:

$$|||(z - P_\epsilon)^{-1} - (z - P)^{-1}||| \leq \pi_\epsilon^\rho \cdot C_z \cdot \|(z - P)^{-1}\|^2$$

where $\rho = \rho(z) \in (0,1)$.

Corollary: (A) - (D) plus mixing implies 1 - 4