The tragedy of the commons via mean-field games

Domènec Ruiz-Balet[1]

[1] Imperial College London

Brest, 18th January 2024

Complexity in Bio-economics of marine fisheries

Mathematics for Bio-Economics and Sustainability of Fisheries.

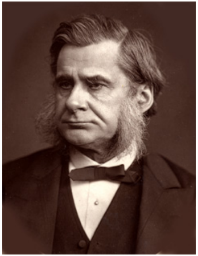
Joint work with **Ziad Kobeissi** and **Idriss Mazari**

Imperial College
London

London, 139 years ago...

I believe then that the cod fishery, the herring fishery, pilchard fishery, the mackerel fishery, and probably all the great sea ***fisheries are inexhaustible****: that is to say that* ***nothing we do seriously affects the number of fish****. And any attempt to regulate these fisheries seems consequently from the nature of the case to be useless*

T.H.Huxley (1884) Inaugural Address, Fisheries Exhibition Lit., 4,1-22, London



The world today: The depletion of the sea

SCIENCE | NEWS

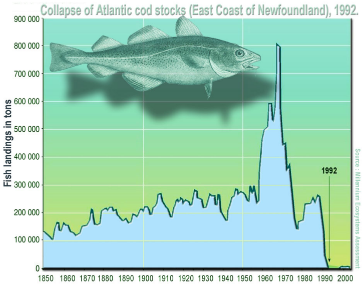
The sea is running out of fish, despite nations' pledges to stop it

Major countries that are promising to curtail funding for fisheries are nevertheless increasing handouts for their seafood industries.

BY **TODD WOODY**

PUBLISHED OCTOBER 8, 2019 • 7 MIN READ

As global fish stocks that feed hundreds of millions of people dwindle, nations are scrambling to finalize by year's end an international agreement to ban government subsidies that fuel overfishing.



The sea is running out of fish, despite nations' pledges to stop it, The National Geographic, 2019

Tragedy of the commons

Definition (≈ Wikipedia)

The **tragedy of the commons** is a situation in which individual users, who have **open access to a resource**, act independently according to their **own self-interest** and **cause depletion of the resource** through their uncoordinated action.

Effects on unregulating land in Ireland

William Forster Lloyd

The tragedy of the commons

Garret Hardin

Tragedy of the commons: objectives

2x2 Game

Typically the tragedy of the commons is understood like a prisoner's dilemma situation.

Objective

Here we want to understand the interplay between spatial effects and game theory.

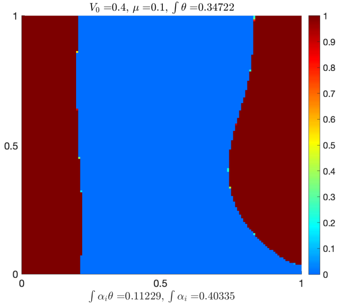
Why Mean-field games to see spatial features?

In a previous work[1] we considered a two player static game on a PDE

$$-\mu\Delta\theta = \theta(K(x)-\theta) - \alpha_1(x)\theta - \alpha_2(x)\theta \quad x\in\Omega$$

$$J_i(\vec{\alpha}) = \int_\Omega \alpha_i \theta_{\alpha_1,\alpha_2} dx$$

Each player was covering an area and determined their "fishing" intensity



[1] Mazari, R-B, Journal of Mathematical Biology 2022

Why Mean-field games to see spatial features?

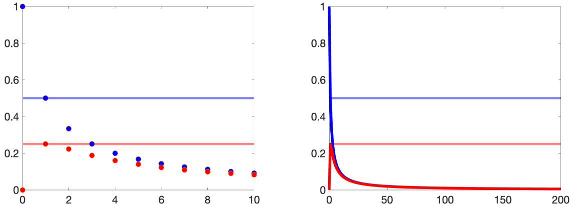
Some geometrical issues found

Competition < Cooperation but not extinction in a 2 player game.

Tragedy of the commons: Extinction when players tend to infinity

There exist a sequence of Nash equilibria $\vec{\alpha}_N^* \in \mathcal{M}^N$, $N \in \mathbb{N}$ such that

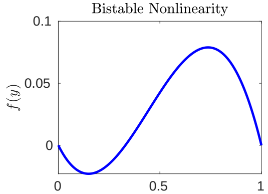
$$\frac{1}{4} = \max_{\vec{\alpha}\in\mathcal{M}^N} \left(\sum_{i=1}^{N} J_i(\vec{\alpha}) \right) > \sum_{i=1}^{N} J_i(\vec{\alpha}_N^*) \underbrace{\longrightarrow}_{N\to+\infty} 0$$



The population model

$$\partial_t \theta = \partial_{xx}\theta + f(\theta) \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+$$

$$\theta(x,0) = \theta_0(x)$$



Allee Effect

When the density of individuals is too low the population decreases

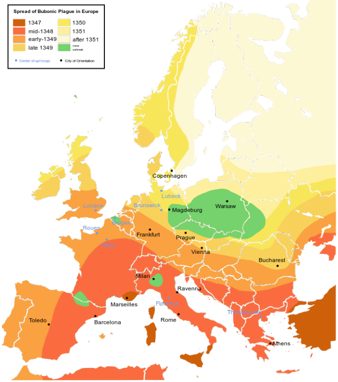
f bistable nonlinearity $f(0) = f(\eta) = f(1) = 0, \quad f'(0), f'(1) < 0, f'(\eta) > 0$

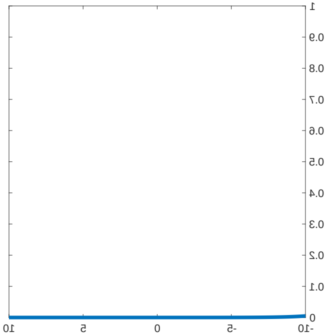
$$\partial_t \theta = f(\theta), \theta(0) = \theta_0$$

If $\theta_0 < \eta$ the population dies,
If $\theta_0 > \eta$ the population grows

Reaction-diffusion equations

$$\partial_t u = \underbrace{f(u)}_{\text{Reaction}} + \underbrace{\mu \Delta u}_{\text{Diffusion}}$$





23

[2] Fisher Kolmogorov Petrovsky and Piskunov

[3] Fife -McLeod

Traveling waves

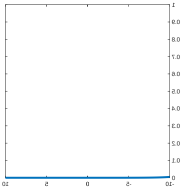
The models show traveling waves, solutions of the form $\theta(x,t) = \Theta(x - ct)$. Moreover, these solutions enjoy generic dynamical attractivity

Theorem (Existence and attractivity)

Let f bistable s.t. $\int_0^1 f(s)ds > 0$, then, there exists a unique speed $c < 0$ and a Traveling wave Θ. Moreover, let

$$\lim_{x \to -\infty} \theta_0(x) < \eta, \qquad \lim_{x \to +\infty} \theta_0(x) > \eta$$

Then there exists x_0, K, w such that $|\theta(x,t) - \Theta(x - ct + x_0)| \leq Ke^{-wt}$



The Fishing problem

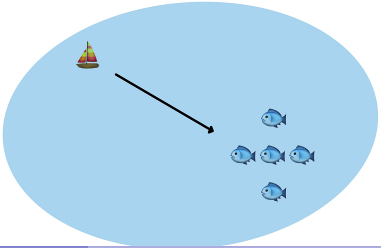
$$\partial_t \theta = \partial_{xx}\theta + f(\theta)$$

Assumption

The **influence of an individual** on the population is **negligible**

$$J(\alpha) = \int_0^{+\infty} \underbrace{e^{-\lambda t}}_{\text{discount factor}} \Bigg(\underbrace{\theta(x_\alpha(t), t)}_{\text{Harvested Fish}} - \underbrace{L(\alpha(t))}_{\text{control cost}} \Bigg) dt$$

$$\dot{x}_\alpha(t) = \alpha(t), \quad x(0) = x_0$$



Mean-field games

Assumption

An **individual fisherman** has a **negligible effect** on the population of fish

HOWEVER, we consider many many fisherman and

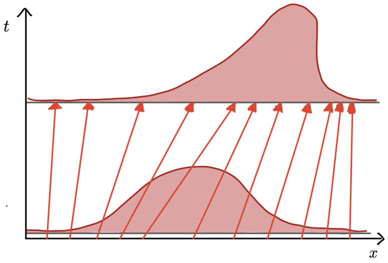
Important

All the fisherman together do have an impact on the population

$$\partial_t \theta - \partial_{xx}\theta = f(\theta) \underbrace{-m(x,t)\theta}_{\text{Total Harvested Fish}}$$

where m are all fishermen following a continuity equation

$$\partial_t m + \partial_x \big(\alpha(x,t) m\big) = 0$$



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$$\partial_t\theta - \partial_{xx}\theta = f(\theta) \quad \underbrace{-m(x,t)\theta}_{\text{Total Harvested Fish}}$$

where m are all fishermen following a continuity equation

$$\partial_t m + \partial_x\big(\alpha(x,t)m\big) = 0$$

But $\alpha(x,t) = \alpha_x(t)$ is not any control, it comes from a selfish optimization

$$J(\alpha_x) = \int_0^{+\infty} e^{-\lambda t}\left(\theta_\alpha(y(t),t) - L(\boldsymbol{\alpha_x(t)})\right) dt \quad \dot{y}_\alpha(t) = \boldsymbol{\alpha_x(t)}, \quad y(0) = x$$

Context of Mean-field games: Larsry and Lions and many others

Derivation of the system

The value function

$$V(x,t) = \sup_{\alpha \in L^\infty(\mathbb{R})} \int_t^{+\infty} e^{-\lambda s} \left(\theta(y(s),s)) - L(\alpha(s))\right) ds, \quad y' = \alpha(t), \quad y(0) = x$$

then V satisfies a **Hamilton-Jacobi** equation:

$$\partial_t V + \lambda V - H(\partial_x V) = \theta(x,t)$$

and the **optimal feedback** is given by

$$\alpha^*(x) = H'(\partial_x V(x))$$

therefore the **fisherman move according** to

$$\partial_t m + \partial_x \left(H'(\partial_x V) m\right) = 0$$

Derivation of the system

The complete Mean-field Game system reads

$$\begin{cases} \partial_t \theta - \partial_{xx}\theta = f(\theta) - m\theta, & \theta(0) = \theta_0 \\ \partial_t m + \partial_x \left(H'(\partial_x V) m\right) = 0, & m(0) = m_0 \\ \partial_t V + \lambda V - H(\partial_x V) = \theta & V(+\infty) = 0 \end{cases}$$

The solutions of the system above correspond to Nash equilibria.

Extinction and Reversed Traveling waves

Extinction

Given θ_0 and m, we say that the population θ extinguishes if

$$\forall x \in \mathbb{R} \lim_{t\to+\infty} \theta(x,t) = 0 \quad \text{where} \quad \partial_t\theta - \partial_{xx}\theta = f(\theta) - m\theta, \quad \theta(0) = \theta_0$$

There are **many possible ways** in which a population can **go extinct** we will try to capture extinction by a "Reversed Traveling Wave"

Reversed Traveling Wave

- If $m = 0$ then

$$\forall x \in \mathbb{R} \lim_{t\to+\infty} \theta_{m(x,t)=0}(x,t) = 1$$

- There exist $\mathcal{M} \in L^1(\mathbb{R})$ and $c > 0$ such that $m(x,t) = \mathcal{M}(x-ct)$ and $\theta(x,t) = \Theta(x-ct)$ are a solution to the system where

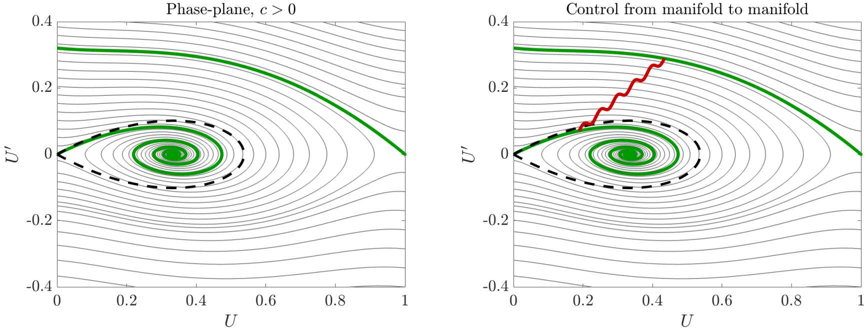
$$\forall x \in \mathbb{R} \lim_{t\to+\infty} \theta_{m(x,t)=\mathcal{M}(x-ct)}(x,t) = 0$$

Reversed traveling waves for the MFG (1)

- First generate a reversed traveling wave for the couple (θ, m).
- Set $\theta(x,t) = \Theta(x-ct)$, $m(x,t) = \mathcal{M}(x-ct)$, $c > 0$ and plug it:

$$\begin{pmatrix}\Theta\\ \Theta'\end{pmatrix}' = \begin{pmatrix}\Theta' \\ -c\Theta' - f(\Theta) + \mathcal{M}\Theta\end{pmatrix}, \begin{cases}\Theta(-\infty) = 0, \Theta'(-\infty) = 0, \\ \Theta(+\infty) = 1, \Theta'(+\infty) = 0\end{cases}$$

Therefore looking for a bilinear type controllability



4

[4] A.Bressan, M.Chiri, N. Salehi, On the optimal control of propagation fronts, M3AS

Reversed traveling waves for the MFG (2)

We need more properties:

- We require that $\mathcal{M} \geq 0$
- Since m follows

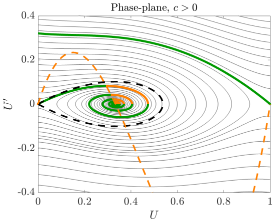
$$\partial_t m + \partial_x \left(H'(\partial_x V) m \right) = 0, \quad m(0) = m_0$$

and we want a traveling wave, we need that

$$H'(\partial_x V) = c \text{ in } \operatorname{supp}(m(\cdot, t)) \quad \text{or} \quad H'(\mathcal{V}') = c \text{ in } \operatorname{supp}(\mathcal{M})$$

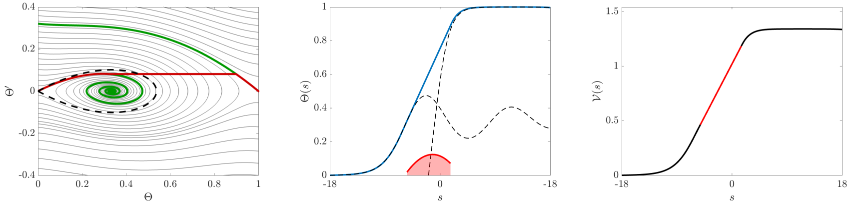
- This will imply that: $\mathcal{V}' = L'(c) = \lambda \Theta'$ in $\operatorname{supp}(\mathcal{M})$
- Hence Θ is linear in $\operatorname{supp}(\mathcal{M})$, hence $\Theta'' = 0$ in $\operatorname{supp}(\mathcal{M})$.

$$\mathcal{M} = \frac{c\Theta' + f(\Theta)}{\Theta}$$



Reversed traveling waves for the MFG (3)

Monotonous Reversed Traveling Waves



Theorem K,M,R-B 2023, Existence of reversed TW

For L quadratic, there exits $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ and every $0 < c \leq c_0(\lambda)$ a reversed traveling waves $(\Theta, \mathcal{M}, \mathcal{V}, \lambda)$ for the MFG exists

Hints of the proof

This result depends on the discount factor λ.

The proof relies on two simple aspects:

1. Fix an admissible reversed traveling wave Θ
2. Compute the conditions for which $\alpha = c$ satisfies first order optimality conditions for any player in the support of m

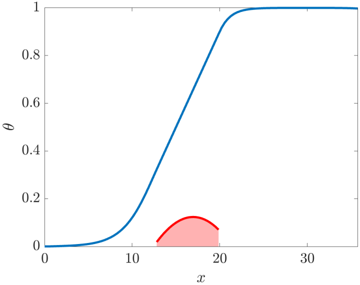
$$DJ(\alpha)[h] = \int_0^{+\infty} \left(p - e^{-\lambda t}\alpha\right) h dt \qquad \begin{cases} -p' = e^{-\lambda t}\theta' \\ p(+\infty) = 0 \end{cases}$$

3. Observe when the functional is strictly concave differentiating once again.

$$D^2 J(\alpha)[h,h] \leq \left(\frac{\|\Theta''\|_{L^\infty}}{\lambda^2} - 1\right) \int_0^{+\infty} e^{-\lambda t} h^2 dt$$

Reversed traveling waves for the MFG (3)

Monotonous Reversed Traveling Waves

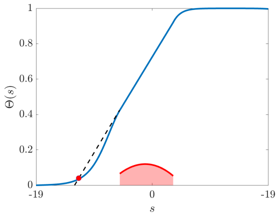


Theorem K,M,R-B 2023, Existence of reversed TW

For any $\lambda > 0$, there exists $\epsilon_0(\lambda) \in (0, c_0(\lambda))$ such that, for any $c \in (0, \epsilon_0(\lambda))$ there exists a monotonous reversed MFG travelling wave with velocity c.

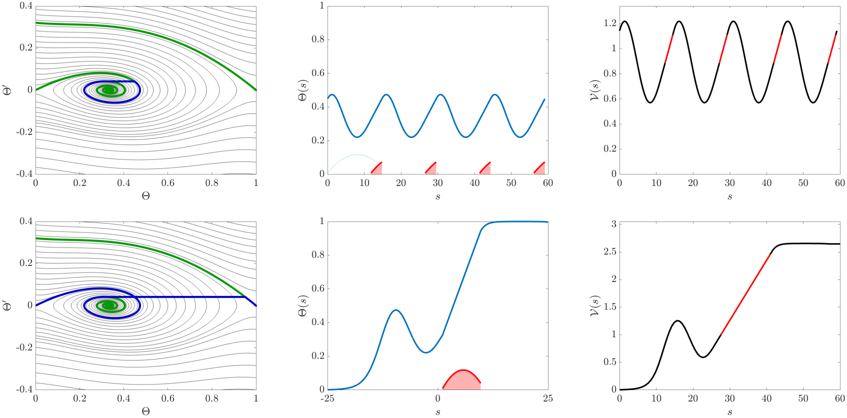
Hints of the proof for the general version

1. The idea for this will be a comparison argument with an auxilary control problem.
2. This auxiliary control problem could be understood as a sort of linearization.
3. We consider ϕ to be affine, and to play the role of Θ. $\phi = \Theta$ in the support of $\mathcal{M}$.
4. The unique optimal control taking ϕ is exactly $\alpha = c$
5. $V_\phi \geq V_\Theta$ for every player in the support of $\mathcal{M}$ where V is the value function.



Other results

Periodic "Reversed" Traveling Waves and nonmonotonous Traveling waves



Cooperation and non-extinction

The total pay-off of all the individuals is given by

$$\mathcal{J}(\alpha) = \int_0^{+\infty} e^{-\lambda t} \int_{\mathbb{R}} m_\alpha(x,t)\Big(\theta_{m_\alpha}(x,t) - L(\alpha(x,t))\Big) dxdt$$

where

$$\partial_t \theta - \partial_{xx}\theta = f(\theta) - m_\alpha \theta, \quad \theta(0) = \Theta$$

$$\partial_t m + \partial_x(\alpha m) = 0, \quad m(0) = \mathcal{M}$$

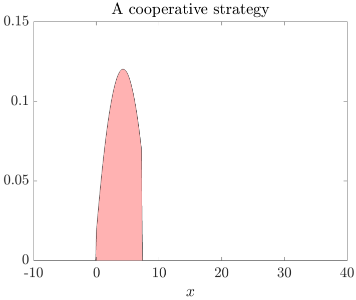
Thm K,M,R-B 2023, Cooperative non-extinguishing strat.

For certain L and λ small, there exists α^* such that

$$\mathcal{J}(\alpha^*) > \mathcal{J}(\alpha_{MFG} = c) \quad \text{and} \quad \forall x \in \mathbb{R}, \begin{cases} \lim_{t\to+\infty} \theta_{m_{\alpha^*}}(x,t) = 1 \\ \lim_{t\to+\infty} \theta_{m_{\alpha_{MFG}=c}}(x,t) = 0 \end{cases}$$

Hints of the proof

1 Choose a common strategy α^* that "separates the players" (decreases the L^∞ norm of m) and evaluate it on the functional $\mathcal{J}(\alpha^*)$



Do it in a way s.t. every player goes at speed $\alpha(x,t) \leq c$!

$$\int_{\mathbb{R}} m(x,t)L(\alpha_{coop}) \leq \int_{\mathbb{R}} m(x,t)L(c)$$

Hints of the proof

2 The reversed traveling wave profile depends on λ. Therefore, we think on a Lagrangian that depends on λ as well so that the traveling wave on θ is fixed as we change λ

$$L_q : \alpha \mapsto \frac{\kappa}{2}|\alpha|^{2q}, \quad q = \frac{\lambda_0}{\lambda}$$

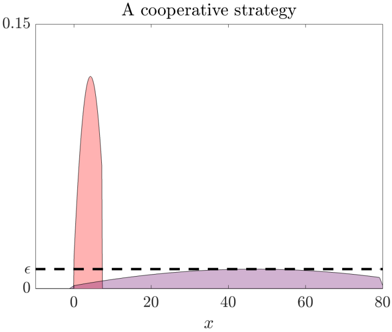
Recall the condition for being an admissible TW candidate:

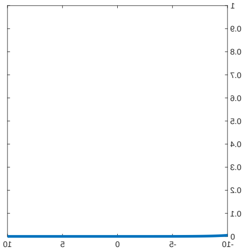
$$L'(c) = \lambda\Theta'$$

Hints of the proof

3 We see that for λ small enough the cooperative strategy is better than the MFG (sub-solution argument plus attractivity of TW).

$$\partial_t\theta - \partial_{xx}\theta = f(\theta) - m\theta \geq f(\theta) - \epsilon\theta$$





Conclusion

Key phenomena for this study

" Reversal or Extinction = the Allee effect + harvesting game "

Mean field game formulation

Infinitely many fisherman each one of them having a negligible impact and acting on their own self interest

Measuring catches

1. The total catch of the traveling wave is constant, but leads to the tragedy of the commons
2. Higher catching rates can be done without extinguishing the resource.
3. This means that by measuring only the catches one cannot assess if the tragedy of the commons is happening or not (Cod collapse)

Conclusion

Prisoner's dilemma (again)

Cooperative strategy vs MFG strategy as a **prisoners dilemma**

Stability?

Deepen understanding on the stability of the traveling waves as well as of the extinction

Solutions?

How can we avoid all this situation?

Acknowledgments

Thank you for your attention

Ziad Kobeissi

Idriss Mazari

[1] Ziad Kobeissi, Idriss Mazari, Domènec Ruiz-Balet, The Tragedy of the commons: an MFG approach to the reversal of traveling waves, preprint.

[2] Idriss Mazari, Domènec Ruiz-Balet, Spatial ecology, optimal control and game theoretical fishing problems, Journal of Mathematical Biology 2022.

If I have some more time...

$$
\begin{cases}
-\mu\Delta u = u\left(k(x) - u\right) - \underbrace{\boldsymbol{\alpha_1(x)}u}_{\text{Player 1}} - \underbrace{\boldsymbol{\alpha_2(x)}u}_{\text{Player 2}} & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\
u > 0 & \text{in } \Omega.
\end{cases}
$$

$$
\max_{\alpha_1 \in \mathcal{M}} J_1(\alpha_1, \alpha_2) = \max_{\alpha_1 \in \mathcal{M}} \int_\Omega \alpha_1 u_{\alpha_1,\alpha_2} dx
$$

$$
\max_{\alpha_2 \in \mathcal{M}} J_2(\alpha_1, \alpha_2) = \max_{\alpha_2 \in \mathcal{M}} \int_\Omega \alpha_2 u_{\alpha_1,\alpha_2} dx
$$

Nash equilibria

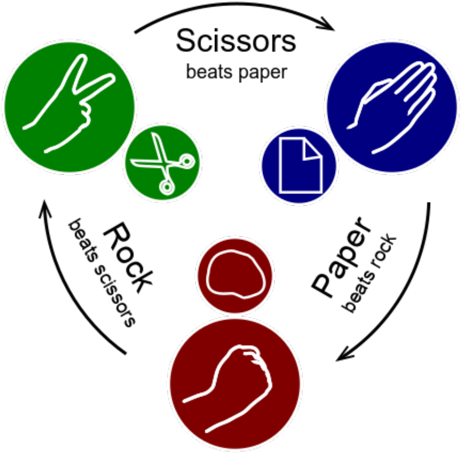
Nash equilibrium for a 2-player game

A Nash equilibrium is a pair of strategies (α_1^*, α_2^*) that satisfies

$$J_1(\alpha_1^*, \alpha_2^*) \geq J_1(\alpha, \alpha_2^*) \quad \forall \alpha \in \mathcal{M}$$

$$J_2(\alpha_1^*, \alpha_2^*) \geq J_2(\alpha_1^*, \alpha) \quad \forall \alpha \in \mathcal{M}$$

Non-existence of Nash equilibria



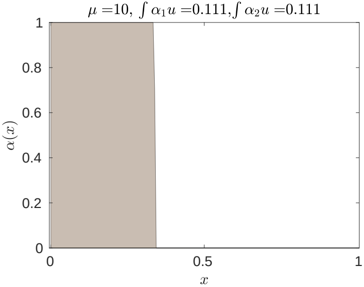
A temptative simulation

We run a **fixed point algorithm**,

$$\alpha_1^{(n+1)} \leftarrow \max_{\alpha \in \mathcal{M}} J_1(\alpha, \alpha_2^{(n)})$$

$$\alpha_2^{(n+1)} \leftarrow \max_{\alpha \in \mathcal{M}} J_2(\alpha_1^{(n)}, \alpha)$$

and **hope** for convergence



- **Convergence $\implies$ existence of Nash eq.**
- The total harvested is equal to $0.222 < \frac{1}{4}$
- $k(x) = 1$ but the **Nash eq is bang-bang**
- For **potential games** the algo converges
- **No numerical guarantees**

Tragedy of the commons

Do there exist Nash equilibria that show a **depletion** of the fishery?

Consider

$$\begin{cases} -\mu\Delta u_{\vec{\alpha}} = u_{\vec{\alpha}}(1 - u_{\vec{\alpha}}) - \left(\sum_{i=1}^{N} \alpha_i(x)\right) u_{\vec{\alpha}} & x \in \Omega \\ \partial_\nu u_{\vec{\alpha}} = 0 & x \in \partial\Omega \end{cases}$$

Where each player is optimizing

$$J_i(\vec{\alpha}) = \int_\Omega \alpha_i u_{\vec{\alpha}} dx$$

Tragedy of the commons

There exist a sequence of Nash equilibria $\vec{\alpha}_N^* \in \mathcal{M}^N$, $N \in \mathbb{N}$ such that

$$\frac{1}{4} = \max_{\vec{\alpha}\in\mathcal{M}^N} \left(\sum_{i=1}^{N} J_i(\vec{\alpha}) \right) > \sum_{i=1}^{N} J_i(\vec{\alpha}_N^*) \underbrace{\longrightarrow}_{N\to+\infty} 0$$

High-diffusivity limit

Assymptotic expansion on μ

Take $|\Omega| = 1$

The functional expands like

$$J_\mu(\alpha) = J^0(\alpha) + \frac{1}{\mu} J_1(\alpha) + O\left(\frac{1}{\mu^2}\right)$$

$$J^0 : \alpha \mapsto \left(\int_\Omega \alpha\right)\left(K_0 - \int_\Omega \alpha\right)$$

$$J^1 : \alpha \mapsto \int_\Omega \alpha v_\alpha$$

The state expands like

$$u_{\alpha,\mu} = \underbrace{\left(K_0 - \int_\Omega \alpha\right)}_{=:M_\alpha} + \frac{v_\alpha}{\mu} + \underset{\mu\to\infty}{O}\left(\frac{1}{\mu^2}\right) \text{ where } \begin{cases} -\Delta v_\alpha - M_\alpha\left(K - \alpha - M_\alpha\right) = 0 & \text{in } \Omega \\ \frac{\partial v_\alpha}{\partial \nu} = 0 \\ \int_\Omega v_\alpha = \frac{1}{M_\alpha^2} \int_\Omega |\nabla v_\alpha|^2. \end{cases}$$

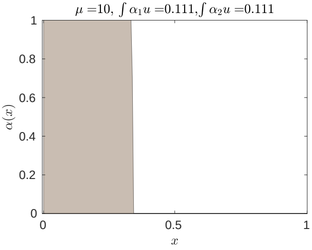
One result

Asymptotic Nash Theorem Mazari-RB 2022

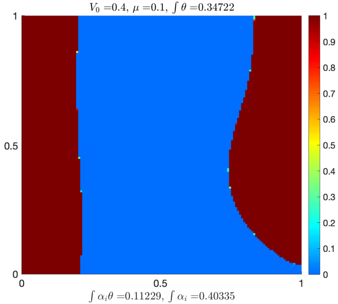
Assume V_1 , $V_2 > \frac{K_0}{4}$, assume K is constant and let

$$\alpha_i^* = \kappa_i \chi_{[0;l_i]} \text{ with } \kappa_i l_i = V_i \ (i = 1, 2).$$

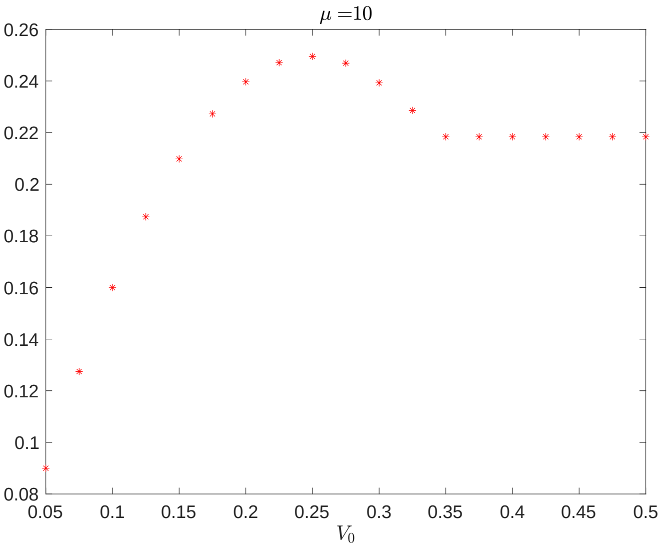
$(\alpha_1^* , \alpha_2^*)$ is a Nash equilibrium in the asymptotic regime



Fragmentation of Nash equilibria?



Optimal regulation?



The tragedy of the commons via mean-field games

Domènec Ruiz-Balet[1]

[1] Imperial College London

Brest, 18th January 2024

Benasque, IX partial differential equations, optimal design and numerics

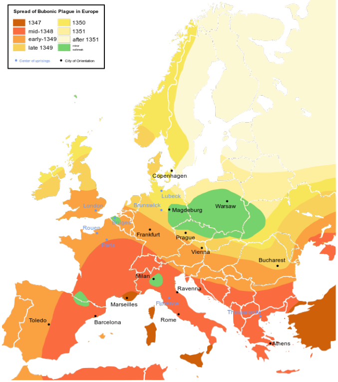
Joint work with **Idriss Mazari**

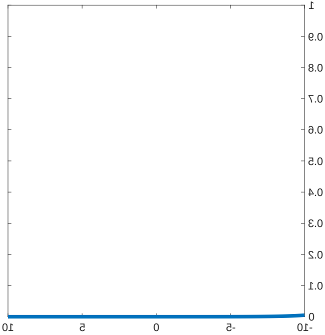
Imperial College London

Elastic Manufacturing Systems

Reaction-diffusion equations

$$\partial_t u = \underbrace{f(u)}_{\text{Reaction}} + \underbrace{\mu \Delta u}_{\text{Diffusion}}$$





[5][6]

[5] Fisher Kolmogorov Petrovsky and Piskunov

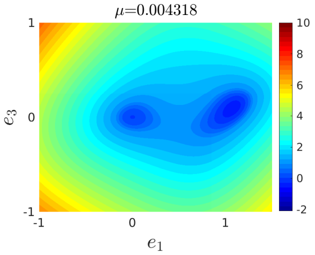
[6] Fife -McLeod

Reaction-diffusion equations

•Gradient structure

$$\partial_t u = -\nabla_u J(u)$$

$$J(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \int_0^{u(x)} f(s)ds \right) dx$$



•Convergence to a steady state

$$\begin{cases} -\mu\Delta u = f(u) & x \in \Omega \\ +\text{Boundary conditions} \end{cases}$$

Current avenues: well-posedness theory and long time behaviour

A monostable bounded domain version

$$\begin{cases} \partial_t \theta - \Delta\theta = \theta(k(x)-\theta) - m\theta, & \theta(0)=\theta_0 \\ \partial_t m + \mathrm{div}_x\,(H'(\nabla V)m) = \sigma\Delta m, & m(0)=m_0 \\ \partial_t V - H(\nabla V) = \theta - \sigma\Delta V & V(T)=0 \end{cases}$$

- **Existence** by fixed points and **Uniqueness** under certain conditions using Lasry-Lions monotonicity condition
- **Long time behaviour**: adaptation of techniques developed by Cardialaguet,Bardi, Kouhkouh and others

Part I: Optimization in Mathematical Ecology

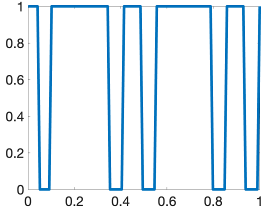
Fires Left These Wallabies Nothing to Eat. Help Arrived From Above, The New York Times, 18 March 2020.

We are interested on maximizing the total population

$$\max_{m \in \mathcal{M}} \int_\Omega u dx, \qquad \mathcal{M} := \left\{ m \in L^\infty(\Omega; [0, \kappa]), \quad \frac{1}{|\Omega|} \int_\Omega m = m_0. \right\}$$

where u follows

$$\begin{cases} -\mu \Delta u = u \left(\underbrace{m}_{\text{Resources}} - u \right) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$



Maximizers of the problem are **Bang-Bang**[7] [8]

Question

How is μ related to the optimal resource distribution?

[7] I.Mazari, G. Nadin, Y. Privat, 2020, & I.Mazari, G. Nadin, Y. Privat, 2021

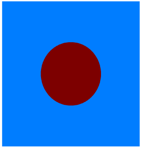
[8] K. Nagahara and E. Yanagida, 2018, Y. Lou, 2008

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\to} +\infty$$

9



$$\begin{cases} -\mu\Delta u = u(m-u) & \text{in } (0,1)^d \\ \partial_\nu u = 0 & \text{on } \{0,1\}^d \end{cases}$$

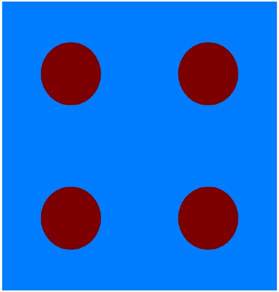
$$\int_{(0,1)^d} u = P$$

[9] I Mazari, D Ruiz-Balet, A fragmentation phenomenon for a nonenergetic optimal control problem: Optimization of the total population size in logistic diffusive models, SIAP 2020

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m^*_\mu\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\to} +\infty$$



Periodize the state and the control

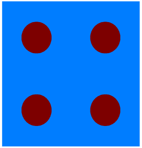
$$\begin{cases} -\mu\Delta v = v(m-v) & \text{in } (-1,1)^d \\ \partial_\nu v = 0 & \text{on } \{-1,1\}^d \end{cases}$$

$$\int_{(-1,1)^d} v = 4P$$

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m^*_\mu\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\longrightarrow} +\infty$$



Rescale back to $(0,1)^d$

$$\begin{cases} -\frac{\mu}{4}\Delta\tilde{u} = \tilde{u}(\tilde{m}-\tilde{u}) & \text{in } (0,1)^d \\ \partial_\nu \tilde{u} = 0 & \text{on } \{0,1\}^d \end{cases}$$

$$\int_{(0,1)^d} \tilde{u} = P$$

Same population but smaller diffusivity

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\to} +\infty$$

- With the **periodization** one can obtain a **lower bound** for the maximum population
- If the maximizers (for any μ) were **uniformly bounded in BV**, then we can make use of a uniform convergence to prove an **upper bound** for the maximum population.

Fragmentation

Theorem (Mazari-RB 2020)

$$\|m_\mu^*\|_{BV((0,1)^d)} \underset{\mu\to 0^+}{\longrightarrow} +\infty$$

Perspectives

- Is there a limit profile when $\mu \to 0$?
- Are there fractal structures when $\mu \to 0$?
- Is there "*some sort*" of recurrence relationship between maximizers of different μ?

Part II: Turnpike without running cost

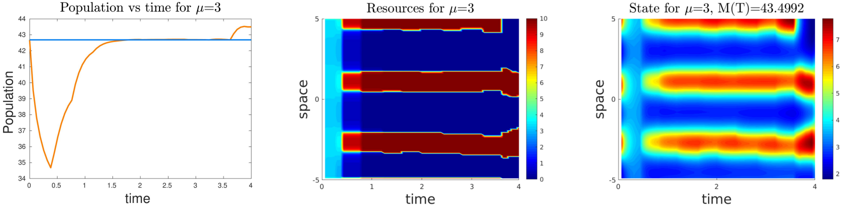
Can we come back to the evolution problem?

Consider

$$\max_{m \in \mathcal{M}_T} \int_\Omega u(T)dx \qquad \begin{cases} \partial_t u - \Delta u = m(x,t)u - u^2 & (x,t) \in \Omega \times (0,T) \\ \partial_\nu u = 0 & (x,t) \in \partial\Omega \times (0,T) \\ u(0) = u_0 > 0 \end{cases}$$

$$\mathcal{M}_T := \left\{ m \in L^\infty(\Omega \times (0,T); [0,\kappa]), \quad \frac{1}{|\Omega|}\int_\Omega m(x,t)dx = m_0 \quad t \in (0,T) \right\}$$

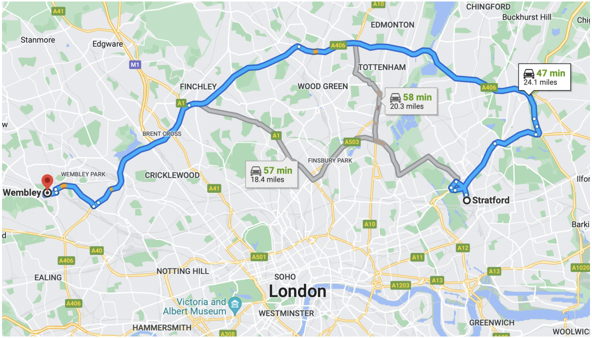
Are the optimal solutions "close" to the elliptic optimisers?



What is Turnpike

The Turnpike principle

*In **large time** horizons the optimal solutions are **near** an optimal solution of an **associated infinite horizon problem***



John Von Neuman

Paul Samuelson

What is Turnpike

The Turnpike principle

*In **large time** horizons the optimal solutions are **near** an optimal solution of an **associated infinite horizon problem***

$$J(u) = \int_0^{T} \|x-x_r\|^2+\|u\|^2 dt+\|x(T)-x_f\|^2 \qquad J_e(u) = \|x - x_r\|^2 + \|u\|^2$$

$$\begin{cases} x' = Ax + Bu \\ x(0) = x_0 \end{cases} \qquad \begin{cases} 0 = Ax + Bu \\ x(0) = x_0 \end{cases}$$

Let (x^*, u^*) be optima for the time evolution problem and (x_e, u_e) be optima for the static case.

$$\|x^*(t) - x_e\|^2 + \|u^*(t) - u_e\|^2 \leq C\left(e^{-t} + e^{-(T-t)}\right)$$

See **Geshkovski, Zuazua 2022** for a recent review.

Linear models first!

We will start with a **linear model** with **bilinear control**

$$\max_{\boldsymbol{m}\in\mathcal{M}_T}\int_\Omega f(u(T))dx$$

where $f:\mathbb{R}^+\to\mathbb{R}^+\backslash\{0\}$

$$\begin{cases}\partial_t u-\Delta u=\boldsymbol{m(x,t)}u & (x,t)\in\Omega\times(0,T)\\ u=0 & (x,t)\in\partial\Omega\times(0,T)\\ u(0)=u_0>0\end{cases}$$

$$\mathcal{M}_T:=\left\{m\in L^\infty(\Omega\times(0,T);[0,\kappa]),\quad \frac{1}{|\Omega|}\int_\Omega m(x,t)dx=m_0\quad t\in(0,T)\right\}$$

In general, we can't assume that there will be steady-states

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In general, we can't assume that there will be steady-states

$$\begin{cases} -\Delta u = m(x)u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

ONLY IF u is eigenfunction with 0 eigenvalue

Associated infinite horizon problem

What could the candidate associated infinite horizon problem be?

The natural candidate would be an **Optimal Eigenvalue**

$$\min_{m\in\mathcal{M}} \lambda_1(m) = \min_{m\in\mathcal{M}} \inf_{v\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 - m(x)v^2 dx}{\int_\Omega v^2 dx}$$

If the control were to be static $m(x,t) = m(x)$ for T large

$$u \sim Ce^{-\lambda_1(m)T} e_1(m)$$

INTUITION:

•The first eigenfunction is positive,

•we want the "fastest" growth

This implies that our turnpike candidate is a SUBSPACE

Optimising the Eigenvalue

$$\begin{cases} -\Delta e_1 - me_1 = \lambda_1 e_1 & x \in \Omega \\ e_1 = 0 & x \in \partial\Omega \end{cases}$$

Naming

$$\xi = \frac{\partial}{\partial m} e_1(m)[h], \quad \lambda_1' = \frac{\partial}{\partial m}\lambda_1(m)[h]$$

Differentiating

$$\begin{cases} -\Delta \xi - m\xi - \lambda_1 \xi = he_1 + \lambda_1' e_1 & x \in \Omega \\ \xi = 0 & x \in \partial\Omega \end{cases}$$

Fredholm alternative

$he_1 + \lambda_1' e_1$ orthogonal to $\mathrm{Ker}(-\Delta - m - \lambda_1) := \mathrm{span}\{e_1\}$

Therefore

$$\lambda_1'(m)[h] = \int_\Omega h(x) e_1(m)^2 dx$$

Optimising the Eigenvalue

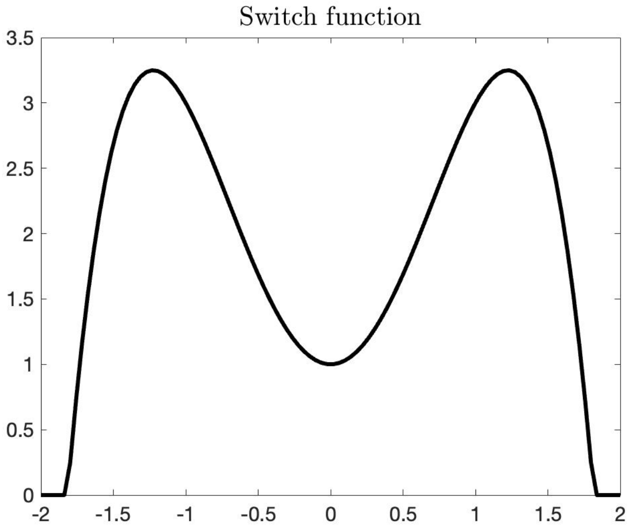
$$\lambda_1'(m)[h] = \int_\Omega h(x) e_1(m)^2 dx$$

$$m \in \mathcal{M} := \{m \in L^\infty(\Omega; [0, \kappa]), \quad \frac{1}{|\Omega|} \int_\Omega m dx = m_0\}$$

Maximising the derivative

$$\max_{h \in \mathcal{M}} \int_\Omega h(x) \underbrace{e_1(m)^2}_{\text{Switch function}} dx$$

Optimality conditions: Bathtub Principle



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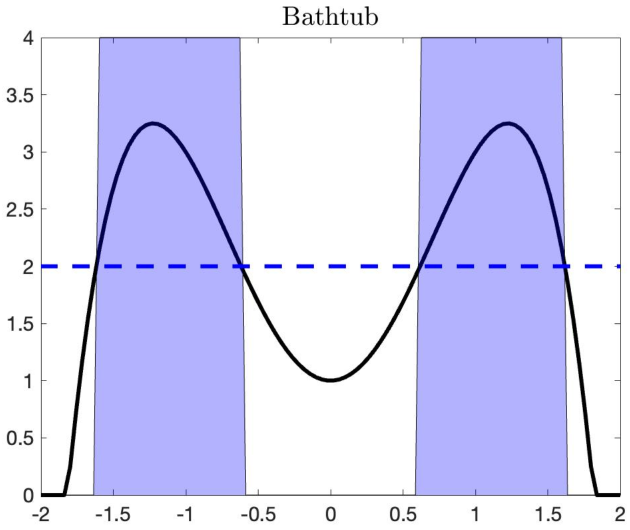

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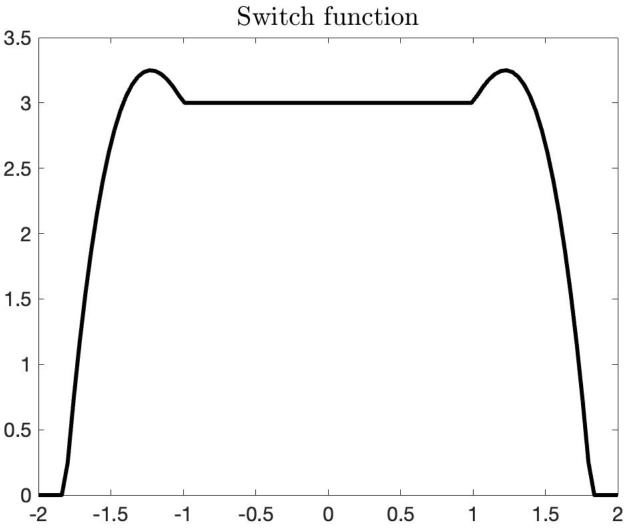

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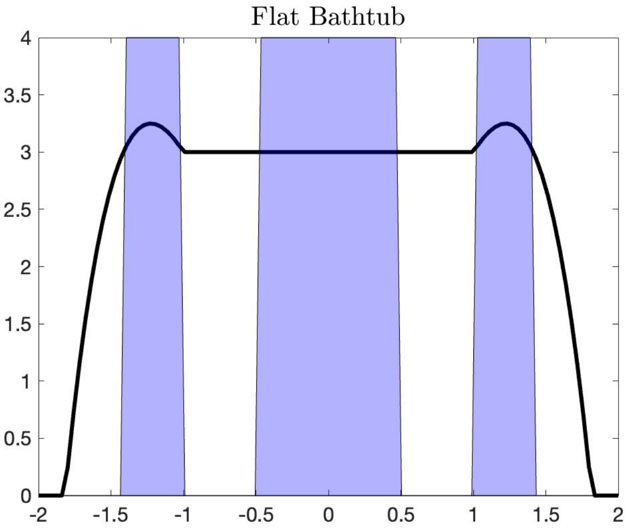

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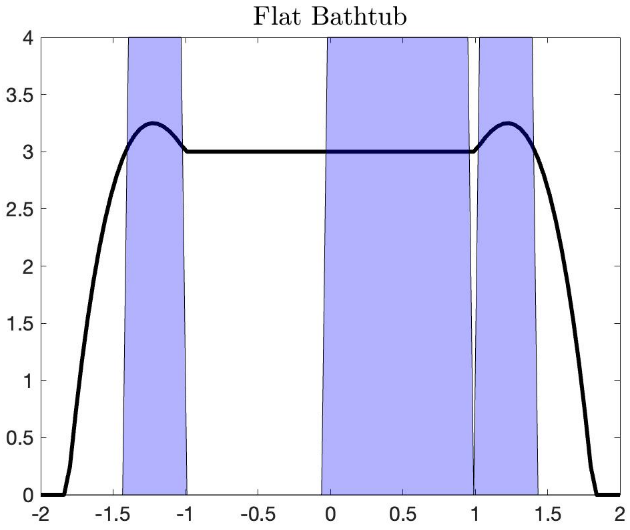

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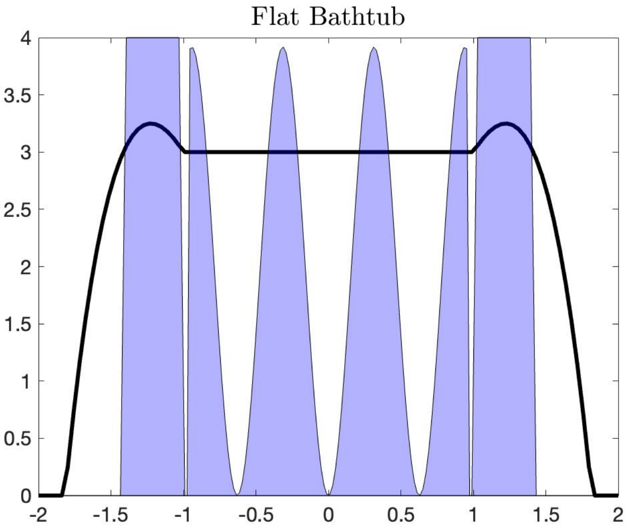


Optimality conditions

The derivative can be expressed as:

$$\max_{h \in \mathcal{M}} \int_\Omega h(x) \underbrace{e_1(m)^2}_{\text{Switch function}} dx$$

The optimal m is a level set of the first eigenfunction

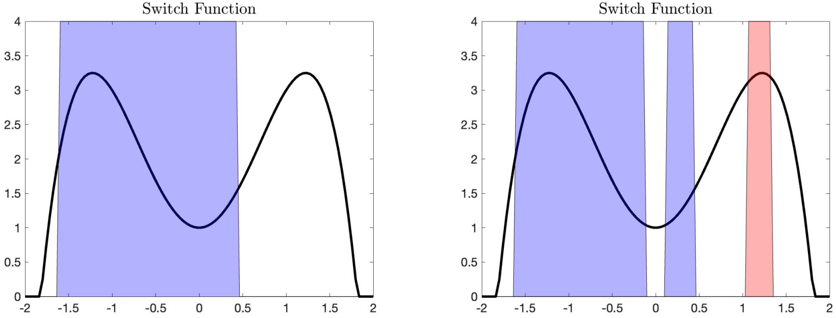
$$\{m = k\} = \{e_1(m)^2 \geq c\}$$
$$|\{m = k\}| = |\{e_1(m)^2 \geq c\}| = m_0$$

Numerical approach

The parameter ϵ will play the role of the **step size**

$$\max_{h\in\mathcal{M}_{\epsilon}} \int_{\{m=0\}} h(x)e_1(m)^2 + \max_{h\in\mathcal{M}_{m_0-\epsilon}} \int_{\{m=\kappa\}} h(x)e_1(m)^2 dx$$

$$\mathcal{M}_{m_0} := \{m \in L^{\infty}(\Omega;[0,\kappa]), \quad \frac{1}{|\Omega|}\int_{\Omega} m dx = m_0\}$$



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If the control were to be static $m(x,t) = m(x)$ for T large

$$u \sim Ce^{-\lambda_1(m)T} e_1(m)$$

INTUITION:

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This implies that our turnpike candidate is a SUBSPACE

Turnpike without running cost

Turnpike Theorem Mazari-RB 2021

There exist $M > 0$ such that, for any $T > 0$ and any solution m_T^* of the parabolic optimisation problem, there holds

$$\int_0^T \inf_{m^* \in I^*} \|m_T^*(t) - m^*\|_{L^1}^2 dt \leq M$$

where I^* is the **optimal spectral set**

Turnpike to the span of the optimal eigenfunction, not to a steady-state!

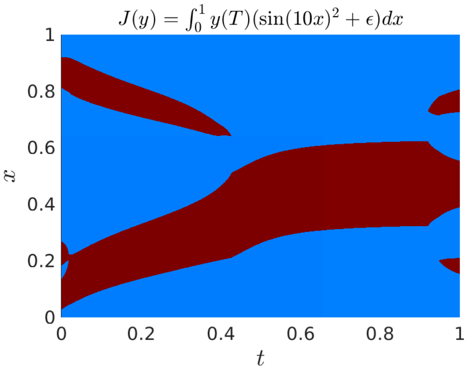
Key:

•Quantitative inequalities

$$\forall m \in \mathcal{M}, \quad \lambda_1(m) - \lambda_1(m^*) \geq C \inf_{m^* \in I^*} \|m - m^*\|_{L^1}^2$$

•Integral Turnpike, not exponential

Simulation



Remarks

- Turnpike helps to accelerate the time-dependant optimisation: **Preconditioning**
- There can be **Turnpike to a subspace**
- We can have **Turnpike without running cost**

Part III: Harvesting problems and Game Theory

SCIENCE | NEWS

The sea is running out of fish, despite nations' pledges to stop it

Major countries that are promising to curtail funding for fisheries are nevertheless increasing handouts for their seafood industries.

BY **TODD WOODY**

PUBLISHED OCTOBER 8, 2019 • 7 MIN READ

As global fish stocks that feed hundreds of millions of people dwindle, nations are scrambling to finalize by year's end an international agreement to ban government subsidies that fuel overfishing.

The sea is running out of fish, despite nations' pledges to stop it, The National Geographic, 2019

The simplest model

Consider

$$\begin{cases} -\mu\Delta u = u(k(x)-u) - \underbrace{\alpha(x)u}_{\text{Harvesting}} & x\in\Omega \\ \partial_\nu u = 0 & x\in\partial\Omega \end{cases}$$

Class of functions

$$\mathcal{M} := \left\{ \alpha \in L^\infty(\Omega;[0,\kappa]), \quad \frac{1}{|\Omega|}\int_\Omega \alpha(x)dx \le V_0 \right\}$$

- Resources $k(x)$ fixed
- Limited number of fishermen in one point
- Limited total number of fishermen

Maximize harvesting

$$\max_{\alpha\in\mathcal{M}} \int_\Omega \alpha(x)u dx$$

Is there a way to understand overfishing through these models?

Some facts

Solutions of

$$\begin{cases} -\mu\Delta u = u(k(x) - u) - \underbrace{\alpha(x)u}_{\text{Harvesting}} & x \in \Omega \\ \partial_\nu u = 0 & x \in \partial\Omega \\ u > 0 & x \in \Omega, \end{cases}$$

do not exist if **zero is linearly stable**
If we force $\|\alpha\|_{L^1} = V_0$ and define

$$H(V_0) = \max_{\|\alpha\|_{L^1} = V_0, \alpha \in \mathcal{M}} \int_\Omega \alpha u dx$$

$\exists K > 0$, such that $H(0) = 0$, $H(K) = 0$ and $H(s) > 0$ if $s \in (0, K)$

Some facts

If we limit ourselves in the case in which $k(x)$ is constant

$$\begin{cases} -\mu\Delta u = u(1-u) - \alpha(x)u & x \in \Omega \\ \partial_\nu u = 0 & x \in \partial\Omega \end{cases}$$

The harvested fish is equal to

$$\int \alpha u_\alpha dx = \int \mu\Delta u_\alpha + u_\alpha(1-u_\alpha)dx = \int u_\alpha(1-u_\alpha)dx$$

Boils down to maximize an integral with a **concave integrand**!

$$\max_{\alpha\in L^\infty(\Omega;[0,\kappa])} \int u_\alpha(1-u_\alpha) \text{ implies } \alpha(x) = \frac{1}{2}$$

$$H(1/2) = \frac{1}{4}$$

Simulations of the Harvesting problem

Differentiating

$$\begin{cases} -\mu\Delta u = u(k(x) - u) - \alpha(x)u & x \in \Omega \\ \partial_\nu u = 0 & x \in \partial\Omega \\ u > 0 & x \in \Omega, \end{cases}$$

with respect to α and introducing the adjoint

$$\begin{cases} -\mu\Delta p - p(k(x) - 2u) + \alpha(x)p = \alpha & x \in \Omega \\ \partial_\nu p = 0 & x \in \partial\Omega \end{cases}$$

One can write

$$DJ(\alpha)[h] = \int_\Omega h(x) \underbrace{u(1-p)}_{\text{Switch function}} dx$$

Simulations of the Harvesting problem

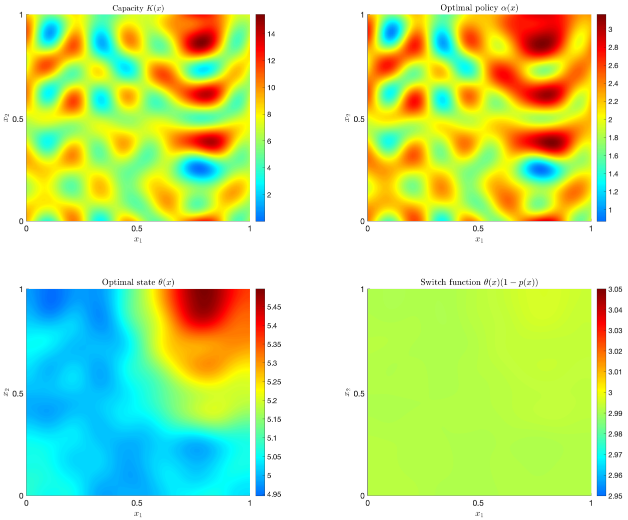


Figure: $V_0 = 0.3$ and $\kappa = 7$

Simulations of the Harvesting problem

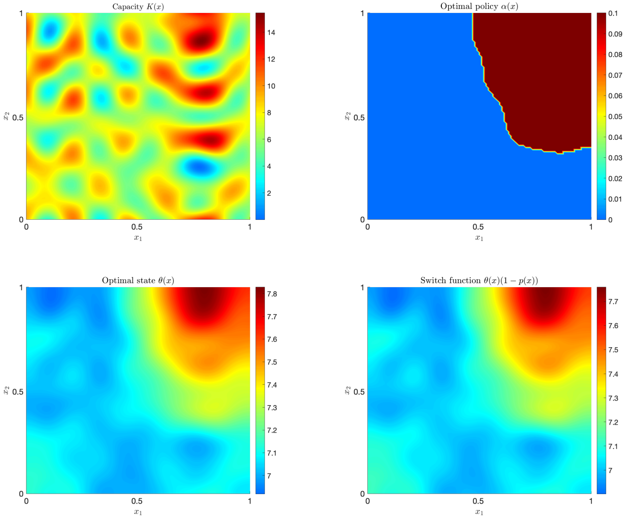


Figure: $V_0 = 0.3$ and $\kappa = 0.1$

Simulations of the Harvesting problem

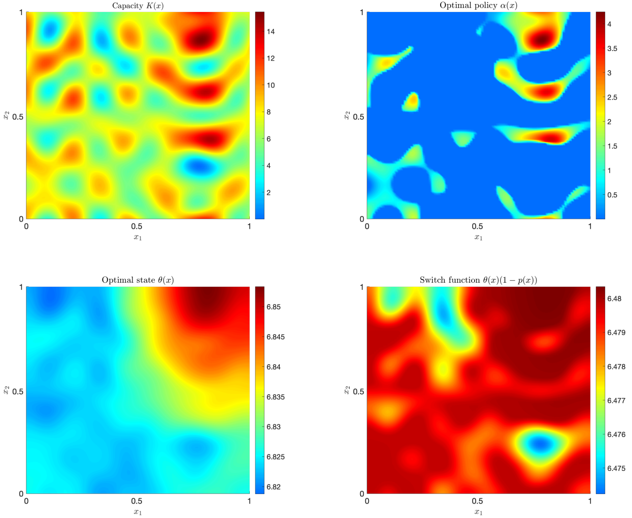


Figure: $V_0 = 0.05$ and $\kappa = 7$.

The model

$$\begin{cases} -\mu\Delta u = u(k(x)-u) - \underbrace{\boldsymbol{\alpha_1(x)}u}_{\text{Player 1}} - \underbrace{\boldsymbol{\alpha_2(x)}u}_{\text{Player 2}} & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

$$\max_{\alpha_1 \in \mathcal{M}} J_1(\alpha_1, \alpha_2) = \max_{\alpha_1 \in \mathcal{M}} \int_\Omega \alpha_1 u_{\alpha_1,\alpha_2} dx$$

$$\max_{\alpha_2 \in \mathcal{M}} J_2(\alpha_1, \alpha_2) = \max_{\alpha_2 \in \mathcal{M}} \int_\Omega \alpha_2 u_{\alpha_1,\alpha_2} dx$$

Nash equilibria

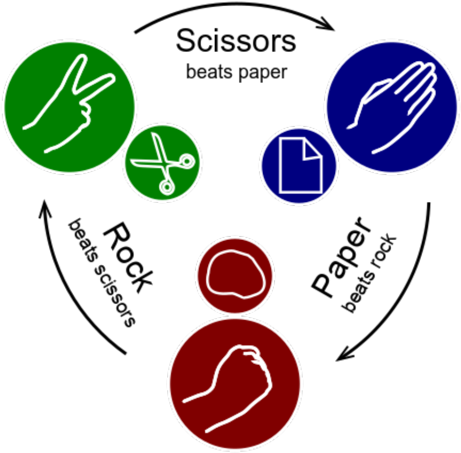
Nash equilibrium for a 2-player game

A Nash equilibrium is a pair of strategies (α_1^*, α_2^*) that satisfies

$$J_1(\alpha_1^*, \alpha_2^*) \geq J_1(\alpha, \alpha_2^*) \quad \forall \alpha \in \mathcal{M}$$

$$J_2(\alpha_1^*, \alpha_2^*) \geq J_2(\alpha_1^*, \alpha) \quad \forall \alpha \in \mathcal{M}$$

Non-existence of Nash equilibria



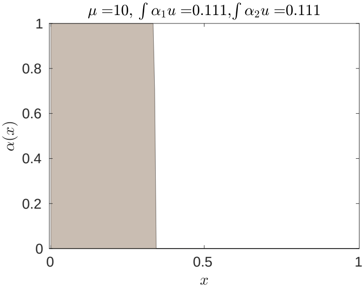
A temptative simulation

We run a **fixed point algorithm**,

$$\alpha_1^{(n+1)} \leftarrow \max_{\alpha \in \mathcal{M}} J_1(\alpha, \alpha_2^{(n)})$$

$$\alpha_2^{(n+1)} \leftarrow \max_{\alpha \in \mathcal{M}} J_2(\alpha_1^{(n)}, \alpha)$$

and **hope** for convergence



- **Convergence $\implies$ existence of Nash eq.**
- The total harvested is equal to $0.222 < \frac{1}{4}$
- $k(x) = 1$ but the **Nash eq is bang-bang**
- For **potential games** the algo converges
- **No numerical guarantees**

Tragedy of the commons

Definition (≈ Wikipedia)

The **tragedy of the commons** is a situation in which individual users, who have **open access to a resource**, act independently according to their **own self-interest** and **cause depletion of the resource** through their uncoordinated action.

Effects on unregulating land in Ireland

William Forster Lloyd

Tragedy of the commons

Do there exist Nash equilibria that show a **depletion** of the fishery?

Consider

$$\begin{cases} -\mu \Delta u_{\vec{\alpha}} = u_{\vec{\alpha}}(1 - u_{\vec{\alpha}}) - \left(\sum_{i=1}^{N} \alpha_i(x) \right) u_{\vec{\alpha}} & x \in \Omega \\ \partial_\nu u_{\vec{\alpha}} = 0 & x \in \partial\Omega \end{cases}$$

Where each player is optimizing

$$J_i(\vec{\alpha}) = \int_\Omega \alpha_i u_{\vec{\alpha}} dx$$

Tragedy of the commons

There exist a sequence of Nash equilibria $\vec{\alpha}_N^* \in \mathcal{M}^N$, $N \in \mathbb{N}$ such that

$$\frac{1}{4} = \max_{\vec{\alpha} \in \mathcal{M}^N} \left(\sum_{i=1}^{N} J_i(\vec{\alpha}) \right) > \sum_{i=1}^{N} J_i(\vec{\alpha}_N^*) \underbrace{\longrightarrow}_{N \to +\infty} 0$$

High-diffusivity limit

Assyptotic expansion on μ

Take $|\Omega| = 1$

The functional expands like

$$J_\mu(\alpha) = J^0(\alpha) + \frac{1}{\mu} J_1(\alpha) + O\left(\frac{1}{\mu^2}\right)$$

$$J^0 : \alpha \mapsto \left(\int_\Omega \alpha\right)\left(K_0 - \int_\Omega \alpha\right)$$

$$J^1 : \alpha \mapsto \int_\Omega \alpha v_\alpha$$

The state expands like

$$u_{\alpha,\mu} = \underbrace{\left(K_0 - \int_\Omega \alpha\right)}_{=:M_\alpha} + \frac{v_\alpha}{\mu} + \underset{\mu\to\infty}{O}\left(\frac{1}{\mu^2}\right) \text{ where } \begin{cases} -\Delta v_\alpha - M_\alpha\left(K - \alpha - M_\alpha\right) = 0 & \text{in } \Omega \\ \frac{\partial v_\alpha}{\partial \nu} = 0 \\ \int_\Omega v_\alpha = \frac{1}{M_\alpha^2} \int_\Omega |\nabla v_\alpha|^2. \end{cases}$$

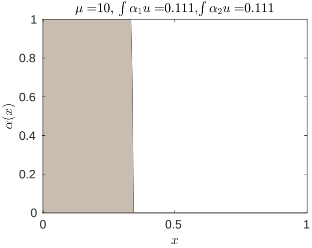
One result

Asymptotic Nash Theorem Mazari-RB 2022

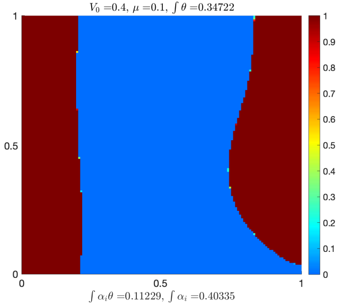
Assume V_1 , $V_2 > \frac{K_0}{4}$, assume K is constant and let

$$\alpha_i^* = \kappa_i \chi_{[0;l_i]} \text{ with } \kappa_i l_i = V_i \; (i = 1, 2).$$

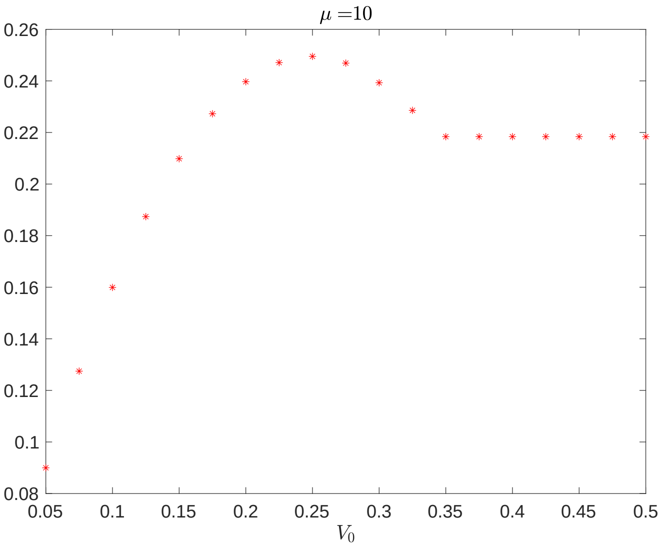
$(\alpha_1^* , \alpha_2^*)$ is a Nash equilibrium in the asymptotic regime



Fragmentation of Nash equilibria?



Optimal regulation?



Perspectives

- How can we **come back to the original problem** without asymptotic regimes?
- How can we **prove the fragmentation of Nash** equilibria?
- Can we find a numerical algorithm that guarantees that an **ϵ-Nash is close to a Nash** equilibria?
- Other game variations and the **incorporation of time**. Can we find new phenomenology?
- Can we quantify the **price of Anarchy**?
- How can we **justify the PDE** from the practical perspective? **Statistics and prediction**, Can a spatio-temporal (PDE) approach enhance **new phenomenology** relevant for ecology not present in simpler models?

Thank you for your attention!

Idriss Mazari

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Imperial College London

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