A two player zerosum game where one player observes a Brownian motion.

common work with Fabien Gensbittel (TSE Toulouse)

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Introduction : Idea of the game

- $0 \leq t \leq T$,
  $m \in \mathcal{P}_2 = \{m \text{ prob. on } \mathbb{R}^d, \text{ s.t. } \int |x|^2 m(dx) < \infty\}$,
- $(B_s^{t,m})_{s\in[t,T]}$ $\mathbb{R}^d$-valued Brownian motion, such that $\mathcal{L}(B_t^{t,m}) = m$,
- Controls: $(u_s, v_s)_{s\in[t,T]} \to U \times V$ compact, metric,
- Payoff: $J(t, m, u., v.) = E\left[\int_t^T f(s, B_s^{t,m}, u_s, v_s)ds\right]$,
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Zero-sum game with asymmetric information:

- Player 1 plays (u_s), tries to minimize the payoff $J(t, m, u., v.)$,
- player 2 plays (v_s), tries to maximize this payoff,
- both players observe the action of their opponent, not the payoff,
- **player 1 observes the Brownian motion, player 2 not.**

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Questions:

1) Is it possible to formalize the game in order to have a **value** ?

$\longrightarrow$ find adapted sets of strategies such that

$$\inf_{\alpha(B^{t,m})} \sup_{\beta} J(t, m, \alpha, \beta) = \sup_{\beta} \inf_{\alpha(B^{t,m})} J(t, m, \alpha, \beta) \ ?$$

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2) If yes, do we have a **characterization** of the value in terms of a PDE ?

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Answer to 1): ???
$\to$ **discrete time games**,
payoff: $E[\sum_{k=1}^{N-1}(t_{k+1}-t_k)f(t_k, B_{t_k}, u_k, v_k)]$,
for $t = t_0 < \ldots < t_N = T$,
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Remark: Approach used in Cardaliaguet- R.-Rosenberg-Vieille (2016), Gensbittel (2016), see also Sorin (2017).

A discrete time game with vanishing stage duration

Fix $(t, m) \in [0, T] \times \mathcal{P}_2$.
For any partition $\pi = \{t = t_1 < \ldots < t_N = T\}$, we define a **$N$-stage game** $\Gamma_\pi(t, m)$:

- at each stage $k \in \{1, \ldots, N-1\}$,
  - player 1 observe $B_{t_k}^{t,m}$
  - both players choose simultaneously a pair of controls $(u_k, v_k) \in U \times V$,
  - the actions are observed after each stage
  - stage payoff : $f(t_k, B_{t_k}, u_k, v_k)$ (not observed)
- total expected payoff :

$$E\left[\sum_{k=1}^{N-1} (t_{k+1} - t_k) f(t_k, B_{t_k}, u_k, v_k)\right].$$

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$$E\left[\sum_{k=1}^{N-1}(t_{k+1}-t_k)f(t_k, B_{t_k}, u_k, v_k)\right].$$

$\longrightarrow$ For each π, the game $\Gamma_\pi(t,m)$ has a value $V_\pi(t,m)$.

Theorem: $V_\pi(t,m)$ converges (up to a subsequence) when $|\pi| \searrow 0$.

An alternative formulation

Let $\mathcal{M}(t,m)$ be the set of processes $(M_s)_{t\leq s\leq T}$ with values in $\mathcal{P}_2$, such that,
for all $s\in[t,T]$, $M_s=\mathcal{L}(B_s^{t,m}|\mathcal{F}_s)$,
for $(\mathcal{F}_s)_{s\in[t,T]}$ filtration such that $(B_s^{t,m})$ is still a $(\sigma(B_r^{t,m}, t\leq r\leq s)\vee\mathcal{F}_s)$-Brownian motion.

Interpretation: if $(\mathcal{F}_s)$ is generated by a control of player 1 in continuous time, M_s is the belief of player 2 on $B_s^{t,m}$.

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Examples:

- $(\mathcal{F}_s)$ is trivial : $\mathcal{L}(B_s^{t,m}|\mathcal{F}_s)=\mathcal{L}(B_s^{t,m})$,
- $(B_s^{t,m})$ is adapted to $(\mathcal{F}_s)$: $\mathcal{L}(B_s^{t,m}|\mathcal{F}_s)=\delta_{B_s^{t,m}}$.

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$$\mathcal{M}(t,m) = \{(M_s)_{t\leq s\leq T} \to \mathcal{P}_2, M_s = \mathcal{L}(B_s^{t,m}|\mathcal{F}_s), (\mathcal{F}_s) \text{ convinient}\}.$$

Suppose that Isaac's assumption holds :
for all $(t,m) \in [0,T] \times \mathcal{P}_2$,

$$\inf_{u\in U} \sup_{v\in V} \int_{\mathbb{R}^d} f(t,x,u,v)m(dx) = \sup_{v\in V} \inf_{u\in U} \int_{\mathbb{R}^d} f(t,x,u,v)m(dx) := H(t,m).$$

Set $V(t,m) := \inf_{M\in\mathcal{M}(t,m)} E[\int_t^T H(s,M_s)ds]$.

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Theorem: For all (t,m),

$$V(t,m) = \lim_{|\pi|\searrow 0} V_\pi(t,m).$$

Characterization of the value : Derivative in the space of probability measures

Proposition [Lyons 201?], [Cardaliaguet-Delarue-Lasry-Lyons 2015]:

Let $L^2_d = \{X \text{ random variables} \to \mathbb{R}^d, E[|X|^2] < \infty\}$.

For $U : \mathcal{P}_2 \to \mathbb{R}$, define $\mathcal{U} : L^2_d \to \mathbb{R}, X \mapsto \mathcal{U}(X) := U(\mathcal{L}(X))$.

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Then (under convenient assumptions), there exists $D_m U : \mathcal{P}_2 \times \mathbb{R}^d \to \mathbb{R}^d$ such that, for all $X, Y \in L_d^2$,

$$\lim_{h \to 0^+} \frac{\mathcal{U}(X + hY) - \mathcal{U}(X)}{h} = E[D_m U(\mathcal{L}(X), Y)Y].$$

Characterization : the equation

payoff $J(t,m,u,v) = \int_t^T E[f(s, B_s^{t,m}, u_s, v_s)]ds$.
Hamiltonian $H(t,m) := \inf_u \sup_v \int_{\mathbb{R}^d} f(t,x,u,v)m(dx)$.

For $U : [0,T] \times \mathcal{P}_2 \to \mathbb{R}$, consider the equation

$$\begin{cases} \partial_t U(t,m) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{div}_x[D_m U](t,m,x)m(dx) + H(t,m) = 0, \\ U(T,m) = 0,\ m \in \mathcal{P}_2, \end{cases} \tag{1}$$

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Proposition: Set $p_s^{t,m} := \mathcal{L}(B_s^{t,m})$. Suppose that H is sufficiently smooth. Then (1) has a unique regular solution:

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Remark: U_0 is the value of a continuous time game where the Brownian motion is observed by nobody.

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Definition: We call a subsolution of (1) a map $U : [0,T] \times \mathcal{P}_2 \to \mathbb{R}$ such that, for all $(t,m) \in [0,T) \times \mathcal{P}_2$, and for all φ sufficiently smooth, such that $\varphi - U$ has a minimum at (t,m),

$$\partial_t \varphi(t,m) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{div}[D_m \varphi](t,m,x)m(dx) + H(t,m) \geq 0.$$

Theorem: The value function V is the largest bounded, continuous subsolution of (1), which is convex in m and satisfies the terminal condition $V(T,\cdot) = 0$.

Open questions and perspectives

- Characterization of V as the solution of a Hamilton-Jacobi equation
- Observation by only the first Player of a controlled diffusion
- Does the continuous time game have a value ?

Thank you for your attention!