# Document de synthèse présenté pour obtenir l'HABILITATION A DIRIGER DES RECHERCHES spécialité Mathématiques Appliquées (english version) 

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## 1 Introduction.

We present here several papers dealing with controlled stochastic differential equations. They roughly can be divided into 3 parts:

1. stochastic viability,
2. stochastic differential equations with respect to non Brownian normal martingales,
3. stochastic differential games.

The first part is devoted to stochastic viability. Given a controlled stochastic differential equation (SDE) and a non empty closed set $K$, we say that $K$ is viable with respect to the SDE if, for all initial condition $x \in K$, there exists a control for which the solution of the SDE remains a.s. in $K$ during all its life. We begin to characterize the viability in terms of viscosity supersolution of some non linear PDE. The result is then generalized to closed sets which depend on time. We consider also the notion of viability before exit time of some open set and of attainability: under which conditions is it possible to control the SDE such that the solution attains in a fixed time a given target? In the case where the set $K$ is not viable, we study its viability kernel, i.e. the set of all points in $K$ from which on a solution to the SDE can stay in $K$. Finally we consider the close notion of invariance, showing that the invariance of a stochastic control system is equivalent to the invariance of the associated doubly controlled deterministic system.
The second part contains two articles. In the first one, we consider backward stochastic differential equations with respect to Azéma's martingale and the associated deterministic equation. The second one is devoted to a controlled stochastic differential system, where not only the parameters of the system but also the jumps of the driving martingale are controlled.
In the third part, we study stochastic differential games. We first characterize the Nash equilibria of non zero sum games, establishing a Folk Theorem in continuous time. Two papers are devoted to zero sum stochastic differential games with incomplete information. In the first one, we characterize the solution of the game as a dual solution in viscosity sense of a non linear PDE, where the notion of dual solution is new. Then, in the case where the lack of information is only on one side, we study the strategy of the fully informed player: to this aim, we introduce an optimization problem over a set of martingale measures. We analyse the optimal measure and its importance in the construction of the optimal strategy.
In order to keep a certain cohesion, the present collection of papers doesn't contain the article $\langle 4\rangle$, which deals with queuing systems, neither the recent work $\langle 17\rangle$, which proves the existence of an optimal control for backward stochastic differential systems. We write [.] for a general reference, and $\langle\cdot\rangle$ for our bibliography.

## 2 Viability.

### 2.1 Introduction.

This section summarizes the articles $\langle 5\rangle,\langle 7\rangle,\langle 8\rangle,\langle 9\rangle,\langle 11\rangle,\langle 15\rangle$. Their commun subject is the viability of controlled stochastic differential systems (resp. its invariance in $\langle 15\rangle$ ).
More precisely we consider on a probability space $(\Omega, \mathcal{F}, P)$ a process $\left(X_{t}, t \geq 0\right)$ with values in
$\mathbb{R}^{n}$ that satisfies a stochastic differential equation (SDE) of the following shape:

$$
\left\{\begin{array}{l}
d X_{t}^{x, u}=b\left(X_{t}^{x, u}, u_{t}\right) d t+\sigma\left(X_{t}^{x, u}, u_{t}\right) d W_{t}, t \in[0, \infty),  \tag{1}\\
X_{0}^{x, u}=x \in \mathbb{R}^{n},
\end{array}\right.
$$

where $W$ is a multidimensional standard Brownian motion, and the admissible control $u$ a process which is adapted to the Brownian filtration $\mathbb{F}$, with values in a compact metric space $U$. We call $\nu:=(\Omega, \mathcal{F}, P, W, \mathbb{F})$ a reference system and denote by $\mathcal{A}_{\nu}$ the set of admissible controls an $\nu$.
Further we consider a non empty closed set $K$ in $\mathbb{R}^{n}$. We define the viability of $K$ for the system (1) as follows:

Definition 2.1 The closed set $K$ is said to be viable for the system (1) if, for all initial condition $x \in K$, there exists a reference system $\nu$ and a control $u \in \mathcal{A}_{\nu}$ such that, $P$-a.s., for all time $t \geq 0$, the solution $X^{x, u}$ of (1) stays in $K: P\left[X_{t}^{x, u} \in K, t \geq 0\right]=1$.

The first reference on viability for deterministic differential systems goes back to Nagumo [86] in 1943. Then the problem has been studied by many authors. The references can be found in the monography of Aubin [6]. The central result can be summarized as follows:
Let be a controlled system

$$
\begin{equation*}
x(t)=f(x(t), u(t)) . \tag{2}
\end{equation*}
$$

and a closed set $K$. Under the assumption that $f$ is of at most linear growth and continuous with respect to $u$ and that $f(x, U)$ is convex and compact, the following assertions are equivalent:

1. The closed set $K$ is viable for (2): for all $x \in K$, there exists a control $u$ such that the trajectory of the solution of (2) stays always in $K$;
2. For all $x \in K$, there exists $u \in U$ such that $f(x, u) \in T_{K}(x)$, where $T_{K}(x)$ represents the Bouligand tangent cone of $K$ at $x$;
3. For all $x \in K$ and all proximal normal $p$ on $K$ at $x, \min _{u \in U}\langle f(x, u), p\rangle \leq 0$.
(for the definition of a Bouligand tangent cone see [6] and for the proximal normal see [37]).

In order to characterize the stochastic viability, Aubin-Da Prato [7], [8] and Gautier-Thibault [54] introduce a stochastic tangent cone, which generalizes the tangent cone of condition 2. The weak point of this approach is that this cone not only depends on the parameters of the diffusion but also of the different trajectories of the Brownian motion. Thus the criterium they obtain has to be satisfied for almost all $\omega$ of the underlying probability space. Some other stochastic characterizations are proposed by Michta [84] and Motyl [85].
The first deterministic characterization of stochastic viability was developped by Buckdahn-Quincampoix-Rascanu [29] for backward SDEs. It is noteworthy that the backward case was solved before the forward case. Indeed, in contrast to a process which starts at time zero from a deterministc value and diffuses potentially in all directions of the space, it seemed easier to oblige to stay in a closed set $K$ a process that goes backward from a random terminal condition to a deterministic value. It is shown in [29] that $K$ is viable for the backward system, if the distance function satisfies on all point of the space an equality depending on the parameters of the system.

The fundamental idea that permitted us in $\langle 5\rangle$ to approach the viability for forward systems is to use the notion of viscosity. Up to this moment, the possibility for the detailed study of the stochastic viability we present here was given.

We present the results in chronological order: we start by the characterization of the stochastic viability with the help of an associated PDE, then we give some generalizations and applications. Further, in the case of a non viable closed set, we study its viability kernel. Finally we consider the notion of invariance.

### 2.2 Characterization of the viability.

The central result and starting point of the papers we present in this section is given by the note au CRAS $\langle 5\rangle$, written in collaboration with R. Buckdahn, S. Peng and M. Quincampoix. We suppose that the parameters $b$ and $\sigma$ satisfy the classical assumptions of uniform continuity and Lipschitzianity in $x$ uniformly in $u$. Moreover we suppose that a Filippov type assumption is satisfied:
(HF) for all $x$, the set $\left\{\left(\frac{1}{2} \sigma \sigma^{*}(x, v), b(x, v)\right), v \in U\right\}$ is convex and compact.
We introduce the second order operator associated to (1):

$$
\mathcal{L}_{(x, v)} \varphi(x)=\langle D \varphi(x), b(x, v)\rangle+\frac{1}{2} \operatorname{tr}\left[D^{2} \varphi(x) \sigma \sigma^{*}(x, v)\right], \varphi \in C^{2}(\mathbb{R}),
$$

and define the distance function to $K$ :

$$
\text { for all } x \in \mathbb{R}^{n}, d_{K}(x)=\inf _{y \in K}|x-y| \text {. }
$$

We get the following theorem:

Theorem $2.1\langle 5\rangle$ The following assertions are equivalent:
(i) $K$ is viable for (1);
(ii) $d_{K}^{2}$ is a viscosity supersolution of the following Hamilton-Jacobi-Bellman equation:

$$
\begin{equation*}
\inf _{v \in U} \mathcal{L}_{(x, v)} \varphi(x)+d_{K}^{2}(x)-C \varphi(x)=0 \tag{3}
\end{equation*}
$$

(where $C>0$ is a constant that depends on the parameters of the system).

The proof of the theorem is simple. The key point is given by the following elementary lemma:

Lemma 2.1 The closed set $K$ is viable if and only if the function $V$ defined by

$$
\begin{equation*}
V(x)=\inf _{u \in \mathcal{A}_{\nu}} E \int_{0}^{\infty} e^{-C s} d_{K}^{2}\left(X_{s}^{x, u}\right) d s \tag{4}
\end{equation*}
$$

vanishes on $K$.

Proof of the Theorem: We first prove that $V$ is a viscosity solution of (3) (see Crandall-IshiiLyons [39] or Fleming-Soner [51]).
Next we suppose that $d_{K}^{2}$ is a supersolution of (3). By the comparision theorem between super-
and subsolutions in viscosity sense of a PDE, we have:

$$
\forall x \in K, V(x) \leq d_{K}^{2}(x)=0 .
$$

Using the lemma, we conclude that $K$ is viable for (1):
The reverse assertion " $K$ viable $\Rightarrow d_{K}^{2}$ supersolution of (3)" is mainly based on Itô's formula and standard estimations for SDEs.

## Remarks:

1. Several works echo this result. Bardi-Goatin [15] and Bardi-Jensen [16] give a geometric characterization using a second order normal cone. In the case where $K$ is convex, Da PratoFrankovska in [40] and [42] give a sufficient and necessary condition based on the derivatives of the distance function.

- Let us mention a variation of Theorem 2.1 developped by Buckdahn-Cardaliaguet-Quincampoix [26] that we will use in the sequel:

Theorem 2.2 ([26]) The following assertions are equivalent:

- $K$ is viable for (1);
- The map $u(x)=1-\mathbf{1}_{K}(x)$ is a discontinuous supersolution in viscosity sense of the following equation:

$$
\begin{equation*}
\inf _{v \in U, \sigma(x, v)^{*} \nabla \varphi(x)=0} \mathcal{L}_{x, v} \varphi=0, \tag{5}
\end{equation*}
$$

- For all function $\varphi \in \mathcal{C}^{2}\left(\mathbb{R}^{n}, \mathbb{R}\right)$ and $x \in \operatorname{Argmax}_{K} \varphi$

$$
\inf _{v \in U, \sigma(x, v)^{*} \nabla \varphi(x)=0} \mathcal{L}_{x, v} \varphi \leq 0 .
$$

2. Theorem 2.1 has been recently generalized by Peng and Zhu [91] to stochastic differential systems with jumps. In [31], Buckdahn, Quincampoix and Tessitore transpose the result for $K$ convex in infinite dimension.

In $\langle 8\rangle$, in collaboration with L.Mazliak, we investigate the possibility to find a control $u$ for which the associated trajectory $X^{x, u}$ has a non zero probability to stay for at least some time in $K$, i.e. there exists a positive stopping time $T$ such that

$$
\begin{equation*}
P\left[X_{t}^{x, u} \in K, \forall t \in \llbracket 0, T \rrbracket\right]>0 . \tag{6}
\end{equation*}
$$

We show that every closed set $K$ which satisfies (6) for some control and some stopping time is automatically viable.
A more difficult question is still open: how to characterize closed sets $K$ for which there exists a reference system $\nu=(\Omega, \mathcal{F}, P, \mathbb{F}, W)$, a control $u \in \mathcal{A}_{\nu}$ and a non vanishing time $T$ such that

$$
\forall t \in \llbracket 0, T \rrbracket, \quad P\left[X_{t}^{x, u} \in K\right]>0 .
$$

### 2.3 Constraints depending on time and viability for backward stochastic differential equations.

The paper $\langle 7\rangle$, written in collaboration with R. Buckdahn, M. Quincampoix et A. Rascanu, presents an unified approach of viability for forward and backward SDEs, where the results are extended to constraints $K$ that can vary in time.

We consider here a decoupled forward-backward system of the following type:

$$
\begin{array}{r}
X_{s}^{t, x}=x+\int_{t}^{s} b\left(r, X_{r}^{t, x}\right) d r+\int_{t}^{s} \sigma\left(r, X_{r}^{t, x}\right) d W_{r}, \\
Y_{s}^{t, x}=H\left(X_{T}^{t, x}\right)+\int_{s}^{T} F\left(r, X_{r}^{t, x}, Y_{r}^{t, x}, Z_{r}^{t, x}\right) d r-\int_{s}^{T} Z_{r}^{t, x} d W_{r}, s \in[t, T], \tag{8}
\end{array}
$$

where the parameters satisfy standard assumptions and, in order to simplify the presentation, $X^{t, x}$ and $Y^{t, x}$ take their values in a same space $\mathbb{R}^{n}$.
We consider further a set on constraints

$$
\mathcal{K}=\{K(t), t \in[0, T]\},
$$

where each $K(t)$ is a closed set in $\mathbb{R}^{n}$, and we say that $\mathcal{K}$ is viable for (7) (resp. (8)) if, for all initial condition $x \in K(t), P\left[X_{s}^{t, x} \in K(s), s \in[t, T]\right]=1$ (resp. if for all application $H$ such that $\left.\forall x \in \mathbb{R}^{n}, H(x) \in K(T), P\left[Y_{s}^{t, x} \in K(s), s \in[t, T]\right]=1\right)$.

The following theorems hold:

Theorem 2.3 Suppose that, for all $x \in \mathbb{R}^{n}$, the map $t \mapsto d_{K(t)}^{2}(x)$ is lower semicontinuous and the sets $K(t)$ are contained in a same compact set of $\mathbb{R}^{n}$. Then the following assertions are equivalent:

- $\mathcal{K}$ is viable for (7);
- The map $(t, x) \mapsto d_{K(t)}^{2}(x)$ is a supersolution in viscosity sense of the PDE

$$
\begin{equation*}
\frac{\partial \varphi(t, x)}{\partial t}+\mathcal{L}_{t, x} \varphi(t, x)-C d_{K(t)}^{2}(x)=0,(t, x) \in[0, T] \times \mathbb{R}^{n} \tag{9}
\end{equation*}
$$

where $C$ is a sufficiently large constant.

Theorem 2.4 Suppose that, for all $x \in \mathbb{R}^{n}$, the map $t \mapsto d_{K(t)}^{2}(x)$ is upper semicontinuous and the sets $K(t)$ are contained in a same compact set of $\mathbb{R}^{n}$. The following assertions are equivalent

1. $\mathcal{K}$ is viable for (8).
2. For all $(x, z) \in \mathbb{R}^{n} \times \mathbb{R}^{n \times n}$, the $\operatorname{map}(t, y) \mapsto d_{K(t)}^{2}(y)$ is a subsolution in viscosity sense of the PDE:

$$
\begin{equation*}
\frac{\partial \varphi(t, y)}{\partial t}+\mathcal{A}_{z}(t, x) \varphi(t, y)+C d_{K(t)}^{2}(y)=0,(t, y) \in[0, T] \times \mathbb{R}^{n} \tag{10}
\end{equation*}
$$

where

$$
\mathcal{A}_{z}(t, x) \varphi(y)=\frac{1}{2} \operatorname{Tr}\left[z \sigma \sigma^{*}(t, x) z^{*} D_{y}^{2} \varphi(y)\right]-\left\langle F\left(t, x, y, z \sigma(t, x), \nabla_{y} \varphi(y)\right\rangle,\right.
$$

and where $C$ is a sufficiently large constant.

## Remarks:

1. In the backward case, it is possible to characterize the viability for a family of closed sets which also depend on the space variable $x$, i.e. to give a necessary and sufficient condition whether if, for all $x \in \mathbb{R}^{n}, H(x) \in K(T, x)$, then, $P$-a.s., for all $s \in[t, T], Y_{s}^{t, x} \in K\left(s, X_{s}^{t, x}\right)$.
2. It is known since Pardoux-Peng [89] that the application $(t, x) \mapsto u(t, x)=Y_{t}$ is a solution in sense of viability of a non linear PDE. Therefore the relation (10) represents also a necessary and sufficient condition wether $\mathcal{K}$ is viable for this PDE.
3. In [26], it is shown that, if a closed set $K$ is viable for a BSDE, then it is convex. In the case where $K$ is varying in time, this results remains of course true.
4. The results applied to a controlled system would give a characterization of the viability for a class of Hamilton-Jacobi equations. This remains to be done.

### 2.4 Viability before exit time.

In $\langle 9\rangle$, written in collaboration with R. Buckdahn, M. Quincampoix, A. Rascanu, we come back to controlled SDEs EDS of type (1):

$$
\left\{\begin{array}{l}
d X_{t}^{x, u}=b\left(X_{t}^{x, u}, u_{t}\right) d t+\sigma\left(X_{t}^{x, u}, u_{t}\right) d W_{t}, t \in[0, \infty),  \tag{11}\\
X_{0}^{x, u}=x \in \mathbb{R}^{n},
\end{array}\right.
$$

and a constraint $K$ which is constant in time. More precisely, we are interested in systems (11) for which there exists a control such that the trajectories stay a.s. in $K$ before leaving a given open set $\mathcal{O}$.

Definition 2.2 Let $\mathcal{O}$ be a nonempty open set in $\mathbb{R}^{n}$. We say that $K$ is viable before exit time from $\mathcal{O}$ if, for all $x \in K \cap \mathcal{O}$, there exists a reference system $\nu=(\Omega, \mathcal{F}, P, \mathbb{F}, W)$ and an admissible control $u \in \mathcal{U}_{\nu}$ such that, for all $t \in \llbracket 0, \tau^{u}(x) \llbracket, X_{t}^{t, u} \in K, P$-a.s., where we set

$$
\tau^{u}(x)=\inf \left\{s \geq 0, X_{s}^{x, u} \notin \mathcal{O}\right\} .
$$

We suppose that, during all the section, the following condition is satisfied:

$$
\forall x \in \mathcal{O}, \pi_{K}(x) \in \mathcal{O},
$$

Setting $\mathcal{O}=\mathbb{R}^{n}$, we can see that the notion of viability before exit time generalize the notion of viability introduced in $\langle 5\rangle$.

The paper starts with the assertion that, in a similar way to $\langle 5\rangle, K$ is viable before exit of $\mathcal{O}$ if and only if the following function vanishes on $K$ :

$$
\tilde{V}(x)=\inf _{u \in \mathcal{A}_{\nu}} E\left[\int_{0}^{\tau^{u}(x)} e^{-C s} d_{K}^{2}\left(X_{s}^{x, u}\right) d s\right], \quad x \in \mathbb{R}^{n}
$$

The difficulty here is that, in contrast to the case where $\mathcal{O}=\mathbb{R}^{n}, \tilde{V}$ is generally not continuous but only upper semicontinuous. Therefore it is not possible to characterize $\tilde{V}$ as a viscosity solution of a PDE. We show that it is a solution in a weaker sense (see Barles [17] for 1st order PDEs, Ishii [64] for 2 nd order PDEs):

Theorem 2.5 The application $\tilde{V}$ is the smallest non negative lower semicontinuous viscosity supersolution of the following HJB equation:

$$
\left\{\begin{array}{l}
\inf _{v \in U}\left\{\frac{1}{2} \operatorname{tr}\left(\sigma \sigma^{*}(x, v) D^{2} \varphi(x)\right)+\langle b(x, v), D \varphi(x)\rangle\right\}+d_{K}^{2}(x)-C \varphi(x)=0  \tag{12}\\
x \in \mathcal{O}
\end{array}\right.
$$

In the proof of Theorem 2.1, only the fact that $V$ is a subsolution is used to characterize the viability. Thus Theorem 2.5 permits us to show that

Theorem 2.6 The closed set $K$ is viable before exit time from $\mathcal{O}$ if and only if $d_{K}^{2}$ is a subsolution in viscosity sense of (12).

If $K$ is a smooth set, then the viability of $K$ before exit time of $\mathcal{O}$ can be directly written as follows:

Corollary 2.1 Let $W \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ such that $(W(x)=0) \Rightarrow(\nabla W(x) \neq 0)$. Then the set

$$
K=\left\{x \in \mathbb{R}^{n}, W(x) \leq 0\right\}
$$

is viable before exit time from $\mathcal{O}$ if and only if

$$
\inf _{v \in U}\left\{\frac{1}{2} \operatorname{tr}\left(\sigma \sigma^{*}(x, v) D^{2} W(x)+\langle b(x, v), D W(x)\rangle\right)\right\}-C W(x) \leq 0, x \in \mathcal{O}
$$

for some constant $C>0$.

In the second part of the paper, we apply this characterization to give a sufficient condition for small time attainability:

Definition 2.3 The set $K$ is small time attainable for the system (11) if it is possible to find a time $T>0$ and an open domain $I \supset K$ such that, for all $x \in I \backslash K$, there exists a control $u$ that permits to $X^{x, u}$ to attain $K P$-a.s. before $T$.

There exists a large litterature in the deterministic case (see for example Bianchini-Stefani [23], Veliov [107], Clarke-Wolenski [36] or Krastanov-Quincampoix [69]). In the stochastic case, Soner and Touzi [98], [99] study target in form of an epigraph.
The theorem we gut here is the following:

Theorem 2.7 Suppose that, for all $T>0$ there exists some $\gamma>0$, such that, with the notations $I=K+\mathcal{B}(0, \gamma T)$ and $h(x, y)=\frac{1}{2}\left(d_{K}(x)-y\right)^{2}$, the following property holds:
For all all $(x, y) \in(I \backslash K) \times \mathbb{R}$, with $y \in\left(0, d_{K}(x)\right]$ and $\varphi \in C^{2}\left(\mathbb{R}^{n} \times \mathbb{R}\right)$ such that $(x, y) \in$ $\operatorname{Argmin}(h-\varphi)$,

$$
\begin{align*}
\inf _{v \in U}\left\{\operatorname{tr}\left[D_{x}^{2} \varphi(x, y) \sigma \sigma^{*}(x, v)\right]\right. & \left.+\left\langle b(x, v), D_{x} \varphi(x, y)\right\rangle\right\}  \tag{13}\\
+ & \gamma\left(d_{K}(x)-y\right)-(C-1) h(x, y) \leq 0
\end{align*}
$$

Then $K$ is small time attainable for the system (11).

Once again it is possible to give a more direct characterization in the case where $K$ is smooth:

Corollary 2.2 Let $W \in C^{2}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)$ such that $(W(x)=0) \Rightarrow(\nabla W(x) \neq 0)$, set

$$
K=\left\{x \in \mathbb{R}^{n}, W(x) \leq 0\right\}
$$

and for $r>0$,

$$
I=\left\{x \in \mathbb{R}^{n}, W(x)<r\right\} .
$$

Suppose that the following condition is satisfied:
there exists $\gamma>0$ and $c>0$ such that, for all $x \in I \backslash K$,

$$
\inf _{v \in U}\left\{\operatorname{tr}\left(D^{2} W(x) \sigma \sigma^{*}(x, v)\right)+\langle b(x, v), D W(x)\rangle\right\}-c W(x) \leq-\gamma
$$

Then $K$ is small time attainable for the system (11).

### 2.5 The viability kernel.

In this paragraph we still consider the same controlled system (1), but are interested now in the case where $K$ is not necessarily viable. In this general case, it is still possible to define its viability kernel:

Definition 2.4 We call viability kernel $N$ of $K$ for the system (1) the following set:

$$
N=\left\{x \in K, \exists \nu \text { and } v \in \mathcal{A}(\nu), \text { such that, P-a.s., } \forall t \geq 0, X_{t}^{x, v} \in K\right\}
$$

The paper $\langle 11\rangle$ with M. Quincampoix is devoted to the study of this viability kernel. First we show that it is closed and viable. Several characterizations follow:

Proposition 2.1 The viability kernel $K$ for the system (1) is

- the largest subset of $K$ which is viable,
- equal to the set $\{x \in K, \bar{V}(x)=0\}$, for

$$
\bar{V}(x)=\inf _{\nu, v \in \mathcal{A}_{\nu}} E\left[\int_{0}^{\infty} e^{-C s}\left(1-\mathbf{1}_{K}\right)\left(X_{s}^{x, v}\right) d s\right]
$$

- the largest closed set $H$ included in $K$ such that $1-\mathbf{1}_{H}$ is a supersolution of (5).

An interesting fact in the deterministic case is that the boundary of the viability kernel $\partial N$ is viable before hitting the boundary of $K$ (see Quincampoix [93] [94]). The result in [93] is obtained as follows: since $N$ is viable, if $\partial N$ would be not viable, we could find a trajectory starting from some $x \in \partial N$ that leaves $\partial N$ after some time and enters inside of $\operatorname{int}(N)$. The parameters of the system being Lipschitz, we can find then for all elements of a small ball around $x$ a trajectory that also enters $\operatorname{Int}(N)$ without leaving $K$. Since this ball contains elements of $K \backslash N$, this is in contradiction with the fact that $N$ is the viability kernel of $K$.
The stochastic case is more difficult, because the trajectories of the processes starting from two close points stay close only in mean. Moreover we will see that the result stays true only under some assumption, and the proof will be more analytic and less intuitive. Let be the following assumption:
(H) for all $(p, A) \in \mathbb{R}^{n} \times \mathcal{S}$ (where $\mathcal{S}$ is the set of symmetric $n \times n$ matrix), $p \neq 0$, the application

$$
x \in \mathbb{R}^{n} \mapsto \inf _{v \in U, \sigma(x, v)^{*} p=0}\left(\langle b(x, v), p\rangle+\frac{1}{2} \operatorname{tr}\left(\sigma(x, v) \sigma(x, v)^{*} A\right)\right)
$$

is continuous (with the convention $\inf \emptyset=+\infty$ ).
Remarks: The assumption (H) requires in particular that, for all $x \in \mathbb{R}^{n}$ and $p \neq 0$, the set $\left\{v \in U, \sigma(x, v)^{*} p=0\right\}$ is non empty. Furthermore, if this set is no empty, the application is naturally lower semicontinuous. Under ( H ), it is also upper semicontinuous.
Assumption (H) seems rather restrictive. Indeed, if the dynamic doesn't depend on any control, $(\mathrm{H})$ is satisfied if and only if $\sigma \equiv 0$. A non trivial example where ( H ) holds is given by the mean curvature motion in Buckdahn-Quincampoix-Cardaliaguet [26]. A similar assumption can also be found in the paper of Soner-Touzi [98],[99].

Theorem 2.8 Suppose that assumption (H) holds. Then, for all $x \in \partial N \backslash \partial K$, there exists $\nu$ and $v \in \mathcal{A}(\nu)$ such that,

$$
P \text {-a.s., } X_{t}^{x, v} \in \partial N, \text { for all } t \leq \tau^{v}(x)=\inf \left\{s \geq 0, X_{s}^{x, v} \notin \operatorname{int}(K)\right\} .
$$

A last section of $\langle 11\rangle$ is devoted to an application of the study of the viability kernel to optimal control with supremum cost.
We consider the dynamic

$$
\left\{\begin{array}{l}
d X_{s}^{t, x, u}=b\left(X_{s}^{t, x, u}, u_{s}\right) d s+\sigma\left(X_{s}^{t, x, u}, u_{s}\right) d W_{s}, s \in[t, T],  \tag{14}\\
X_{t}^{t, x, u}=x,
\end{array}\right.
$$

and the value function

$$
W(t, x)=\inf _{\nu, u \in \mathcal{A}_{\nu}}\left({\left.\operatorname{ess}-\sup _{\Omega} g\left(X_{T}^{t, x, u}\right)\right) . . . .}^{t}\right.
$$

We are interested in the epigraph of $W$. Indeed, this epigraph can be written as the viability kernel of a controlled SDE in an enlarged space:

Theorem 2.9 The epigraph of $W$ is the viability kernel of

$$
\mathcal{K}=[0, T] \times \mathbb{R}^{n} \times \mathbb{R},
$$

with target $T \times E p i(g)$, for the dynamic

$$
\left\{\begin{array}{l}
d S_{s}^{t, x, u}=d s \\
d X_{s}^{t, x, u}=b\left(X_{s}^{t, x, u}, v_{s}\right) d s+\sigma\left(X_{s}^{t, x, u}, u_{s}\right) d W_{s} \\
d Y_{s}^{t, x, u}=0, s \in[t, T] \\
S_{t}^{t, x, u}=t, X_{t}^{t, x, u}=x, Y_{t}^{t, x, u}=y
\end{array}\right.
$$

i.e., for all $(t, x, y) \in[0,+\infty) \times \mathbb{R}^{n} \times \mathbb{R}$, P-p.s.,

$$
\left(s, X_{s}^{t, x, u}, y\right) \in[0, T] \times \mathbb{R}^{n} \times \mathbb{R}, \text { on }\left\{s \leq \tau^{u}(x)\right\}
$$

with

$$
\begin{aligned}
\tau^{u}(x) & =\inf \left\{r \geq t,\left(r, X_{r}^{t, x, u}, y\right) \in\{T\} \times \operatorname{Epi}(g)\right\} \\
& =\left\{\begin{array}{l}
T \text { if } g\left(X_{T}^{t, x, u}\right) \leq y \\
+\infty \text { else. }
\end{array}\right.
\end{aligned}
$$

Using the different characterizations of the viability kernel, we deduce from this theorem different characterizations of $W$ :

Corollary 2.3 1. $W$ is the smallest lower semicontinuous application $\varphi$ which satisfies
(a) $\varphi(T, x)=g(x), x \in \mathbb{R}^{n}$;
(b) $1-\mathbf{1}_{E p i(\varphi)}$ is a supersolution of

$$
\begin{equation*}
\psi_{t}+\inf _{v \in U, \sigma(x, v)^{*} \nabla_{x} \psi(t, x, y)}\left(\left\langle\nabla_{x} \psi(t, x, y), b(x, v)\right\rangle+\frac{1}{2} \operatorname{tr}\left[D_{x x}^{2} \psi(t, x, y) \sigma \sigma^{*}(x, v)\right]\right)=0 \tag{15}
\end{equation*}
$$

2. Under the assumption $(H), 1-\mathbb{1}_{E p i(W)}$ is a discontinuous viscosity solution of (15) ${ }^{1}$.
3. The application $W$ is the smallest lower semicontinuous supersolution of

$$
\begin{equation*}
\varphi_{t}+\inf _{v \in U, \sigma(x, v)^{*} \nabla_{x} \varphi(t, x)=0} \mathcal{L} \varphi(t, \cdot)=0 \tag{16}
\end{equation*}
$$

4. Under $(H), W$ is a discontinuous viscosity solution of (16).
5. Suppose that $(H)$ is satisfied and that $g$ is bounded and uniformly continuous. Then $W$ is uniformly continuous with respect to $x$ and the unique discontinuous viscosity solution of (16) with boundary condition $W(T, x)=g(x), x \in \mathbb{R}^{n}$.

### 2.6 Invariance.

We still consider the same controlled system (1) and a nonempty closed set $K$.

Definition 2.5 We say that $K$ is invariant with respect to the system (1) if, for all $x \in K$, $\nu$, $u \in \mathcal{A}_{\nu}$ and all $t \geq 0, P\left[X_{t}^{x, u} \in K\right]=1$.

[^0]If the stochastic differential equation doesn't depend on any control, the notions of viability and invariance coincide. A well known idea in this case (see Doss [44], Sussmann [103], Aubin-Doss [9]) is to associate to the stochastic system (1) a system of ordinary differential equations(EDOs):

$$
\begin{align*}
& \left.x^{\prime}(t)=\tilde{b}(x(t)), u(t)\right)+\sigma(x(t), u(t)) v(t), t \geq 0  \tag{17}\\
& x(0)=x
\end{align*}
$$

where $\tilde{b}$ denotes the Stratonovich drift

$$
\tilde{b}(x, u)=b(x, u)-\frac{1}{2} \sum_{i=1}^{d}\left\langle D_{x} \sigma^{i}(x, u), \sigma^{i}(x, u)\right\rangle,
$$

and $\sigma^{i}(x, u)$ the $i$ th column of $\sigma(x, u)$.
In $\langle 15\rangle$ written in collaboration with R. Buckdahn, M. Quincampoix et J. Teichmann, we are interested in this approach. More precisely we propose a new proof of a result of Da PratoFrankowska [41], that shows that, under minimal assumptions, the invariance of $K$ for the system (1) is equivalent to invariance for (17).

We remark that the EDO (17) can be interpreted as a doubly controlled system. The notion of invariance for (17) is defined in the following way:

Definition 2.6 We say that $K$ is invariant w.r.t. (17) if, for all $x \in K$, all $u \in L^{\infty}([0,+\infty), U)$ and $v \in L_{l o c}^{1}\left([0,+\infty), \mathbb{R}^{d}\right)$ and all $t \geq 0, x^{x, u, v}(t) \in K$, where $x^{x, u, v}$ is the solution of (17).

We finally introduce the differential operator associated to (17):

$$
\mathcal{L}_{x, u}^{\prime} \varphi=\langle\tilde{b}(x, u), D \varphi(x)\rangle .
$$

Theorem 2.10 The following assertions are equivalent:

1. $K$ is invariant w.r.t. (1);
2. For all $\varphi \in C^{2}$ et $x \in \operatorname{Argmax}_{K} \varphi$,

$$
\left\{\begin{array}{l}
\sup _{u \in U} \mathcal{L}_{x, u} \varphi \leq 0 \\
\left\langle\sigma^{i}(x, u), D \varphi(x)\right\rangle=0, \forall i \in\{1, \ldots, d\}, \forall u \in U
\end{array}\right.
$$

3. For all $\varphi \in C^{2}$ et $x \in \operatorname{Argmax}_{K} \varphi$,

$$
\left\{\begin{array}{l}
\sup _{u \in U} \mathcal{L}_{x, u}^{\prime} \varphi \leq 0 \\
\left\langle\sigma^{i}(x, u), D \varphi(x)\right\rangle=0, \forall i \in\{1, \ldots, d\}, \forall u \in U \\
\text { the matrix } \left.\left.A_{\varphi, x}=\left(a_{i j}\right), \text { with } a_{i j}=\left\langle\sigma^{i}(x, u), D_{x}\right\rangle \sigma^{i}(\cdot, u), D \varphi(\cdot)\right\rangle(x)\right\rangle \\
\text { is symmetric and positive semidefinite positive; }
\end{array}\right.
$$

4. $K$ is invariant for (17);
5. For all $\varphi \in C^{2}$ et $x \in \operatorname{Argmax}_{K} \varphi$,

$$
\sup _{u \in U, v \in \mathbb{R}^{d}}\left\{\mathcal{L}_{x, u}^{\prime} \varphi(x)+\langle\sigma(x, u) v, D \varphi(x)\rangle\right\} \leq 0
$$

In contrast to [41], our approach is probabilistic. A key point is the equivalence of invariance for constant controls. Therefore we can reduce the problem to a problem without controls and apply a convergence scheme of type Wong-Zakai, where the equation (1) is the limit of equations of type (17) (see Lipster-Shiryaev [77]). This permits us to show that, if the set $K$ is invariant for (1), it is also invariant for (17). The principal tool for the reverse is a stochastic Taylor expansion (see Lyons-Victoir [79] or Baudouin [24]).

Remark: Bardi and Cesaroni [14] prove the equivalence of the viability of stochastic and deterministic system under the assumption that the boundary of $K$ is a $C^{2}$-hypersurface. It is an open question wether this equivalence holds true without this assumption. It seems impossible to adapt our approach to viability, because the link with a system without control fails, and there is no convergence scheme for controlled SDEs.

## 3 Normal martingales, backward stochastic differential equations and control.

In this chapter we present the two papers $\langle 6\rangle$ and $\langle 13\rangle$. Their common point is to study stochastic differential equations where the Brownian motion is replaced by some other well chosen normal martingales, and to deduce stochastic representations for solutions of deterministic functional equations of a new type.

### 3.1 Introduction.

Definition 3.1 1) A real martingale $\left(X_{t}, t \geq 0\right)$ is said to be normal if its predictable quadratic variation satisfies, for all $t \geq 0$,

$$
\langle X, X\rangle_{t}=t .
$$

2) A normal martingale ( $X_{t}, t \geq 0$ ) satisfies a structure equation if $\left(X_{t}, t \geq 0\right)$ and its quadratic variation $[X, X]_{t}$ are linked by an equation of the following type:

$$
[X, X]_{t}=t+\int_{0}^{t} \phi_{s} d X_{s}, t \geq 0
$$

where $\left(\phi_{t}, t \geq 0\right)$ is a predictable process.
The main examples are

- the Brownian motion: $\phi_{t} \equiv 0$,
- the compound Poisson process: $\phi_{t} \equiv a, a \in \mathbb{R}$,
- Azéma's martingale: $\phi_{t}=-X_{t-}$, on which we come back later.

Structure equations were introduced by Emery in [48] to study the chaotic representation property. Other references on the subject are for example [50], Attal-Emery [5], Pham [92] or 〈1〉. An important property of martingales which are solution of a structure equation is that they have the predictable representation property.

### 3.2 Backward stochastic differential equation with Azéma's martingale.

This section resumes the paper $\langle 6\rangle$.

There are several ways to define Azéma's martingale. The most natural in our framework is to use a structure equation [48]:

Definition 3.2 We call Azéma's martingale a martingale ( $\mu_{t}, t \geq 0$ ) which satistifies the structure equation

$$
\begin{equation*}
d[\mu, \mu]_{s}=d s-\mu_{s-} d \mu_{s}, s \geq 0 \tag{18}
\end{equation*}
$$

The first definition of Azéma's martingale in Azéma [11] and Azéma-Yor [12] was its explicit construction based on a Brownian motion:
Let $\left(B_{s}, s \geq 0\right)$ be a one-dimensional standard Brownian motion and, for all $s \geq 0, G_{s}=\sup \{r \leq$ $\left.s, B_{r}=0\right\}$, with $\sup \emptyset=0$, its last zero before $s$. Then the process $\left(\operatorname{sign}\left(B_{s}\right) \sqrt{2\left(s-G_{s}\right)}, s \geq 0\right)$ ( with the convention $\operatorname{sign}(0)=0$ ) satisfies the structure equation (18).
Azéma's martingale is also equal, up to a multiplicatif constant, to the projection of the Brownian motion on the filtration generated by the processes $\left(G_{s}, s \geq 0\right)$ and $\left(\operatorname{sign}\left(B_{s}\right), s \geq 0\right)$.

We need here to make start our martingale at an arbitrary time $t \geq 0$ from an arbitrary point $x \in \mathbb{R}$. Therefore we introduce, for all $(t, x) \in[0, \infty) \times \mathbb{R}$, the process $\left(\mu_{s}^{t, x}, s \geq t\right)$, solution of

$$
\left\{\begin{array}{l}
d\left[\mu^{t, x}, \mu^{t, x}\right]_{s}=d s-\mu_{s-}^{t, x} d \mu_{s}^{t, x}, s \geq t,  \tag{19}\\
\mu_{t}^{t, x}=x .
\end{array}\right.
$$

Let $\left(\mathcal{F}_{s}^{t, x}, s \geq t\right)$ be the filtration generated by $\left(\mu_{s}^{t, x}, s \geq t\right)$ completed and right continuous.

The existence and uniqueness of solutions of BSDEs with respect to martingales which have the predictable representation property was established by Karoui and Huang in [46].
Here we need just to fix the notations: for $T>0$ a deterministic terminal time and some constant $K>0$, we introduce the spaces $H^{2}=H^{2}(K, T, t, x)$ and $H^{\prime 2}=H^{\prime 2}(K, T, t, x)$ of real-valued, $\mathcal{F}^{t, x}$-predictable and càdlàg processes $U$ which satisfy

$$
\forall s \geq 0, U_{s \wedge t}=U_{t}, U_{s \vee T}=U_{T} \text { et } E\left[\sup _{s \in[t, T]} e^{K s} U_{s}^{2}\right]<\infty,
$$

(resp. real-valued $\mathcal{F}^{t, x}$-predictable processes $V$ such that

$$
\left.\forall s \geq 0, V_{s \wedge t}=V_{s \vee T}=0 \text { and } E\left[\int_{t}^{T} e^{K s} V_{s}^{2}\right]<\infty .\right)
$$

Theorem 3.1 Under standard assumptions on $f$ and $g$, the following BSDE has a unique solution $(Y, Z)$ in the space $H^{2} \times H^{\prime 2}$ :

$$
\begin{equation*}
Y_{s}=g\left(\mu_{T}^{t, x}\right)+\int_{s}^{T} f\left(r, \mu_{r}, Y_{r}, Z_{r}\right) d r+\int_{s}^{T} Z_{r} d \mu_{r}^{t, x}, s \in[t, T] . \tag{20}
\end{equation*}
$$

The comparison theorem for solutions of classical BSDEs is not always true for non Brownian martingales. We prove that it holds for Azéma's martingale, under a supplementary assumption which controls the size and the direction of the jumps.

As in the Brownian case, for all $(t, x) \in[0, T] \times \mathbb{R}$, the random variable $Y_{t}$ is $P$-a.s equal to a deterministic value $u(t, x)$.

Theorem 3.2 The application $(t, x) \mapsto u(t, x)$ is continuous, of almost polynomial growth and a viscosity solution of the following equation:

$$
\left\{\begin{array}{l}
f\left(t, x, u(t, x), \frac{u(t, x)-u(t, 0)}{x}\right)+\frac{\partial u}{\partial t}(t, x)+\frac{u(t, 0)-u(t, x)+x \frac{\partial u}{\partial x}(t, x)}{x^{2}}=0,(t, x) \in[0, T] \times \mathbb{R}^{*},  \tag{21}\\
f\left(t, 0, u(t, 0), \frac{\partial u}{\partial x}(t, 0)\right)+\frac{\partial u}{\partial t}(t, 0)+\frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}}(t, 0)=0 \\
u(T, x)=g(x), x \in \mathbb{R} .
\end{array}\right.
$$

(where the notion of viscosity solution has been generalized in a close way to that of PDEs with integral term in [19]).

By a proof inspired from Barles [18], we get the following comparison theorem and its corollary:
Theorem 3.3 Let $u$ and $v$ be a sub- (resp. a super) solution of (21). If one of the two maps is of class $C^{1,2}$ and both satisfy, for $a>0$ sufficiently small,

$$
\lim _{|x| \rightarrow \infty} u(t, x) e^{-a x^{2}}=\lim _{|x| \rightarrow \infty} v(t, x) e^{-a x^{2}}=0, \text { uniformly in } t,
$$

then $u \leq v$.
Corollary 3.1 If (21) admits a solution $v \in C^{1,2}$ which satisfies $\lim _{|x| \rightarrow \infty} v(t, x) e^{-a x^{2}}=0$ for $a>0$ sufficiently small, uniformly in $t$, then this solution is unique.

It is clear that, in the general case, the solution of (21) isn't of class $C^{1,2}$. And indeed, we are not able to prove an uniqueness result in the general case, and encounter the same difficulty in the next chapter. There exists a large litterature about integro-differential equations (see for example Barles-Buckdahn-Pardoux [19], Alvarez-Tourin [1], Amadori-Karlsen-La Chioma [2] or Jacobsen-Karlsen [65], Barles-Chassaigne-Imbert [20] and Barles-Imbert [21]). But, even if these papers concern large classes of operators, our operators never form a part of these classes. However, we can prove an uniqueness result in the case where $f \equiv 0$ : If $f \equiv 0$, the functional equation (21) is much simplier:

$$
\left\{\begin{array}{l}
\frac{\partial u}{\partial t}(t, x)+\frac{u(t, 0)-u(t, x)+x \frac{\partial u}{\partial x}(t, x)}{x^{2}}=0,(t, x) \in[0, T] \times \mathbb{R}^{*}  \tag{22}\\
\frac{\partial u}{\partial t}(t, 0)+\frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}}(t, 0)=0 \\
u(T, x)=g(x), x \in \mathbb{R}
\end{array}\right.
$$

Theorem 3.4 For all continuous function $g$ with at least polynomial growth, there exists a unique viscosity solution of (22).

Remark: Using the BSDE associated to (22), the Feynman-Kač formula and the explicit expressions of certain known distributions (see Revuz-Yor [96] or [25]), this solution can be explicitely computed.
The existence of a solution follows from 3.2. The proof of the uniqueness is quite involved. It is strongly based on Theorem 3.3.

In the second part of the paper, we are interested in an asymmetric martingale which we introduced in an anterior work $\langle 1\rangle$ (see also $\mathrm{Hu}[63]$ ), namely the solution of the following structure equation:

$$
\left\{\begin{array}{l}
d[X, X]_{s}=d s-X_{s-}^{+} d X_{s}, s \geq t, \\
X_{t}=x .
\end{array}\right.
$$

Its particularity is to behave like Azéma's martingale if it is positive and like a Brownian motion else. As in the first part, we can prove the existence of a solution of an asymmetric functional equation, that is a nonlinear PDE on $[0, T] \times(-\infty, 0]$ and of type $(21)$ on $[0, T] \times(0, \infty)$.

### 3.3 Structure equations and control.

In this section we present the paper $\langle 13\rangle$ written in collaboration with R. Buckdahn and J. Ma. Here it is in a system of controlled SDEs that we replace the Brownian motion by a normal martingale, which is solution of a structure equation. We will make depend this structure equation from a control. Roughly speaking, this means that the controller of the system will be able, at each moment, to influence, in addition to the parameters of the system, also the frequency of the jumps of the driving martingale.

The first part of the paper consists in the explicit resolution of a family of structure equations. P.A. Meyer, in [82], showed that all structure equation of Markovian form

$$
d[X, X]_{t}=d t+f\left(X_{t-}\right) d X_{t}
$$

admits a solution, if $f$ is continuous. For the study of particular structure equations, we can cite of course Emery [48] but also Kurtz-Protter [73], $\langle 1\rangle$ and Pham [92]. Structure equations in dimension 2 are studied in Attal-Emery [4], [5] and Kurtz [72].
The novelty here is that we leave the Markovian frame.

Let us consider a $d$-dimensional Brownian-motion $B$ and a Poisson measure $\mu$ on $\mathbb{R}_{+} \times \mathbb{R}^{*}$. We suppose that $B$ and $\mu$ are independent and that the Lévy measure of $\mu$, denoted by $\nu$, satisfies the assumption

$$
\int_{\mathbb{R}^{*}}\left(1 \wedge|x|^{2}\right) \nu(d x)<\infty .
$$

We consider now the following structure equation:

$$
\left\{\begin{array}{l}
d\left[X^{i}, X^{i}\right]_{t}=d t+u_{t}^{i} d X_{t}, 1 \leq i \leq d,  \tag{23}\\
d\left[X^{i}, X^{j}\right]_{t}=0,1 \leq i<j \leq d, t \geq 0,
\end{array}\right.
$$

where $\left(u_{t}=\left(u_{t}^{1}, \ldots, u_{t}^{d}\right), t \geq 0\right)$ is a predictable process for the filtration $\mathbb{F}$ generated by $B$ and $\mu$, with values in $U^{d}$, a non empty compact set of $\mathbb{R}^{d}$. This type of structure equation can be solved explicitely:

Proposition 3.1 The equation (23) has at least a solution of the following type:

$$
\begin{equation*}
X_{t}=\int_{0}^{t} \alpha_{s} d B_{s}+\int_{0}^{t} \int_{\mathbb{R}^{*}} \beta_{s}(x) \tilde{\mu}(d x d s), t \geq 0 \tag{24}
\end{equation*}
$$

where $(\alpha, \beta)$ is a couple of $\mathbb{F}$-predictable processes and $\tilde{\mu}$ is the compensated Poisson measure $\mu$.

It is not difficult to build counterexample which shows that there is no uniqueness, even in law.

This result allows us to insert this new family of martingales in a control problem.
To establish later a dynamic programming principle, we introduce, for all $t \in[0, T]$, the filtration $\mathbf{F}^{t}$, which is trivial before $t$ and generated by $\left(B_{r}-B_{t}\right)$ and $\left(\nu(A), A \in \mathcal{B}\left([t, r] \times \mathbb{R}^{*}\right), r \in[t, T]\right.$ after. Let be the following structure equation:

$$
\begin{cases}d\left[X^{i}, X^{i}\right]_{s}=d s+u_{s}^{i} d X_{s}^{i}, & 1 \leq i \leq d, s \in[0, T]  \tag{25}\\ d\left[X^{i}, X^{j}\right]_{s}=0, & 1 \leq i<j \leq d, s \in[0, T] \\ X_{s}-X_{t} \text { independent of } \mathcal{F}_{t}, & s \geq t\end{cases}
$$

with $\left(u_{t}=\left(u_{t}^{1}, \ldots, u_{t}^{d}\right), s \in[0, T]\right) \mathbf{F}^{t}$-predictable with values in $U^{d}$, compact in $\mathbb{R}^{d}$.
We also introduce a second compact set $U_{1}$ in $\mathbb{R}$, where will live the controls which are classically involved in the SDE. We set $\bar{U}=U_{1} \times U^{d}$.

As before, the equation (25) has a solution which is not always unique even in law. This is why we call here control a triplet $(\pi, u, X)$, where the process $(\pi, u)$ is $\mathbf{F}^{t}$-predictable with values in $\bar{U}$ and $X$ is a solution of (25) for this $u$. For all fixed $t \in[0, T], \mathcal{U}(t)$ denotes the set of controls defined in this way.

For all $(t, y) \in[0, T] \times \mathbb{R}^{m}$ and $a=(\pi, u, X) \in \mathcal{U}(t)$, we consider now the dynamic

$$
\left\{\begin{array}{l}
Y_{s}^{t, y, a}=y+\int_{t}^{s} b\left(Y_{r}^{t, y, a}, \pi_{r}, u_{r}\right) d r+\int_{t}^{s} \sigma\left(Y_{r-}^{t, y, a}, \pi_{r}, u_{r}\right) d X_{r}, s \in[t, T]  \tag{26}\\
Y_{s}^{t, y, a}=y, s \in[0, t)
\end{array}\right.
$$

and the value function

$$
\begin{equation*}
V(t, y)=\inf _{a \in \mathcal{U}(t)} E\left[g\left(Y_{T}^{t, y, a}\right)\right],(t, y) \in[0, T] \times \mathbb{R}^{m} \tag{27}
\end{equation*}
$$

where, as usual, $b, \sigma$ and $g$ are supposed to be uniformly continuous, Lipschitz in $y$, uniformly in $(\pi, u)$, and $g$ is continuous and bounded.
We prove that $V$ is continuous and satisfy a dynamic programming principle. Moreover it satisfies the following theorem:

Theorem 3.5 The value function $V$ is a viscosity solution of the following equation:

$$
\left\{\begin{array}{cl}
-\frac{\partial}{\partial t} V(t, y)-\inf _{(\pi, u) \in \bar{U}} \mathcal{L}_{\pi, u}[V](t, y)=0, & (t, y) \in[0, T] \times \mathbb{R}^{m}  \tag{28}\\
V(T, y)=g(y), & y \in \mathbb{R}^{m}
\end{array}\right.
$$

where $\mathcal{L}_{\pi, u}$ is of the form

$$
\begin{align*}
\mathcal{L}_{\pi, u}[\varphi](t, y)= & \nabla_{y} \varphi(t, y) b(y, \pi, u)+\sum_{i=1}^{d}\left\{\mathbb{1}_{\left\{u^{i}=0\right\}} \frac{1}{2}\left(D_{y y}^{2} \varphi(t, y) \sigma^{i}(y, \pi, u), \sigma^{i}(y, \pi, u)\right)\right. \\
& \left.+\mathbb{1}_{\left\{u^{i} \neq 0\right\}} \frac{\left.\varphi\left(t, y+u^{i} \sigma^{i}(y, \pi, u)\right)\right)-\varphi(t, y)-u^{i} \nabla_{y} \varphi(t, y) \sigma^{i}(y, \pi, u)}{\left(u^{i}\right)^{2}}\right\},
\end{align*}
$$

(again, the definition of viscosity solution is generalyzed in an analogue way to [19].)

The difficulty to prove some uniqueness result for (28) comes from the fact that the Hamiltonian associated to this problem,

$$
\begin{gathered}
\mathcal{H}(t, y, v, p, S)=\inf _{(\pi, u)}\left(p b(y, \pi, u)+\sum_{i=1}^{d}\left\{\mathbb{1}_{\left\{u^{i}=0\right\}} \frac{1}{2}\left(S \sigma^{i}(y, \pi, u), \sigma^{i}(y, \pi, u)\right)\right.\right. \\
\left.\left.+\mathbb{1}_{\left\{u^{i} \neq 0\right\}} \frac{\left.\varphi\left(t, y+u^{i} \sigma^{i}(y, \pi, u)\right)\right)-\varphi(t, y)-u^{i} p \sigma^{i}(y, \pi, u)}{\left(u^{i}\right)^{2}}\right\}\right)
\end{gathered}
$$

is not continuous in $(p, S)$. We get around this problem by by introducing, for $d=1$, the following assumption:
(HC) There exists a compact set $U_{0} \subset \mathbb{R}$ such that

1. $0 \notin U_{0}$,
2. $U=U_{0}$ ou $U=\{0\} \cup U_{0}$.

Theorem 3.6 Under the assumption ( $H C$ ), the value function $V:[0, T] \times \mathbb{R}^{m}$ defined in (27) is the unique viscosity solution of (28) in the class of continuous and bounded functions.

We are not able to solve the problem of uniqueness without assumption (HC). In the other hand, another perspective which probably would not rise difficulties is to consider a non linear cost, i.e. to replace the system (26) by a decoupled forward-backward system.

## 4 Stochastic differential games.

### 4.1 Introduction.

In this section, we present the 4 papers $\langle 10\rangle,\langle 12\rangle,\langle 14\rangle,\langle 16\rangle$, treating on stochastic differential games. These games are based on the following dynamic:

$$
\left\{\begin{array}{l}
d X_{s}=f\left(X_{s}, u_{s}, v_{s}\right) d s+\sigma\left(X_{s}, u_{s}, v_{s}\right) d W_{s}, s \in[t, T]  \tag{30}\\
X_{t}=x \in \mathbb{R}^{n}
\end{array}\right.
$$

where $W$ is a $d$-dimensional standard Brownian motion and $u$ and $v$ are two processes with values in some compact metric spaces $U$ and $V$. The processes $u$ and $v$ represent the controls played by Player 1 and 2, respectively.
Each player needs to maximize some payoff

$$
\begin{equation*}
J_{i}(t, x, u, v)=E\left[g_{i}\left(X_{T}\right)+\int_{t}^{T} l_{i}\left(s, X_{s}, u_{s}, v_{s}\right) d s\right], i \in\{1,2\} \tag{31}
\end{equation*}
$$

The parameters $g_{i}$ and $l_{i}$ will vary according to the different problems, in particular wether we consider a zero-sum or a nonzero-sum game. They will be known or unknown by the players and, to simplify the problems, will vanish successively.

The first references on stochastic differential games are Friedman [53] and Varaya [106] in the seventies of last century. Their approach consists in finding Markovian optimal controls $u=u(t, x)$, which can be obtained by solving the associated Hamilton-Jacobi-Bellman equation. Several authors continue in this way (for example Bensoussan-Frehse [22] and Mannucci [80]). Conversely, using plainely the stochastic character, Elliott [47] gives another interpretation of stochastic differential games, by letting intervene the players through Girsanov transformations. This approach is taken up and considerably simplified by the use of BSDEs in the papers of Hamadène et al. (for example Hamadène-Lepeltier [61], Hamadène-Lepeltier-Peng [62], Hamadène [58]). A weak point of all these approaches is that the diffusion term has to be uniformly elliptic and cannot contain any control. Finally a third way is opened in 1989 by Fleming-Souganidis [52] (see also Nisio [87]). For a nonzero sum game, they define rigorousely all possible actions of the players and establish directly the existence and characterization of the value of the game, through a dynamic programming principle. Unlike the other approaches, this doesn't need to suppose the existence of a Markovian optimal control, and permits so to raise the restrictive assumptions we mensioned above. It is way we follow in the papers we present here.
In $\langle 10\rangle$ and $\langle 12\rangle$, we study a nonzero-sum game. We prove the existence of a Nash equilibrium payment, give a characterization of it and make a link to the approach of Hamadène-LepeltierPeng.
In $\langle 14\rangle$, we are interested in a nonzero-sum game, where the players have only a partial knowledge of the parameters of the value funtion. More precisely, these parameters are chosen randomly in a set of functions $\left(g_{k l}, l_{k l},(k, l) \in\{1, \ldots, I\} \times\{1, \ldots, J\}\right.$, then each player is informed on one of the two indices of the chosen parameter. Finally one player has to minimize, the other to maximize the resulting payoff. We show that, under Isaacs' condition, the game has a value and give a characterization, in terms of viscosity solutions of a new type. In $\langle 16\rangle$, in ordrer to go more into details, we simplify a lot the game: we consider a game without dynamic, where only one player has a lack of information $(J=1)$. This permits us to analyse the behavior of the fully informed player: we show that it is narrowly linked to a certain martingale with values in the simplex $\Delta(I)$, whose role is to manage the amount of information used (and thus revealed) along the game.

Let us precise the common frame of all the presented papers:

As usal, $T>0$ denotes a deterministic final time, $\mathbb{F}^{t}=\left(\mathcal{F}_{t, s}^{t}, t \leq s \leq T\right)$ is the filtration generated by the increments $B_{s}-B_{t}$ of the Brownian motion, and $\mathcal{U}(t)$ and $\mathcal{V}(t)$ are the sets of admissible controls for the two players.
The assumptions on the parameters of the game guaranty a strong solution of the system (30): $f:[0, T] \times \mathbb{R}^{n} \times U \times V \rightarrow \mathbb{R}^{n}, \sigma:[0, T] \times \mathbb{R}^{n} \times U \times V \rightarrow \mathbb{R}^{n \times d}, g: \mathbb{R}^{n} \rightarrow \mathbb{R}$ and $l:[0, T] \times \mathbb{R}^{n} \times U \times V \rightarrow \mathbb{R}$ are measurable, bounded, continous, Lipschitz in $(t, x)$ (or in $x$ for $g$ ), uniformly in $(u, v)$.
For all $t \in[0, T], x \in \mathbb{R}^{d}$ and $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$, we denote by $X^{t, x, u, v}$ the solution of (30). We introduce Isaacs' condition:
(HI) For all $(t, x, A, p) \in[0, T] \times \mathbb{R}^{n} \times \mathcal{S} \times \mathbb{R}^{n}$,

$$
\begin{aligned}
\sup _{u \in U} \inf _{v \in V} & \left\{\langle p, f(t, x, u, v)\rangle+\frac{1}{2}\left(A \sigma \sigma^{*}(t, x, u, v)\right)\right\} \\
& =\inf _{v \in V} \sup _{u \in U}\left\{\langle p, f(t, x, u, v)\rangle+\frac{1}{2}\left(A \sigma \sigma^{*}(t, x, u, v)\right)\right\} .
\end{aligned}
$$

A fundamental notion in game theory is the notion of strategy. Indeed, each player has to adjust his actions at each moment of the game. Thus, these actions have to be applications that, at each moment, reply to the control of the opponent player and the observed dynamic by the control he consider to be the best adapted.
Ther are several ways to define strategies. We use here the following:

Definition 4.1 A strategy for Player 1 starting at time $t$ is a Borel-measurable application $\alpha:[t, T] \times \mathcal{C}\left([t, T], \mathbb{R}^{n}\right) \times L^{2}([t, T], V) \rightarrow U$ for which there exists $\delta>0$ such that, for all $s \in[t, T], f, f^{\prime} \in \mathcal{C}\left([t, T], \mathbb{R}^{n}\right)$ and $g, g^{\prime} \in L^{2}([t, T], V)$, if $f=f^{\prime}$ and $g=g^{\prime}$ a.s. on $[t, s]$, then $\alpha(\cdot, f, g)=\alpha\left(\cdot, f^{\prime}, g^{\prime}\right)$ on $[t, s+\delta]$.
We define strategies for Player 2 in a symmetric way and denote by $\mathcal{A}(t)$ (resp. $\mathcal{B}(t)$ the set of strategies for Player 1 (resp. Player 2).

## Remarks:

1. In [52], a strategy is an application that answers to a control of the opponent player by another control, i.e. it associates to a stochastic process another stochastic process. This rises strong technical difficulties and need heavy assumptions on the sets of strategies. After adopting this definition in our first paper $\langle 10\rangle$, we have changed from $\langle 11\rangle$ on to 4.1 . Beside the fact that the new definition is much simplier and also simplier to handle, it seems more pertinent to us, that the players only can observe and react to some trajectories and not to the hole processes.
2. It is worthy to mension that the strategies we defined in 4.1 are strategies with delay. This property is crucial to make each couple of strategies "playable", as it is formulated in the next lemma. We remark that this delay isn't not necessarily the same for all strategies.

Lemma 4.1 For all $(t, x) \in[0, T] \times \mathbb{R}^{n}$ and all $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, there exists a unique couple of controls $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ which satisfy $P$-a.s.

$$
\begin{equation*}
(u, v)=\left(\alpha\left(\cdot, X^{t, x, u, v}, v .\right), \beta\left(\cdot, X^{t, x, u, v}, u .\right)\right) \text { a.s. on }[t, T] \text {. } \tag{32}
\end{equation*}
$$

Notation 4.1 For all $(t, x) \in[0, T] \times \mathbb{R}^{d}$ and $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, we set

$$
X^{t, x, \alpha, \beta}=X^{t, x, u, v} \text { and } J_{i}(t, x, \alpha, \beta)=J_{i}(t, x, u, v),
$$

where $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$ is fixed by the relation (32).

Finally let us recall the well understoud case of zero-sum games $g:=g_{2}=-g_{1}$ and $l:=l_{2}=-l_{1}$. We set

$$
J(t, x, u, v)=E\left[g\left(X_{T}\right)+\int_{t}^{T} l\left(s, X_{s}, u_{s}, v_{s}\right) d s\right] .
$$

Player 1 has to minimize the value function $J$, Player 2 has to minimize it.
Following Fleming-Souganidis [52], under Isaacs' condition (HI), the game has a value:

$$
\left.W(t, x):=\inf _{\alpha \in \mathcal{A}(t)} \sup _{v \in \mathcal{V}(t)} J(t, x, \alpha(v), v)\right)=\sup _{\beta \in \mathcal{B}(t)} \inf _{u \in \mathcal{U}(t)} J(t, x, u, \beta(u))
$$

The function $W$ is the unique viscosity solution of the following nonlinear PDE:

$$
\left\{\begin{array}{l}
w_{t}+\inf _{u \in U} \sup _{v \in V} H\left(t, x, D w, D^{2} w, u, v\right)=0, t \in[0, T] \\
w(T, x)=g(x)
\end{array}\right.
$$

with

$$
H(t, x, p, A, u, v)=\langle p, f(t, x, u, v)\rangle+\frac{1}{2}\left(A \sigma(t, x, u, v) \sigma^{*}(t, x, u, v)\right)+l(t, x, u, v)
$$

### 4.2 Nash Equilibria for nonzero-sum stochastic differential games.

In this section, we present the results of the paper $\langle 10\rangle$ written in collaboration with R . Buckdahn and P. Cardaliaguet, and of the paper $\langle 12\rangle$.
We consider a game defined by the dynamic (30) and the value functions introduced by (31), where, to simplify, we omit the integral term:

$$
J_{i}(t, x, u, v)=E\left[g_{i}\left(X_{T}\right)\right], i \in\{1,2\}
$$

Each player $i$ has to maximize his payoff $J_{i}$. We are interested to the Nash equilibria of this game.
Concerning the litterature on nonzero-sum stochastic differential games, among the authors cited before, we will come back more precisely on some works of Hamadène et al. [61], [62]. There are also the papers of Ghosh and Kumar (for example [55]) based on fixe point arguments, and Kushner [70], [71] who treates numerical approximations. Finally a recent and important courrant are mean field games introduced by Lasry and Lions [74] (see also [75],[76], Buckdahn, Djehiche, Li and Peng [27] and [28]).
Let's remark that only 2-player games are considered here, but the results can be easily generalized to an arbitrary number of players.

We recall the notion of Nash equilibria:

Definition 4.2 $A$ Nash equilibrium at some point $(t, x)$ is a couple $(\bar{\alpha}, \bar{\beta}) \in \mathcal{A}(t) \times \mathcal{B}(t)$ such that, for all $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$, it holds that

$$
\begin{equation*}
J_{1}(t, x, \bar{\alpha}, \bar{\beta}) \geq J_{1}(t, x, \alpha, \bar{\beta}) \text { and } J_{2}(t, x, \bar{\alpha}, \bar{\beta}) \geq J_{2}(t, x, \bar{\alpha}, \beta) \tag{33}
\end{equation*}
$$

In our framework, such equilibria not always exist. This motivates the following definition:
Definition 4.3 $A$ Nash equilibrium payment (NEP) at $(t, x)$ is a couple $\left(e_{1}, e_{2}\right) \in \mathbb{R}^{2}$ such that, for all $\epsilon>0$, there exists a couple of strategies $\left(\alpha_{\epsilon}, \beta_{\epsilon}\right) \in \mathcal{A}(t) \times \mathcal{B}(t)$ which satisfies, for all $(\alpha, \beta) \in \mathcal{A}(t) \times \mathcal{B}(t)$,

1. $J_{1}\left(t, x, \alpha_{\epsilon}, \beta_{\epsilon}\right) \geq J_{1}\left(t, x, \alpha, \beta_{\epsilon}\right)-\epsilon$ et $J_{2}\left(t, x, \alpha_{\epsilon}, \beta_{\epsilon}\right) \geq J_{2}\left(t, x, \alpha_{\epsilon}, \beta\right)-\epsilon$,
2. for $i=1,2,\left|J_{i}\left(t, x, \alpha_{\epsilon}, \beta_{\epsilon}\right)-e_{i}\right| \leq \epsilon$.

The principal result of $\langle 10\rangle$ is the existence of such NEPs and their characterization:
Theorem 4.1 Under Isaacs' condition (HI), for any initial condition $(t, x)$, there exists a NEP.
To formulate a characterization of a NEP, we have to introduce $W_{1}$ and $W_{2}$, the values of the two zero-sum games

$$
W_{1}(t, x)=\inf _{\beta \in \mathcal{B}(t)} \sup _{u \in \mathcal{U}(t)} J_{1}(t, x, u, \beta(u))
$$

and

$$
W_{2}(t, x)=\inf _{\alpha \in \mathcal{A}(t)} \sup _{v \in \mathcal{V}(t)} J_{2}(t, x, \alpha(v), v) .
$$

Theorem 4.2 Still under Isaacs' condition, a couple $\left(e_{1}, e_{2}\right) \in \mathbb{R}^{2}$ is a NEP at $(t, x)$ if and only if, for all $\epsilon>0$, there exists a couple of controls $\left(u_{\epsilon}, v_{\epsilon}\right) \in \mathcal{U}(t) \times \mathcal{V}(t)$ such that

1. for all $s \in[t, T]$ and $i \in\{1,2\}$,

$$
P\left\{E\left[g_{i}\left(X_{T}^{\epsilon}\right) \mid \mathcal{F}_{t, s}\right] \geq W_{i}\left(s, X_{s}^{\epsilon}\right)-\epsilon\right\} \geq 1-\epsilon,
$$

with $X^{\epsilon}=X^{t, x, u_{\epsilon}, v_{\epsilon}}$,
2. for $i \in\{1,2\}$,

$$
\left|E\left[g_{i}\left(X_{T}^{\epsilon}\right)\right]-e_{i}\right| \leq \epsilon .
$$

These results were established in the deterministic framework by Kononenko [68] and Tolwinski-Haurie-Leitmann [104]. Their techniques cannot be generalized to the stochastic case. Therefore our approach is completely different.
Close to the Folk Theorem for repeated games, Theorem 4.2 heuristically says that there is a Nash equilibrium if and only if, at each moment, if one of the players deviates, the other can punish him by minimizing its value function, We will use Theorem 4.2 in the sequel to identify NEPs as the equilibrium payments obtained in the framework of Hamadène et al. [61], [62].

## Comparison with the Nash equilibria of Hamadène et al.

The approach of Hamadène et al. consists in making intervene the players by a change of probability. Since their introduction by Elliott [47] until the recent developments on mixed games [59] (see also Karatzas-Zamfirescu [67]), this approach was developped absolutely independently to the others. Therefore it seemed interesting to us to make some link.
In [62], the following SDE is considered:

$$
X_{s}^{t, x}=x+\int_{t}^{s} \sigma\left(r, X_{r}^{t, x}\right) d B_{r}, s \in[t, T], x \in \mathbb{R}^{n}
$$

where $B$ is still a $d$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and $\sigma$ is supposed to be strictly elliptic.
The controls of the players are introduced via a change of probability: for all $u$ and $v$ controls with values in $U$ (resp. $V$ ), some probability $P^{u, v}$ is defined by

$$
\begin{align*}
\frac{d P^{u, v}}{d P}=\exp \left\{\int_{t}^{T}\right. & \sigma^{-1}\left(s, X_{s}\right) f\left(s, X_{s}, u_{s}, v_{s}\right) d B_{s}  \tag{34}\\
& \left.-\frac{1}{2} \int_{t}^{T}\left|\sigma^{-1}\left(s, X_{s}\right) f\left(s, X_{s}, u_{s}, v_{s}\right)\right|^{2} d s\right\}
\end{align*}
$$

and one defines

$$
B_{s}^{u, v}=B_{s}-B_{t}-\int_{t}^{s} \sigma^{-1}\left(r, X_{r}\right) f\left(r, X_{r}, u_{r}, v_{r}\right) d r .
$$

By Girsanovs Theorem, $B^{u, v}$ is a $P^{u, v}$-Brownian motion and $X^{t, x}$ can be written as a solution of the controlled SDE

$$
d X_{s}^{t, x}=f\left(s, X_{s}^{t, x}, u_{s}, v_{s}\right) d s+\sigma\left(s, X_{s}^{t, x}\right) d B_{s}^{u, v}, s \in[t, T] .
$$

The payoffs are defined here as follows:

$$
J^{i}(t, x, u, v)=E^{u, v}\left[g_{i}\left(X_{T}\right)+\int_{t}^{T} h_{i}\left(s, X_{s}, u_{s}, v_{s}\right) d s\right], i \in\{1,2\},(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)
$$

and a Nash equilibrium is a couple of controls $\left(u^{*}, v^{*}\right) \in \mathcal{U}(t) \times \mathcal{V}(t)$ which satisfies, for all $(u, v) \in \mathcal{U}(t) \times \mathcal{V}(t)$,

$$
\begin{equation*}
J^{1}\left(t, x, u^{*}, v^{*}\right) \geq J^{1}\left(t, x, u, v^{*}\right) \text { and } J^{2}\left(t, x, u^{*}, v^{*}\right) \geq J^{2}\left(t, x, u^{*}, v\right) . \tag{35}
\end{equation*}
$$

For $i \in\{1,2\}$, let be the following Hamiltoniens

$$
G_{i}(t, x, p, u, v)=\langle p, f(t, x, u, v)\rangle+h_{i}(t, x, u, v),
$$

and the assumption ( HH ):
(HH) a) there exist two borelian maps $u^{*}, v^{*}$ from $[0, T] \times \mathbb{R}^{3 n}$ to $U \times V$ such that, for all $(t, x, p, q, u, v) \in[0, T] \times \mathbb{R}^{3 n} \times U \times V$,

$$
G_{1}\left(t, x, u^{*}(t, x, p, q), v^{*}(t, x, p, q)\right) \geq G_{1}\left(t, x, u, v^{*}(t, x, p, q)\right)
$$

and

$$
G_{2}\left(t, x, u^{*}(t, x, p, q), v^{*}(t, x, p, q)\right) \geq G_{2}\left(t, x, u_{1}^{*}(t, x, p, q), v\right) .
$$

b) the map that, to $(p, q) \in \mathbb{R}^{2 n}$ associates

$$
\left(G_{1}\left(t, x, u^{*}(t, x, p, q), v^{*}(t, x, p, q)\right), G_{2}\left(t, x, u^{*}(t, x, p, q), v^{*}(t, x, p, q)\right)\right.
$$

is continuous.
Under (HH), the following result holds:

Theorem 4.3 ([62]) There are two couples of processes $\left(W^{1}, Z^{1}\right)$ and $\left(W^{2}, Z^{2}\right)$ which satisfy, for $i \in\{1,2\}, s \in[t, T], P$-a.s.,

$$
\begin{equation*}
W_{s}^{i}=g_{i}\left(X_{T}\right)+\int_{s}^{T} G_{i}\left(r, X_{r}, Z_{r}^{i}, u^{*}\left(r, X_{r}, Z_{r}^{1}, Z_{r}^{2}\right), v^{*}\left(r, X_{r}, Z_{r}^{1}, Z_{r}^{2}\right)\right) d r-\int_{s}^{T} Z_{r}^{i} d X_{r}, \tag{36}
\end{equation*}
$$

and the couples of controls $\left(u^{*}, v^{*}\right):=\left(u^{*}\left(r, X_{r}, Z_{r}^{1}, Z_{r}^{2}\right), v^{*}\left(r, X_{r}, Z_{r}^{1}, Z_{r}^{2}\right)\right)$ are Nash equilibriums in the sense of (35).

Remark: The probabilities $P$ and $P^{u, v}$ are always equivalent, but it can happen that the filtration $\mathcal{F}^{u, v}$ generated by $B^{u, v}$ is strictly smaller than that generated by $B$. In particular, in this case, arbitrary controls $u, v$ are not necessarily adapted to $\mathcal{F}^{u, v}$. An example for such a situation is given by $\sigma=I, f(t, x, u, v)=u$, and $u$ given by the example of Tsirel'son [105] (see also [96]). But, using a result of Bahlali [13], we can show that, under the present assumptions, the equations (36) admit strong solutions. In this case the two filtrations coincide.

Another preliminary result is that the values of the zerosum game of Hamadène-Lepeltier introduced in [60] coincide with $W_{1}$ and $W_{2}$ in Fleming-Souganidis [52]:

Proposition 4.1 Under the assumption (HH), we have, for all $(t, x) \in[0, T] \times \mathbb{R}^{n}$,

$$
\begin{align*}
& W_{1}(t, x)=\sup _{u \in \mathcal{U}(t)} \inf _{v \in \mathcal{V}(t)} J^{1}(t, x, u, v)=\inf _{v \in \mathcal{V}(t)} \sup _{u \in \mathcal{U}(t)} J^{1}(t, x, u, v),  \tag{37}\\
& W_{2}(t, x)=\inf _{u \in \mathcal{U}(t)} \sup _{v \in \mathcal{V}(t)} J^{2}(t, x, u, v)=\sup _{v \in \mathcal{V}(t)} \inf _{u \in \mathcal{U}(t)} J^{2}(t, x, u, v) .
\end{align*}
$$

Finally we show that the Nash equilibria in [62] identified by Theorem 4.2 give raise to a NEP in our sense:

Theorem 4.4 Under the assumption (HH), $\left(J^{1}\left(t, x, u^{*}, v^{*}\right), J^{2}\left(t, x, u^{*}, v^{*}\right)\right)$ is a Nash equilibrium in the sense defined by 4.3.

### 4.3 Differential games with lack of information.

### 4.3.1 Description of the game.

This section resumes the papers $\langle 14\rangle$ and $\langle 16\rangle$ written in collaboration with P. Cardaliaguet. We are interested here in a stochastic differential zerosum-game with still the same dynamic (30):

$$
\left\{\begin{array}{l}
d X_{s}=f\left(X_{s}, u_{s}, v_{s}\right) d s+\sigma\left(X_{s}, u_{s}, v_{s}\right) d W_{s}, s \in[t, T], \\
X_{t}=x \in \mathbb{R}^{n} .
\end{array}\right.
$$

To simplify, we suppose here again that there is no integral cost in the payoff (31). However, the results of this section stay true even if the integral term is not equal to zero. This will be important for the last section.
The particularity here is to consider a hole family of applications ( $g_{i j}, i \in\{1, \ldots I\}, j \in\{1, \ldots, J\}$ ) which permit to define $I \times J$ different payoffs

$$
J_{i j}(t, x, u, v)=E\left[g_{i j}\left(X_{T}^{t, x, u, v}\right)\right],
$$

among which will be randomly chosen the payoffs the players have to optimize. To this aim, are given to probabilities $p \in \Delta(I)$ and $q \in \Delta(J)$ (where, for all $K \in \mathbb{N}^{*}, \Delta(K)$ denotes the set of probabilities on $\{1, \ldots, K\}$ ). The game is played in two steps:

1. The couple of $(i, j)$ is randomly chosen according to the probability $p \otimes q$. Then index $i$ is communicated to Player 1, index $j$ to Player 2.
2. Player 1 has to minimize the payoff $J_{i j}(t, x, u, v)$, Player 2 has to maximize it. More precisely, the players have to optimize the average

$$
\sum_{i, j} p_{i} q_{j} J_{i j}\left(t, x, u^{i}, v^{j}\right) .
$$

We will show that, under Isaacs' condition, the game has a value, and we characterize it as the unique solution in a dual sense of a Hamilton-Jacobi-Isaacs equation.

The subject of this paper is close to the idea of insider trading (see for example Amendinger-Becherer-Schweizer [3] or Corcuera-Imkeller-Kohatzu-Higa [38]), but our approach is very different. We take our inspiration in the repeated games with incomplete information. These were introduced by Aumann and Maschler in the sixties of last century (see [10] or [100]) and were treated by a large number of authors, for example De Meyer-Rosenberg [43], Mertens-Zamir [81], Rosenberg-Solan-Vieille [97] or Sorin [100]. In continuous time, the subject was treated for the first time only in 2006 by Cardaliaguet in [32]. This paper considered the determinstic case.

In opposition to classical games, the player have to use here random strategies. This can be explained by the fact that, if a player use immediately and plainely its knowledge, he reveals it also at the same moment to the opponent player and looses his advantage. It is obvious that the opposite strategy consisting by never revealing, hence by never using it, is also not optimal. Therefore the player will have to navigate between the two extrema. This is possible only by introducing randomness. We give following definition:

Definition 4.4 $A$ random strategy for Player 1 is a couple $\left(\left(\Omega_{\alpha}, \mathcal{F}_{\alpha}, \mathbf{P}_{\alpha}\right), \alpha\right)$, where $\left(\Omega_{\alpha}, \mathcal{F}_{\alpha}, \mathbf{P}_{\alpha}\right)$ is a probability space and $\alpha$ a random variable with values in the set of strategies $\mathcal{A}(t)$.

It is again possible to make play two random strategies together and to make sense to the following definition:
For $(\hat{\alpha}, \hat{\beta})=\left(\alpha_{i}, \beta_{j}\right)_{1 \leq i \leq I, 1 \leq j \leq J} \in \mathcal{A}_{r}(t)^{I} \times \mathcal{B}_{r}(t)^{J}$ and $(p, q) \in \Delta(I) \times \Delta(J)$, we set

$$
J^{p, q}(t, x, \hat{\alpha}, \hat{\beta})=E_{\alpha, \beta}\left[\sum_{i, j} p_{i} q_{j} J_{i j}\left(t, x, u_{i}, v_{j}\right)\right],
$$

where the expectation is taken with respect to the probability $P_{\alpha} \otimes P_{\beta}$ and where $\left(u_{i}, v_{j}\right)$ is the unique couple of random controls associated $\omega$-wise by Lemma 4.1 to ( $\alpha_{i}, \beta_{j}$ ).

### 4.3.2 The value of a game with incomplete information.

We define the upper- and lower value functions by:

$$
\begin{aligned}
& V^{+}(t, x, p, q):=\inf _{\hat{\alpha} \in \mathcal{A}_{r}(t)^{I}} \sup _{\hat{\beta} \in \mathcal{B}_{r}(t)^{J}} J^{p, q}(t, x, \hat{\alpha}, \hat{\beta}), \\
& V^{-}(t, x, p, q):=\sup _{\hat{\beta} \in \mathcal{B}_{r}(t)^{J}} \inf _{\hat{\alpha} \in \mathcal{A}_{r}(t)^{I}} J^{p, q}(t, x, \hat{\alpha}, \hat{\beta}) .
\end{aligned}
$$

In opposition to a game with perfect information, these value functions don't satisfy a dynamic programming principle (PPD). Indeed, each player collects informations on the knowledge of his
opponent during all the game. This implies that he would not be able to play in an optimal way if he would start from zero on at some arbitrary time, as if a PPD holds. Therefore it is not possible to characterize the value functions as classical viscosity solutions of a Hamilton-Jacobi equation.

An important property of $V^{+}$and $V^{-}$is:

Proposition 4.2 The value functions $V^{+}$and $V^{-}$are convex in $p$ and concave in $q$.

This motivates the introduction of Fenchel transformates:

$$
\begin{aligned}
V^{-*}(t, x, \hat{p}, q) & :=\sup _{p \in \Delta(I)}\left\{\langle\hat{p}, p\rangle-V^{-}(t, x, \hat{p}, q)\right\},(t, x, \hat{p}, q) \in[0, T] \times \mathbb{R}^{n} \times \mathbb{R}^{I} \times \Delta(J), \\
V^{+\sharp}(t, x, p, \hat{q}) & :=\inf _{q \in \Delta(J)}\left\{\langle\hat{q}, q\rangle-V^{+}(t, x, p, \hat{q})\right\},(t, x, p, \hat{q}) \in[0, T] \times \mathbb{R}^{n} \times \Delta(I) \times \mathbb{R}^{J} .
\end{aligned}
$$

The following theorem holds:
Theorem 4.5 Under Isaacs' condition (HI), for all $(t, x, p, q) \in[0, T] \times \mathbb{R}^{n} \times \Delta(I) \times \Delta(J)$, the game has a value $V(t, x, p, q):=V^{+}(t, x, p, q)=V^{-}(t, x, p, q)$. It is the unique viscosity solution in dual sense of

$$
\begin{equation*}
w_{t}+H\left(t, x, D w, D^{2} w, p, q\right)=0 \tag{38}
\end{equation*}
$$

with terminal condition

$$
V^{+}(T, x, p, q)=V^{-}(T, x, p, q)=\sum_{i=1}^{I} \sum_{j=1}^{J} p_{i} q_{j} g_{i j}(x) \quad \forall(x, p, q) \in \mathbb{R}^{n} \times \Delta(I) \times \Delta(J),
$$

where

$$
\begin{equation*}
H(s, y, \xi, A, p, q)=\inf _{u \in U} \sup _{v \in V}\left\{\langle b(t, x, u, v), p\rangle+\frac{1}{2} \operatorname{Tr}\left(A \sigma \sigma^{*}(t, x, u, v)\right)\right\} . \tag{39}
\end{equation*}
$$

and where $V$ is a dual solution of (38) if
(i) $V$ is convex with respect to $p$ and concave with respect to $q$,
(ii) $V^{-*}$ is a viscosity subsolution of the Hamilton-Jacobi-Isaacs equation

$$
w_{t}+\inf _{v \in V} \sup _{u \in U}\left\{\langle b(t, x, u, v), p\rangle+\frac{1}{2} \operatorname{Tr}\left(A \sigma \sigma^{*}(t, x, u, v)\right)\right\}=0
$$

(iii) $V^{+\sharp}$ is a supersolution of

$$
w_{t}+\sup _{u \in U} \inf _{v \in V}\left\{\langle b(t, x, u, v), p\rangle+\frac{1}{2} \operatorname{Tr}\left(A \sigma \sigma^{*}(t, x, u, v)\right)\right\}=0 .
$$

In Cardaliaguet [33], the author proposes an equivalent formulation of 4.5: he shows that a function is a solution in a dual sense of (38) if and only if it is solution in a direct sense of an obstacle problem:

Theorem 4.6 ([33]) Let $w: \mathbb{R}^{N} \times \Delta(I) \times \Delta(J) \rightarrow \mathbb{R}$ bounded, continuous application, that is convex in $p$ and concave in $q$.
$w$ is a subsolution (resp. supersolution) in a dual sense of (38) if and only if it is a viscosity sub(resp. super-) solution of the equation

$$
\begin{equation*}
\min \left\{\max \left\{w+H\left(x, D w, D^{2} w, p, q\right) ;-\lambda_{\min }\left(\frac{\partial^{2} w}{\partial p^{2}}\right)\right\} ;-\lambda_{\max }\left(\frac{\partial^{2} w}{\partial q^{2}}\right)\right\}=0 \tag{40}
\end{equation*}
$$

where $\lambda_{\min }(A)$ (resp. $\left.\lambda_{\max }(A)\right)$ is the smallest (resp. largest) eigenvalue of the matrix $A$.
(for precise definitions see [33]).
Corollary 4.1 Under (HI), $V:=V^{-}=V^{+}$is the unique viscosity solution of (40).

## Remarks:

1. This corollary follows from the theorems 4.5 and 4.6. It is the characterization of the value function that we will keep for the sequel, because its formulation is simplier and easier to handle than 4.5. Therefore it would be interesting to find a direct proof of it. This is an open question.
2. Theorem 4.5 and Corollary 4.1 stay true if the payments are more general:

$$
\begin{equation*}
J_{i j}(t, x, u, v)=E\left[\int_{t}^{T} l_{i j}\left(s, X_{s}^{t, x, u, v}, u_{s}, v_{s}\right) d s+g_{i j}\left(X_{T}^{t, x, u, v}\right)\right], \tag{41}
\end{equation*}
$$

provided that the Hamiltonian defined by (39) is replaced by

$$
\begin{aligned}
H(t, x, \xi, A, p, q)= & \inf _{u \in U} \sup _{v \in V}\{\langle b(t, x, u, v), p\rangle \\
& \left.+\frac{1}{2} \operatorname{Tr}\left(A \sigma \sigma^{*}(t, x, u, v)\right)-\sum_{i j} l_{i j}(t, x, u, v) p_{i} q_{j}\right\}
\end{aligned}
$$

and Isaacs' condition and the equations satisfied by the Fenchel transformates are adapted.

### 4.3.3 Game without dynamic: analysis of the strategy of the informed player.

## Introduction.

In $\langle 16\rangle$, we consider the case where only Player 2 has a lack of information, and we are interested in the strategy of Player 1, in particular in the way he reveals his information along the game. We will see that his strategy is strongly related to a martingale $\mathbf{p}$ living on the simplex $\Delta(I)$. This martingale is known in the framework of repeated games under the name of "martingale of posteriors" (see [100]). The crucial point here is the equivalence between the game and a certain optimization problem on the set of probabilities which optimum is precisely attained on the law of the martingale $\mathbf{p}$. This will permit us to construct explicitely the strategy of the informed player. Then we will analyse this martingale. Finally we shall see some examples.

We consider here a much simplified game, without any dynamic, where the payoffs are as follows:

$$
\int_{t_{0}}^{T} l_{i}\left(s, u_{s}, v_{s}\right) d s, i \in\{1, \ldots, I\}
$$

where $\left(u_{s}\right)$ and $\left(v_{s}\right)$ denote the controls used by Player 1 (resp.Player 2).
As in the previous section, we fix some probability $p \in \Delta(I)$, choose an index $i$ at random among the set $\{1, \ldots, I\}$ according to $p$ and communicate it to Player 1. Player 2 only knows the probability $p$ according to which the index was chosen.
Isaacs' condition is here as follows:

$$
\left(\mathrm{HI}^{\prime}\right) \text { For all }(t, p) \in[0, T] \times \Delta(I), H(t, p):=\inf _{u \in U} \sup _{v \in V} \sum_{i=1}^{I} p_{i} l_{i}(t, u, v)=\sup _{v \in V} \inf _{u \in U} \sum_{i=1}^{I} p_{i} l_{i}(t, u, v)
$$

The results of the last section apply here and lead to the following theorem:

Theorem 4.7 Under Isaacs assumption (HI'), the game has a value

$$
\begin{aligned}
V(t, p) & =\inf _{\left(\alpha_{i}\right) \in\left(\mathcal{A}_{r}(t)\right)^{I}} \sup _{\beta \in \mathcal{B}_{r}(t)} \sum_{i=1}^{I} p_{i} \mathbf{E}_{\alpha_{i} \beta}\left[\int_{t}^{T} \ell_{i}\left(s, \alpha_{i}(s), \beta(s)\right) d s\right] \\
& =\sup _{\beta \in \mathcal{B}_{r}\left(t_{0}\right)} \inf _{\left(\alpha_{i}\right) \in\left(\mathcal{A}_{r}(t)\right)^{I}} \sum_{i=1}^{I} p_{i} \mathbf{E}_{\alpha_{i} \beta}\left[\int_{t}^{T} \ell_{i}\left(s, \alpha_{i}(s), \beta(s)\right) d s\right]
\end{aligned}
$$

which is the unique viscosity solution of the Hamilton Jacobi equation

$$
\begin{equation*}
\min \left\{w_{t}+H(t, p) ; \lambda_{\min }\left(\frac{\partial^{2} w}{\partial p^{2}}\right)\right\}=0,(t, p) \in[0, T] \times \Delta(I) \tag{42}
\end{equation*}
$$

## An equivalent optimization problem.

We introduce the following notations: Let $\mathbf{D}(t)$ be the set of càdlàg functions from $\mathbb{R}$ to $\Delta(I)$ which are constant on $(-\infty, t)$ and $[T,+\infty)$, and $t \mapsto \mathbf{p}(t)$ the coordinate mapping on $\mathbf{D}(t)$. Then, for $p \in \Delta(I)$, we denote by $\mathbf{M}(t, p)$ the set of probabilities $\mathbf{P}$ on $\mathbf{D}(t)$, under $\mathbf{P}$, $\mathbf{p}$ is a martingale that satisfies

$$
\forall s<t, \mathbf{p}(s)=p, \text { and } \forall s \geq T, \mathbf{p}(s) \in\left\{e_{i}, i=1, \ldots, I\right\} \mathbf{P} \text {-a.s. }
$$

where $\left\{e_{1}, \ldots, e_{I}\right\}$ is the canonical basis of $\mathbb{R}^{I}$.
The principal result is:

## Theorem 4.8

$$
\begin{equation*}
\mathbf{V}(t, p)=\min _{\mathbf{P} \in \mathbf{M}(t, p)} \mathbf{E}_{\mathbf{P}}\left[\int_{t}^{T} H(s, \mathbf{p}(s)) d s\right] \quad \forall(t, p) \in[0, T] \times \Delta(I) \tag{43}
\end{equation*}
$$

Remark: In the introduction, we mensioned an optimization problem over a set of martingales. But for technical reasons, it is easier to think in terms of martingales measures on the canonical set of càdlàg paths rather than in terms of martingales. Therefore the process $\mathbf{p}$ is an optimal martingale under the optimal martingale measure.

We have two proofs of this result: a constructive one, by discretization, and one which is based on a dynamic programming principle.
Since the set $\mathbf{M}(t, p)$ is compact for the topology of Meyer-Zheng (see Meyer-Zheng [83]) and the Hamiltonian $H$ is continuous and bounded, we have easily an existence result:

Proposition 4.3 For all $(t, p)$, there exists at least one optimal martingale measure for the problem (43).

Let $\overline{\mathbf{P}}$ denote such an optimal measure.

## The optimal strategy.

As already mensioned, we will use Theorem 4.8 to construct an optimal strategy for the informed Player 1:
Following the definition of $\mathbf{M}(t, p)$, the sets $E_{i}=\left\{\mathbf{p}(T)=e_{i}\right\}$ form a partition of $\mathbf{D}(t)$. Therefore we get, for all index $i=1, \ldots, I$, a probability measure $\overline{\mathbf{P}}_{i}$ by setting

$$
\forall A \text { mesurable, } \overline{\mathbf{P}}_{i}(A)=\overline{\mathbf{P}}\left[A \mid E_{i}\right]
$$

Then we set

$$
\bar{u}(s)=u^{*}(s, \mathbf{p}(s)), s \in[t, T]
$$

where $u^{*}=u^{*}(s, p)$ is a measurable selection of $\operatorname{Argmin}_{u \in U}\left(\max _{v \in V} \sum_{i=1}^{I} p_{i} l_{i}(s, u, v)\right)$. Finally let $\bar{u}_{i}$ be the random control $\bar{u}$ under the probability $\overline{\mathbf{P}}_{i}$.

Theorem 4.9 The strategy which consist in playing the random control $\left(\bar{u}_{i}\right)_{i=1, \ldots, I}$ is optimal for $V(t, p)$ :

$$
\begin{equation*}
V(t, p)=\sup _{\beta \in \mathcal{B}_{r}(t)} \sum_{i=1}^{I} p_{i} \mathbf{E}_{\bar{u}_{i}}\left[\int_{t}^{T} l_{i}\left(s, \bar{u}_{i}(s), \beta\left(\bar{u}_{i}\right)(s)\right) d s\right] \tag{44}
\end{equation*}
$$

Remark: The optimal behavior of the noninformed player was analysed by Souquière [102].

## Analysis of the optimal martingale measure.

Now that we have shown that the measure $\overline{\mathbf{P}}$ is crucial for the elaboration of optimal strategies of the informed player, we analyse the behavior of the process $\mathbf{p}$ under $\overline{\mathbf{P}}$.
To simplify the presentation, we suppose here that the value function $V$ is of class $C^{1,2}$.
Therefore, the fact that $V$ is a solution of the obstacle problem

$$
\min \left\{w_{t}+H(t, p) ; \lambda_{\min }\left(\frac{\partial^{2} w}{\partial p^{2}}\right)\right\}=0,(t, p) \in[0, T] \times \Delta(I)
$$

means that, for all couple $(t, p)$, at least one of the following conditions is satisfied:

- $V_{t}=-H(t, p)$,
- $\lambda_{\min }\left(\frac{\partial^{2} V}{\partial p^{2}}\right)=0$.

This motivates us to have a closer look at the set

$$
\mathcal{H}:=\left\{(t, p) \in[0, T] \times \Delta(I) \mid V_{t}=-H(t, p)\right\}
$$

and its link to the optimal martingale measure.

Theorem 4.10 Let $\overline{\mathbf{P}}$ be an optimal martingale measure for (43).
Then, for all $s \in[t, T], \overline{\mathbf{P}}-p . s .$,

1. $(s, \mathbf{p}(s)) \in \mathcal{H}$,
2. $V(s, \mathbf{p}(s))-V\left(s, \mathbf{p}(s-)=\left\langle\frac{\partial V}{\partial p}(s, \mathbf{p}(s-)), \mathbf{p}(s)-\mathbf{p}(s-)\right\rangle\right.$.

Remark: Recall that $V$ is convex in $p$. Therefore, for all $s \in[0, T], \overline{\mathbf{P}}$-a.s.,

$$
V(s, \mathbf{p}(s))-V\left(s, \mathbf{p}(s-) \geq\left\langle\frac{\partial V}{\partial p}(s, \mathbf{p}(s-)), \mathbf{p}(s)-\mathbf{p}(s-)\right\rangle\right.
$$

Thus condition 2. implies that, on the segment $[\mathbf{p}(s-), \mathbf{p}(s)], \overline{\mathbf{P}}$-a.s., $V$ is affine. Therefore, if $\mathbf{p}$ jumps, it steps over a part of the domain of $[0, T] \times \Delta(I)$ where the second part of the obstacle problem is satisfied:

$$
\lambda_{\min }\left(\frac{\partial^{2} V}{\partial p^{2}}\right)=0
$$

The counterpart of Theorem 4.10 can be announced as follows:
Theorem 4.11 (Verification theorem)
For $(t, p) \in[0, T] \times \Delta(I)$, let $\mathbf{P} \in \mathbf{M}(t, p)$ such that for all $s \in[t, T] \mathbf{P}$-a.s.,

1. $(s, \mathbf{p}(s)) \in \mathcal{H}$,
2. $V(s, \mathbf{p}(s))-V\left(s, \mathbf{p}(s-)=\left\langle\frac{\partial V}{\partial p}, \mathbf{p}(s)-\mathbf{p}(s-)\right\rangle\right.$,
3. $\mathbf{P}$ is purely discontinuous.

Then $\mathbf{P}$ is optimal in problem (43).
Remark: The function $V$ is supposed here to be of class $C^{1,2}$. Therefore we can apply directly Itô's formula, and the verification theorem follows, under the condition that the following holds

$$
\mathbf{E}_{\overline{\mathbf{P}}}\left[\int_{t_{0}}^{T} \frac{\partial^{2} V}{\partial \hat{p}^{2}}\left(s, \mathbf{p}\left(s^{-}\right)\right) d\left\langle\mathbf{p}^{c}\right\rangle_{s}\right]=0
$$

where $\mathbf{p}^{c}$ denotes the continuous part of the martingale $\mathbf{p}$. Indeed, this term vanishes if $\mathbf{p}$ is purely discontinuous. But we ignore whether this relation stays true without condition 3.
In the next paragraph, we will give examples of situations where at least one optimal martingale measure is purely discontinuous, others where there exist a continuous optimal martingale measure.

## Examples:

1. The autonomious case:

$$
l_{i}=l_{i}(u, v), i=1, \ldots, I
$$

It is proved in [101] that

$$
V(t, p)=(T-t) \operatorname{Vex} H(p)
$$

This is also what Aumann-Maschler formula states for repeated games wih incomplete information on one side (see [10]). The optimal martingale is here as follows: starting on $(-\infty, p)$ from $p$, it jumps directly at time $t$ to some points $p^{1}, \ldots, p^{I}$, remains constant on $[t, T)$ and joins at time $T$ the extrema $e_{1}, \ldots, e_{I}$.
2. $I=2$.

The elements of $\Delta(2)$ are the couples $(p, 1-p)$, with $p \in[0,1]$. These probabilities are completely determined by the parameter $p \in[0,1]$, and the set $\mathcal{H}$ can be identified with some closed subset of $[0, T] \times[0,1]$.
We consider here the following assumption:
$\left(H_{12}\right)$ There exists $h_{1}, h_{2}:[0, T] \rightarrow[0,1]$ continuous, $h_{1} \leq h_{2}$, $h_{1}$ decreasing, $h_{2}$ increasing, such that

1. $\operatorname{Vex} H(t, p)=H(t, p) \Leftrightarrow p \in\left[0, h_{1}(t)\right] \cup\left[h_{2}(t), 1\right]$,
2. $\frac{\partial^{2} H}{\partial p^{2}}(t, p)>0 \quad \forall(t, p)$ with $p \in\left[0, h_{1}(t)\right) \cup\left(h_{2}(t), 1\right]$.

Proposition 4.4 Under assumption $\left(H_{12}\right)$,

$$
\begin{equation*}
V(t, p)=\int_{t}^{T} \operatorname{Vex} H(s, p) d s \quad \forall(t, p) \in[0, T] \times \Delta(I) \tag{45}
\end{equation*}
$$

and

$$
\mathcal{H}=\left\{(t, p) \in[0, T] \times[0,1] \mid p \in\left[0, h_{1}(t)\right] \cup\left[h_{2}(t), 1\right]\right\}
$$

In particular $V$ is of class $\mathcal{C}^{1,2}$.

Proposition 4.5 Under the assumption $\left(H_{12}\right)$, there is a unique optimal martingale measure $\overline{\mathbf{P}}$. Under this mesure, $\mathbf{p}$ is a purely discontinuous martingale and satisfies:

$$
\mathbf{p}\left(s^{-}\right)=p \forall s \in\left[t, t^{*}\right] \overline{\mathbf{P}} \text {-a.s., where } t^{*}=\inf \left\{s \geq t \mid p_{0} \in\left[h_{1}(s), h_{2}(s)\right]\right\}
$$

et

$$
\mathbf{p}(s) \in\left\{h_{1}(s), h_{2}(s)\right\} \quad \forall s \in\left[t^{*}, T\right) \overline{\mathbf{P}}-a . s .
$$

In particular,

$$
\begin{equation*}
\overline{\mathbf{P}}\left[\mathbf{p}(s)=h_{1}(s) \mid \mathbf{p}(r)=h_{1}(r)\right]=\frac{h_{2}(s)-h_{1}(r)}{h_{2}(s)-h_{1}(s)} \quad \forall t^{*} \leq r \leq s<T \tag{46}
\end{equation*}
$$

Remarks: 1. We don't know if, for $I=2$, the assumption $\left(H_{12}\right)$ is necessary to get the uniqueness of an optimal martingale measure, neither if there exist non purely discontinuous optimal martingale measures. Here is some sufficient condition for the existence of a purely discontinuous optimal martingale measure:

Theorem 4.12 Suppose that there is some non decreasing map $K:[0, T] \rightarrow[0,+\infty)$ such that

$$
\begin{equation*}
\forall s, t \in[0, T) \text { with } s \leq t, \forall p \in \mathcal{H}(s), \exists p^{\prime} \in \mathcal{H}(t) \text { with }\left|p^{\prime}-p\right| \leq K(t)-K(s) \tag{47}
\end{equation*}
$$

Then, for any initial position $(t, p)$, there exists a martingale measure $\mathbf{P} \in \mathbf{M}(t, p)$ under which the process $\mathbf{p}$ satisfies the conditions 1.-3. of Theorem 4.11.
2. The discret approximation of the optimal martingale in the first proof of 4.8 takes here the following form:
For all subdivision $\left\{t_{k}=t+\frac{k(T-t)}{n}, k=1, \ldots, n\right\}$, we set:

- $\pi_{0}^{n}=p$,
- for all $k=0, \ldots, n$, if $\pi_{k-1}^{n}$ is defined,
- we set $\pi_{k}^{n}=\pi_{k-1}^{n}$ on the set $\left\{\left(t_{k}, \pi_{k-1}^{n}\right) \in \mathcal{H}\right\}$,
- on $\left\{\left(t_{k}, \pi_{k-1}^{n}\right) \notin \mathcal{H}\right\} \pi_{k}^{n}$ takes its values in the set $\left\{h_{1}\left(t_{k}\right), h_{2}\left(t_{k}\right)\right\}$, and the conditional distribution of the jumps is uniquely determined by the fact that $\pi^{n}$ is a martingale.

Then we construct a martingale on continuous time, by setting $\mathbf{p}^{n}=p$ on $(-\infty, t)$ and $\mathbf{p}_{s}^{n}=\pi_{k+1}^{n}$ on $\left[t_{k}, t_{k+1}\right)$.
An interesting problem is to consider the same construction in a more abstract frame:
Let $\mathcal{H}$ a closed set in $[0, T] \times[0,1]$ which contains the set $[0, T] \times\{0,1\}$. In the same way as above, we can define a sequence of martingales $\left(\mathbf{p}^{n}\right)_{n \geq 1}$, replacing the couple $\left(h_{1}\left(t_{k}\right), h_{2}\left(t_{k}\right)\right)$ by the narrowest neighbors of $\pi_{k-1}^{n}$ in $\mathcal{H}\left(t_{k}\right)$. The sequence of distributions of these martingales contains at least one converging subsequence. Several questions can be raised:

- Is this limit always unique, as under the assumption $\left(H_{12}\right)$ ?
- Are the limit measures always purely discontinuous ?
but also
- Do the limit measures depend on the discretization?
- Are they Markovian ?
- It is possible to caracterize these martingale measures without speaking about the sequence from which they come from?

We close this section with an example that makes a link to some well known martingale.
Example 4.1 We set $T=\frac{1}{4}, h_{1}(s)=\frac{1}{2}-\sqrt{s}, h_{2}(s)=\frac{1}{2}+\sqrt{s}, s \in[0, T], t=0$ and $p=\frac{1}{2}$. The process $\mathbf{p}$, under the optimal martingale measure is, up to a constant, Azéma's martingale with parameter 2: under $\overline{\mathbf{P}},\left(X_{t}:=\mathbf{p}(t)-\frac{1}{2}, t \in[0, T]\right)$ satisfies the structure equation

$$
d[X]_{t}=d t-2 X_{t-} d X_{t}, t \in[0, T], \quad X_{0}=0
$$

(voir Emery [48])
3. In higher dimensions.

When the example for $I=2$ is extended to higher space dimensions, Proposition 4.4 remains true. Concerning the optimal measures, two interesting phenomena occur: as for $I=2$, there exists a purely discontinuous optimal martingale measure, but simultanuously, there are also optimal martingale measures under which $\mathbf{p}$ is continuous.
For arbitrary $I \geq 2$, we introduce the following assumption:
$\left(H_{K}\right)$ There exists a smoothly evolving and increasing family of open family convex subsets $(K(t))_{t \in[0, T]}$, whose closure is contained in the interior of in $\Delta(I)$ such that, for all $t \in[0, T]$,

1. $\operatorname{Vex} H(t, p)=H(t, p) \Leftrightarrow p \notin K(t)$,
2. $H(t, \cdot)$ is affine on $K(t)$,
3. $\frac{\partial^{2} H}{\partial p^{2}}(t, p)$ is definite positive for $p \notin \overline{K(t)}$.

We get:

Proposition 4.6 Under the asumption $\left(H_{K}\right)$,

$$
V(t, p)=\int_{t}^{T} \operatorname{VexH}(s, p) d s \quad \forall(t, p) \in[0, T] \times \Delta(I)
$$

and

$$
\mathcal{H}=\{(t, p) \in[0, T] \times \Delta(I) \mid p \notin K(t)\}
$$

In particular, $\mathbf{V}$ is of class $\mathcal{C}^{1,2}$.

Proposition 4.7 Under the assumption $\left(H_{K}\right)$, any optimal martingale measure $\overline{\mathbf{P}}$ has the following structure:

$$
\mathbf{p}\left(s^{-}\right)=p \quad \forall s \in\left[t, t^{*}\right] \quad \text { and } \quad \mathbf{p}(s) \in \partial K(s) \quad \forall s \geq t^{*}, \overline{\mathbf{P}} \text {-a.s. },
$$

ò̀ $t^{*}=\sup \{s \geq t \mid p \notin K(s)\}$. Moreover there exists an optimal martingale measure under which $\mathbf{p}$ is purely discontinuous.
If $(K(t))_{t \in[0, T]}$ has a positive minimal curvature and if $p \notin K(t)$, then there is also an optimal martingale measure under which $\mathbf{p}$ is continuous.

## Remarks:

1. If the family of sets $t \rightarrow \partial K(T-t)$ is moving according to its mean curvature, then there exists, for all $p \in \partial K(t)$, an optimal martingale measure under which $\mathbf{p}$ satisfies

$$
d \mathbf{p}(t)=\sqrt{2}(I-\nu(t, \mathbf{p}(t)) \otimes \nu(t, \mathbf{p}(t))) d W_{t},
$$

where $I$ is the identity matrix of size $(I-1),\left(W_{t}\right)$ a $(I-1)$-dimensional Brownian motion conditionned to live in the hyperplan generated by $\Delta(I)$ and $\nu(t, p)$ denotes the unit outward normal to $K(t)$ at $p \in \partial K(t)$ (see Buckdahn-Cardaliaguet-Quincampoix [26] , Soner-Touzi [98]).
2. A large number of questions remain open, in particular the ones we already have mensioned concerning the optimal martingale measures. Furthermore the above analysis was done in the very narrow framework of games without dynamic and with lack of information only at one side. The characterization of the value function through martingale measures for a game with dynamic and lack of information at both sides has still to be done.

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[^0]:    ${ }^{1}$ The notion of discontinuous viscosity solution we use here is the following: $\psi$ is a discontinuous viscosity solution if its upper semicontinuous envelope is a supersolution of (15), its lower semicontinuous envelope a subsolution (see [64]).

