Transient random walk in $\mathbb{Z}^2$ with stationary orientations

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Abstract. *In this paper, we extend a result of Campanino and Pétritis [5]. We study a random walk in $\mathbb{Z}^2$ with random orientations. We suppose that the orientation of the kth floor is given by ξ_k, where $(\xi_k)_{k\in\mathbb{Z}}$ is a stationary sequence of random variables. Once the environment fixed, the random walk can go either up or down or can stay in the present floor (but moving with respect to its orientation). This model was introduced by Campanino and Pétritis in [5] when the $(\xi_k)_{k\in\mathbb{Z}}$ is a sequence of independent identically distributed random variables. In [10], Guillotin-Plantard and Le Ny extend this result to a situation where the orientations of the floors are independent but chosen with stationary probabilities (not equal to 0 and to 1). In the present paper, we generalize the result of [5] to some cases when $(\xi_k)_k$ is stationary. Moreover we extend slightly a result of [10].*

1 Introduction

Random walks in random environment in $\mathbb{Z}^d$ have been studied by many authors. For a general reference on this subject, we refer to chapter 6 of the book of Hughes [14] . Random walks with random orientations have been less studied. However these two subjects are not far from each other. Indeed, random walks with random orientations can be viewed as a degenerate case of random walks in random environment in the sense that transition probabilities are allowed to be null. But this difference is significant. Moreover random walks in $\mathbb{Z}^2$ with random orientations can also be viewed as a question of oriented percolation (see section 12.8 of the Book of Grimmett [9]).

The present paper contains an extension of the model introduced by Campanino and Pétritis in [5] in another direction than the one chosen by Guillotin-Plantard and Le Ny in [10]. But our result will also apply to random walks of the form studied in [10]. Now, let us present the different models introduced in [5], in [10] and in the present paper with their common ideas and their differences. Let us construct a random walk $(M_n = (\tilde{X}_n, \tilde{Y}_n))_{n\geq 0}$ in $\mathbb{Z}^2$ with random orientations as follows. Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary sequence of centered random variables with values in $\{-1; 1\}$. The orientations of the kth horizontal floor of $\mathbb{Z}^2$ is given by ξ_k. Once the environment fixed, the random walk $(M_n = (\tilde{X}_n, \tilde{Y}_n))_n$ will be such that $M_0 = (0, 0)$ and such that the distribution of $M_{n+1} - M_n$ conditioned to $\sigma(M_k; k = 0, ..., n)$ is uniform on $\{(0, 1); (0, -1); (\xi_{\tilde{Y}_n}, 0)\}$.

In [5], Campanino and Pétritis prove the transience of the random walk $(M_n)_n$ when $(\xi_k)_{k\in\mathbb{Z}}$ is sequence of independent identically distributed random variables. Moreover, they point out the fact that the random walk $(M_n)_{n\geq 0}$ is recurrent in the 'alternate' case where ξ_k only depends on the parity of k. Hence the behaviour of this random walk depends on the randomness of the orientations $(\xi_k)_{k\in\mathbb{Z}}$.

In [10], Guillotin-Plantard and Le Ny give a first generalization of the work of Campanino and Pétritis. They envisage the case when the orientations of the floors are taken independently with stationary probabilities. More precisely, they consider the following situation : Let $(f_k)_{k\in\mathbb{Z}}$ be a stationary sequence of random variables with values in $[0; 1]$ and with expectation equal to $\frac{1}{2}$ defined on some probability space $(M, \mathcal{F}, \nu)$. Let us consider the probability space given by $(\Omega_1 := M \times [0; 1]^{\mathbb{Z}}, \mathcal{F}_1 := \mathcal{F} \otimes (\mathcal{B}([0; 1]))^{\otimes\mathbb{Z}}, \nu_1 :=$

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$\nu \otimes (\lambda)^{\otimes \mathbb{Z}})$, where λ is the Lebesgue measure on $[0;1]$. We define $(\tilde{\xi}_{k,f_k})_{k\in\mathbb{Z}}$ on this space as follows :

$$\tilde{\xi}_{k,f_k}(\omega,(z_m)_{m\in\mathbb{Z}}) := 2.\mathbf{1}_{\{z_k \le f_k(\omega)\}} - 1.$$

This means that, once a realization of $(f_k)_k$ given, the horizontal floors are oriented independently; the kth floor being oriented to the right with probability f_k. We will use this notation $\tilde{\xi}_{k,f_k}$ later in the paper. In [10], Guillotin-Plantard and Le Ny prove that, if $(\xi_k)_k = \left(\tilde{\xi}_{k,f_k}\right)_k$, then the corresponding random walk $(M_n)_n$ is transient under the following condition : $\int_M \frac{1}{\sqrt{f_0(1-f_0)}}\, d\nu < +\infty$ (this implies that $0 < f_0 < 1$ a.s.).

Let us notice that the $(\xi_k)_k$ studied in [10] is stationary. Conversely, if $(\xi_k)_k$ is stationary, then it can be described by the approach of [10] by taking $f_k := \mathbf{1}_{\{\xi_k=1\}} = \frac{1}{2}(\xi_k + \mathbf{1})$. But the method of [10] cannot be applied to a function f_0 that can be equal to 0 or 1 with a non-null probability.

In this paper, we are interested in the case when $(\xi_k)_{k\in\mathbb{Z}}$ is a stationary sequence of random variables satisfying some strong decorrelation properties. We state our main result in section 2 and prove it in section 3. Examples are given in section 2 and detailed in the appendix. Our examples satisfy a strong mixing condition. We complete this paper with a short discussion in section 4 about the model envisaged by Guillotin-Plantard and Le Ny. We prove that their result remains true if the condition $\int_M \frac{1}{\sqrt{f_0(1-f_0)}}\, d\nu < +\infty$ is replaced by $\int_M \frac{1}{[f_0(1-f_0)]^p}\, d\nu < +\infty$, for some $p > 0$.

2 Main result, examples, strong mixing property

Theorem 1. *Let $(\xi_k)_{k\in\mathbb{Z}}$ be a stationary sequence of centered random variables with values in $\{-1;1\}$ such that :*

1. *we have : $\sum_{p\ge 0}\sqrt{1+p}\,|\mathbb{E}[\xi_0\xi_p]| < +\infty$ and $c_0' := \sup_{N\ge 1} N^{-2}\sum_{k_1,k_2,k_3,k_4=0,\dots,N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]| < +\infty$.*
2. *There exist some $C > 0$, some $(\varphi_{p,s})_{p,s\in\mathbb{N}}$ and some integer $r \ge 1$ such that for all positive integers p and s, we have $\varphi_{p+1,s} \le \varphi_{p,s}$, such that we have $\lim_{s\to+\infty} s^6\varphi_{rs,s} = 0$ and such that, for all integers n_1, n_2, n_3, n_4 with $0 \le n_1 \le n_2 \le n_3 \le n_4$, for all real numbers $\alpha_{n_1},\dots,\alpha_{n_2}$ and $\beta_{n_3},\dots,\beta_{n_4}$, we have :*

$$\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\xi_k}, e^{i\sum_{k=n_3}^{n_4}\beta_k\xi_k}\right)\right| \le C\left(1+\sum_{k=n_1}^{n_2}|\alpha_k| + \sum_{k=n_3}^{n_4}|\beta_k|\right)\varphi_{n_3-n_2,n_4-n_3}.$$

Then the random walk $(M_n)_n$ is transient.

This result is proved in section 3. We will see in its proof that this question is linked with $\sum_{k=0}^{n-1}\xi_{S_k}$ where $(S_m)_{m\ge 0}$ is a simple symmetric random walk on $\mathbb{Z}$ independent of $(\xi_k)_{k\in\mathbb{Z}}$. Let us give some examples of stationary sequences $(\xi_k)_{k\in\mathbb{Z}}$ to which this result applies.

Theorem 2. *[(α-mixing condition)] Let $(g_k)_{k\in\mathbb{Z}}$ be a stationary sequence of bounded real-valued random variables defined on some probability space $(\Omega,\mathcal{A},\mathbb{P})$ satisfying the following α-mixing condition :*

$$\sup_{n\ge 1} n^6\alpha_n < +\infty, \quad \text{with} \quad \alpha_n := \sup_{p\ge 0;\ m\ge 0}\ \sup_{A\in\sigma(g_{-p},\dots,g_0)}\ \sup_{B\in\sigma(g_n,\dots,g_{n+m})} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

Then :

(a) If g_k takes its values in $\{-1;1\}$, if $\int_M g_k\, d\nu = 0$ and if $(\xi_k := g_k)_{k\in\mathbb{Z}}$, then $(M_n)_n$ is transient.

(b) If g_k takes its values in $[0;1]$, if $\int_M g_k\, d\nu = \frac{1}{2}$ and if $(\xi_k := \tilde{\xi}_{k,g_k})_{k\in\mathbb{Z}}$, then $(M_n)_n$ is transient.

We will prove that the hypotheses of theorem 1 are satisfied in the general context of strongly mixing dynamical systems. We say that $(M, \mathcal{F}, \nu, T)$ is an invertible dynamical system if $(M, \mathcal{F}, \nu)$ is a probability space endowed with an invertible bi-measurable transformation $T : M \to M$.

Definition 3. *We say that an invertible dynamical system $(M, \mathcal{F}, \nu, T)$ is* **strongly mixing** *if there exists $c_0 > 0$, there exist two real sequences $(\varphi_n)_{n\geq 0}$ and $(\kappa_m)_{m\geq 0}$ and, for any function $g : M \to \mathbb{C}$, there exist $K_g^{(1)} \in [0; +\infty]$ and $K_g^{(2)} \in [0; +\infty]$ such that, for all bounded functions $g, h : M \to \mathbb{C}$:*

1. *for all integer $n \geq 0$, we have : $|Cov_\nu(g, h \circ T^n)| \leq c_0 \left(\|g\|_\infty \|h\|_\infty + \|h\|_\infty K_g^{(1)} + \|g\|_\infty K_h^{(2)} \right) \varphi_n$;*
2. *for all integer $m \geq 0$, we have : $K_{g\circ T^{-m}}^{(1)} \leq c_0 K_g^{(1)}$ and : $K_{h\circ T^m}^{(2)} \leq c_0 K_h^{(2)}(1 + \kappa_m)$;*
3. *we have : $K_{g\times h}^{(1)} \leq \|g\|_\infty K_h^{(1)} + \|h\|_\infty K_g^{(1)}$ and : $K_{g\times h}^{(2)} \leq \|g\|_\infty K_h^{(2)} + \|h\|_\infty K_g^{(2)}$;*
4. *the sequence $(\varphi_n)_{n\geq 0}$ is decreasing, the sequence $(\kappa_m)_{m\geq 0}$ is increasing and there exists an integer $r \geq 1$ such that : $\sup_{n\geq 1} n^6(1 + \kappa_n)\varphi_{rn} < +\infty$.*

Proposition 4. *Let $(M, \mathcal{F}, \nu, T)$ be a* **strongly mixing** *dynamical system. Let the sequence (ξ_k) be of one the two following kinds :*

(a) $\xi_k = f \circ T^k$ with $f : M \to \{-1; 1\}$ a ν-centered function such that $K_f^{(1)} + K_f^{(2)} < +\infty$. We suppose that there exists some real number $c_1 > 0$ such that, for any real number α, we have : $K_{\exp(i\alpha f)}^{(1)} + K_{\exp(i\alpha f)}^{(2)} \leq c_1|\alpha|$.

(b) $\xi_k = \tilde{\xi}_{k,f\circ T^k}$ with $f : M \to [0; 1]$ such that $\int_M f\, d\nu = \frac{1}{2}$ and such that there exists some $c_1 > 0$ such that, for any $a, b \in \mathbb{C}$, we have $K_{af+b}^{(1)} + K_{af+b}^{(2)} \leq c_1|a|$.

Then $(\xi_k)_k$ satisfies the hypothesis of theorem 1.

Proposition 4 is proved in appendix A. Theorem 2 will appear as a direct consequence (see appendix B). Our strong mixing property is satisfied by a large class of dynamical systems (endowed with some metric) with $K_f^{(1)}$ and $K_f^{(2)}$ dominated by the Hölder constant of f of order η. Interesting examples are given by hyperbolic or quasi-hyperbolic dynamical systems. We quickly give some examples of such dynamical systems. In the case of the billiard transformation, because of the discontinuity of the transformation, our class of allowed functions will contain discontinuous functions.

Examples 2.1. *1. Let $(M, \mathcal{F}, \nu, T)$ where T is an ergodic algebraic automorphism of the torus or a diagonal transformation on a compact quotient of $Sl_{d_0}(\mathbb{R})$ by a discrete group. Let $\eta > 0$. According to [16], the strong mixing property holds with $K_g^{(1)}$ some η-Hölder constant of g along the unstable manifolds and with $K_h^{(2)}$ some η-Hölder constant of h along the stable-central manifolds and with $\varphi_n = \alpha^n$ for some $\alpha \in (0, 1)$ and $\kappa_m = m^\beta$ for some $\beta \geq 0$. Moreover $K_g^{(1)}$ and $K_g^{(2)}$ are dominated by the Hölder constant of order η of g.*

2. Let $(M, \mathcal{F}, \nu, T)$ where T is the Sinai billiard transformation (in $\mathbb{T}^2$) with C^3-convex scatterers and with finite horizon and where ν is the T invariant measure absolutely continuous with respect to the Lebesgue measure [19]. Let $m_0 \in \mathbb{Z}_+$ and $\eta > 0$. According to [6] (theorem 4.3), the strong mixing property holds with $\varphi_n = \alpha^n$ for some $\alpha \in (0, 1)$ and $\kappa_m = m^\beta$ for some $\beta \geq 0$, $K_g^{(1)}$ being some Hölder constant of g along the $T^{-m_0}(\gamma^u)$'s (where the γ^u's are the unstable curves) and $K_h^{(2)}$ being some Hölder constant of h along the $T^{m_0}(\gamma^s)$'s (where the γ^s's are the stable curves). The quantities $K_h^{(1)}$ and $K_h^{(2)}$ will be dominating by $C_h^{(\eta,m_0)} = \sup_{C\in\mathcal{C}_m} \sup_{x,y\in C,\ x\neq y} \frac{|h(x)-h(y)|}{\max(d(T^k(x),T^k(y));\ k=-m,\dots,m)^\eta}$, where $\mathcal{C}_m$ is a set of open subsets of M on which T^m and T^{-m} are C^1.

The first example is a direct consequence of [16]. The second example is a consequence of [6]. In appendix C, we give a precise definition of $K_f^{(1)}$ and of $K_f^{(2)}$ for these examples (and a definition of $\mathcal{C}_m$ for the Sinai billiard). For these systems, we can say a little more :

Theorem 5. *Let $\eta \in (0,1)$ and let $(M, \mathcal{F}, \nu, T)$ be a* **strongly mixing** *dynamical system (endowed with some metric) such that there exists $\alpha \in (0,1)$ and $\beta \geq 0$ such that $\varphi_n = \alpha^n$ and $\kappa_m = m^\beta$ and such that $K_h^{(1)}$ and $K_h^{(2)}$ are both dominated by the η-Hölder constant of h. Then :*

(A) If $(\xi_k := \tilde{\xi}_{k,g_0 \circ T^k})_{k\in\mathbb{Z}}$ with $g_0 : M \to [0;1]$ a Hölder continuous function (of order η) such that $\int_M g_0 \, d\nu = \frac{1}{2}$, then $(M_n)_n$ is transient.

(B) If $(\xi_k = 2\mathbf{1}_A \circ T^k - 1)_{k\in\mathbb{Z}}$ with $\nu(A) = 1/2$ and with A such that there exist $c_A > 0$ and $\zeta > 0$ such that, for every $\varepsilon \in]0;1[$, we have : $\nu(\{x \in M \ : \ d(x,A) < \varepsilon\}) \leq c_A \varepsilon^\zeta$, then $(M_n)_n$ is transient.

Conclusion *(A)* of theorem 5 follows directly from proposition 4. Conclusion *(B)* of theorem 5 is proved in appendix D.

3 Proof of theorem 1

Let us define $T_0 := 0$ and, for all $n \geq 1$: $T_{n+1} := \inf\{k > T_n \ : \ \tilde{Y}_k \neq \tilde{Y}_{k-1}\}$. According to lemma 2.5 of [5], we have the following result :

Lemma 6. *If $(M_{T_n})_{n\geq 0}$ is transient, then $(M_n)_{n\geq 0}$ is transient*

Now, still following [5], we construct a realization of $(M_{T_n})_n$. Let us consider a symmetric random walk $(S_n)_n$ on $\mathbb{Z}$ independent of $(\xi_k)_{k\in\mathbb{Z}}$. For any integer $m \geq 1$ and any integer k, we define :

$$N_m(k) := Card\{j = 0, ..., m \ : \ S_j = k\}.$$

Let us also consider a sequence of independent random variables $(\zeta_i^{(y)})_{i\geq 1, y\in\mathbb{Z}}$ with geometric distribution with parameter $\frac{1}{3}$, and independent of $((\xi_y)_{y\in\mathbb{Z}}, (S_p)_{p\geq 1})$.

Lemma 7. *The process $(X_n, S_n)_{n\geq 1}$ with $X_n := \sum_{y\in\mathbb{Z}} \xi_y \sum_{i=1}^{N_{n-1}(y)} \zeta_i^{(y)}$ has the same distribution as $(M_{T_n})_{n\geq 1}$.*

In this lemma, $\zeta_i^{(y)}$ corresponds to the duration of the stay at the yth horizontal floor during the ith visit to this floor. According to the Borel-Cantelli lemma, it suffices to prove that : $\sum_{n\geq 1} \mathbb{P}(\{(X_n, S_n) = (0,0)\}) < +\infty$. We follow the scheme of the proof of [5]. The difference will be in our way of estimating $I_n^{(1)}$ and in the introduction of the sets U_n. We will consider δ_1, δ_2, δ_3, and γ such that : $0 < \delta_1 < 2\delta_2$, $\delta_1 + (\frac{27}{2} + 16)\delta_2 < \frac{1}{8}$, $\delta_3 > 0$, $\frac{1}{4} - 3\delta_2 < \delta_3 < \frac{1}{4} - \frac{5}{2}\delta_2 - \delta_1$, $\frac{\delta_3}{2} - 2\delta_2 < \beta < \frac{\delta_3}{2} - \delta_2$, $\max(\delta_1, \delta_2) < \gamma < \frac{1}{2} - 22\max(\delta_1, \delta_2)$. The idea is that δ_1, δ_2, $\frac{1}{4} - \delta_3$ and $\frac{1}{8} - \beta$ are positive numbers very close to zero. As in [5, 10], let us define : $A_n := \{\omega \in \Omega \ : \ \max_{\ell\in\mathbb{Z}} N_{n-1}(\ell) \leq n^{\frac{1}{2}+\delta_2}$ and $\max_{k=0,...,n} |S_k| < n^{\frac{1}{2}+\delta_1}\}$. Moreover, we define : $U_n := \{\omega \in A_n \ : \ \forall x, y \in \mathbb{Z}, \ |N_{n-1}(x) - N_{n-1}(y)| \leq \sqrt{|x-y| n^{\frac{1}{2}+\gamma}}\}$. The sketch of the proof is the following :

1. As in proposition 4.1 of [5], we have : $\sum_{n\geq 1} \mathbb{P}(\{X_n = 0 \text{ and } S_n = 0\} \setminus A_n) < +\infty$. Actually we have : $\sum_{n\geq 1} \mathbb{P}(\{S_n = 0\} \setminus A_n) < +\infty$.
2. We will see in lemma 8 of the present paper that we have : $\sum_{n\geq 1} \mathbb{P}(A_n \setminus U_n) < +\infty$. Therefore, we have : $\sum_{n\geq 0} \mathbb{P}(\{X_n = 0 \text{ and } S_n = 0\} \setminus U_n) < +\infty$;
3. Let us define $B_n := \{\omega \in U_n \ : \ \left|\sum_{y\in\mathbb{Z}} \xi_y N_{n-1}(y)\right| > n^{\frac{1}{2}+\delta_3}\}$. As in proposition 4.3 of [5], we have : $\sum_{n\geq 0} \mathbb{P}(B_n \cap \{X_n = 0 \text{ and } S_n = 0\}) < +\infty$. It remains to prove that : $\sum_{n\geq 0} \mathbb{P}(U_n \cap \{X_n = 0 \text{ and } S_n = 0\} \setminus B_n) < +\infty$.

(a) As in lemma 4.5 of [5], there exists a real number $C > 0$ such that :

$$\sup_{\omega \in U_n \setminus B_n} \mathbb{P}\left(\{X_n = 0\} | (S_p)_{p\geq 1}, (\xi_k)_{k\in\mathbb{Z}}\right) \leq C\sqrt{\frac{\ln(n)}{n}}.$$

(b) We will prove that there exists some $\tilde{\delta} > 0$ and some $C' > 0$ such that :

$$\forall \omega \in U_n, \quad \mathbb{P}\left(U_n \setminus B_n | (S_p)_p\right)(\omega) \leq C' n^{-\tilde{\delta}}.$$

i. This probability is bounded by $c' n^{\frac{1}{2}+\delta_3} I_n(\omega)$ with $I_n(\omega) = I_n^{(1)}(\omega) + I_n^{(2)}(\omega)$ and

$$I_n^{(1)}(\omega) := \int_{\{|t| \leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}} \xi_y N_{n-1}(y)(\omega)} \Big| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt$$

and

$$I_n^{(2)}(\omega) := \int_{\{|t| > n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}} N_{n-1}(y)(\omega)} \Big| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt.$$

ii. We will prove that $n^{\frac{1}{2}+\delta_3} \sup_{U_n} I_n^{(1)} = O(n^{-\delta})$ for some $\delta > 0$ (see our lemma 9);

iii. On the other hand, following [5], we have :

$$n^{\frac{1}{2}+\delta_3} I_n^{(2)} \leq \int_{\{|s| > n^{\delta_2}\}} e^{-\frac{s^2}{2}}\, ds \leq 2n^{-\delta_2} e^{-\frac{n^{2\delta_2}}{2}}$$

(c) We have $\mathbb{P}(S_n = 0) \leq C'' n^{-\frac{1}{2}}$.

(d) Hence we have : $\mathbb{P}\left(U_n \cap \{X_n = 0 \text{ and } S_n = 0\} \setminus B_n\right) \leq C''' n^{-1-\tilde{\delta}} \sqrt{\ln(n)}$.

We have to prove that points 2 and 3(b)(ii) are true with our choices of parameters. Indeed, all the other points are true for any positive $\delta_1, \delta_2, \delta_3$ and for any sequence of random variables $(\xi_k)_{k\in\mathbb{Z}}$ independent of $(S_p)_p$. We notice that, for any integer $n \geq 1$, we have : $\sum_{j=0}^{n-1} \xi_{S_j} = \sum_{k\in\mathbb{Z}} \xi_k N_{n-1}(k)$. In our proof, we need some real numbers δ_1, δ_2, δ_3, δ_4, β, γ and $\varepsilon > 0$. We will suppose that :

$\delta_1 > 0$, $\delta_2 > 0$, $\delta_1 + (\frac{27}{2} + 16)\delta_2 < \frac{1}{8}$, $\delta_3 > 0$, $\delta_1 < \delta_4 < \frac{1}{4} - \delta_3 - \frac{5}{2}\delta_2$, $\frac{1}{4} - 3\delta_2 < \delta_3 < \frac{1}{4} - \frac{5}{2}\delta_2$, $\frac{5}{3}\delta_2 < \frac{1}{2}\delta_3$, $\frac{\delta_3}{2} - 2\delta_2 < \beta < \frac{\delta_3}{2} - \delta_2$, $\frac{5}{2}\delta_3 > \frac{1}{2} + 6\delta_2 + \delta_1$, $\max(\delta_1, \delta_2) < \gamma < \frac{1}{2} - 22\max(\delta_1, \delta_2)$ and :

$$n^{\delta_1 + 11\delta_2} \sum_{m \geq \frac{(r+1)n^\beta}{2}} |\mathbb{E}[\xi_0 \xi_m]| = O(n^{-\varepsilon}).$$

(we have : $\sum_{m\geq N} |\mathbb{E}[\xi_0\xi_m]| \leq N^{-\frac{1}{2}} \sum_{m\geq N} \sqrt{m} |\mathbb{E}[\xi_0\xi_m]|$). All these inequalities are true with the following choices of parameters :

$$\delta_1 = \frac{1}{3000},\ \delta_2 = \frac{1}{500},\ \delta_3 = \frac{1}{4} - \frac{11}{4}\delta_2 = 489/2000,\ \delta_4 = 1/2500,\ \beta = \frac{\delta_3}{2} - \frac{3}{2}\delta_2 = 477/4000,\ \gamma = \frac{1}{4}.$$

Lemma 8. *We have :* $\sum_{n\geq 1} \mathbb{P}\left(A_n \setminus U_n\right) < +\infty$.

Proof. Let us consider any $x, y \in \mathbb{Z}$ with $x \neq y$ and $|x - y| \leq 3n^{\frac{1}{2}+\delta_1}$. For any integer $j \geq 1$, we define the time $\tau_j(x)$ of the jth visit of $(S_p)_p$ to x and the number $\mathcal{N}_j(x, y)$ of visits of $(S_p)_p$ to y between the times $\tau_j(x)$ and $\tau_{j+1}(x)$. According to [15, 20] (see [15] lemma 2), for any integer $p \geq 1$, there exists $K_p > 0$ such that, for any $x' \neq y,'$ we have : $\mathbb{E}[(\mathcal{N}_j(x', y'))^p] \leq K_p |x' - y'|^{p-1}$. According to [15], on the set $\{\tau_1(x) \leq \tau_1(y)\}$, we have :

$$(N_{n-1}(x) - N_{n-1}(y)) = \sum_{j=1}^{N_{n-1}(x)} (1 - \mathcal{N}_j(x, y)) + \sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)} \mathbf{1}_{\{S_k = y\}}.$$

Let p be any positive integer. We have :

$$(N_{n-1}(x)-N_{n-1}(y))^{2p}\mathbf{1}_{\{\tau_1(x)\leq\tau_1(y)\}}\leq 2^{2p}\left[\left(\sum_{j=1}^{N_{n-1}(x)}(1-\mathcal{N}_j(x,y))\right)^{2p}+\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)}\mathbf{1}_{\{S_k=y\}}\right)^{2p}\right].$$

But, on A_n, since we have $N_{n-1}(x)\leq n^{\frac{1}{2}+\delta_2}$, we get :

$$\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)}\mathbf{1}_{\{S_k=y\}}\right)^{2p}\leq\left(\mathcal{N}_{N_{n-1}(x)}(x,y)\right)^{2p}\leq\sum_{j=1}^{\left\lfloor n^{\frac{1}{2}+\delta_2}\right\rfloor}(\mathcal{N}_j(x,y))^{2p}.$$

Hence we have:

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)}\mathbf{1}_{\{S_k=y\}}\right)^{2p}\mathbf{1}_{A_n}\right] &\leq n^{\frac{1}{2}+\delta_2}K_{2p}|x-y|^{2p-1}\\ &\leq K_{2p}3^{p-1}|x-y|^p\left(n^{\frac{1}{2}+\max(\delta_1,\delta_2)}\right)^p.\end{aligned}$$

Moreover, on A_n, we have :

$$\left(\sum_{j=1}^{N_{n-1}(x)}(1-\mathcal{N}_j(x,y))\right)^{2p}\leq\max_{k=1,\dots,\left\lfloor n^{\frac{1}{2}+\delta_2}\right\rfloor}\left(\sum_{j=1}^{k}(1-\mathcal{N}_j(x,y))\right)^{2p}.$$

Since $\left(\sum_{j=1}^{k}(1-\mathcal{N}_j(x,y))\right)_{k\geq1}$ is a martingale (see [15] lemma 2), according to a maximal inequality, we have :

$$\left\|\max_{k=1,\dots,\left\lfloor n^{\frac{1}{2}+\delta_2}\right\rfloor}\left(\sum_{j=1}^{k}(1-\mathcal{N}_j(x,y))\right)^2\right\|_{L^p}\leq\frac{p}{p-1}\max_{k=1,\dots,\left\lfloor n^{\frac{1}{2}+\delta_2}\right\rfloor}\left\|\left(\sum_{j=1}^{k}(1-\mathcal{N}_j(x,y))\right)^2\right\|_{L^p}.$$

Hence we have :

$$\mathbb{E}\left[\left(\sum_{j=1}^{N_{n-1}(x)}(1-\mathcal{N}_j(x,y))\right)^{2p}\mathbf{1}_{A_n}\right]\leq\left(\frac{p}{p-1}\right)^p\max_{k=1,\dots,\left\lfloor n^{\frac{1}{2}+\delta_2}\right\rfloor}\mathbb{E}\left[\left(\sum_{j=1}^{k}(1-\mathcal{N}_j(x,y))\right)^{2p}\right].$$

Let us write $\mathcal{M}^{2p}_{\nu_1,\dots,\nu_l}=\frac{(2p)!}{\prod_{i=1}^{l}\nu_i!}$. For any $k=1,\dots,\left\lfloor n^{\frac{1}{2}+\delta_2}\right\rfloor$, since the $\mathcal{N}_{j_m}$'s are independent and since $\mathbb{E}\left[1-\mathcal{N}_j(x,y)\right]=0$, we have :

$$\mathbb{E}\left[\left(\sum_{j=1}^{k}(1-\mathcal{N}_j(x,y))\right)^{2p}\right]=\sum_{l=1}^{2p}\sum_{\nu_1+\dots+\nu_l=2p;\ \min_i\nu_i\geq1}\mathcal{M}^{2p}_{\nu_1,\dots,\nu_l}\sum_{j_1<\dots<j_l}\prod_{m=1}^{l}\mathbb{E}\left[(1-\mathcal{N}_{j_m}(x,y))^{\nu_m}\right],$$

$$\begin{aligned}&\leq\sum_{l=1}^{2p}\sum_{\nu_1+\dots+\nu_l=2p;\ \min_i\nu_i\geq2}\mathcal{M}^{2p}_{\nu_1,\dots,\nu_l}\sum_{1\leq j_1<\dots<j_l\leq k}\prod_{m=1}^{l}\left(2^{\nu_m}\mathbb{E}\left[1+(\mathcal{N}_{j_m}(x,y))^{\nu_m}\right]\right)\\ &\leq\sum_{l=1}^{2p}\sum_{\nu_1+\dots+\nu_l=2p;\ \min_i\nu_i\geq2}\mathcal{M}^{2p}_{\nu_1,\dots,\nu_l}\sum_{1\leq j_1<\dots<j_l\leq k}\prod_{m=1}^{l}\left(2^{\nu_m}(1+K_{\nu_m}|x-y|^{\nu_m-1}\right)\\ &\leq\tilde{C}_p\sum_{l=1}^{2p}|x-y|^{2p-l}(n^{\frac{1}{2}+\delta_2})^l\leq2p3^p\tilde{C}_p|x-y|^p(n^{\frac{1}{2}+\max(\delta_1,\delta_2)})^p.\end{aligned}$$

Hence we get : $\mathbb{E}\left[(N_{n-1}(x)-N_{n-1}(y))^{2p}\mathbf{1}_{A_n}\right] \leq \tilde{C}'_p|x-y|^p(n^{\frac{1}{2}+\max(\delta_1,\delta_2)})^p$. Therefore, according to the Markov inequality, for any integer $p \geq 1$, we have :

$$\mathbb{P}\left(A_n \setminus U_n\right) \leq \sum_{x,y=-\left\lceil n^{\frac{1}{2}+\delta_1}\right\rceil}^{\left\lceil n^{\frac{1}{2}+\delta_1}\right\rceil} \mathbb{P}\left(A_n \cap \left\{|N_{n-1}(x)-N_{n-1}(y)| > \sqrt{|x-y|n^{\frac{1}{2}+\gamma}}\right\}\right)$$

$$\leq \sum_{x,y=-\left\lceil n^{\frac{1}{2}+\delta_1}\right\rceil}^{\left\lceil n^{\frac{1}{2}+\delta_1}\right\rceil} \frac{\mathbb{E}[(N_{n-1}(x)-N_{n-1}(y))^{2p}\mathbf{1}_{A_n}]}{|x-y|^p(n^{\frac{1}{2}+\gamma})^p} \leq c_p\left(5n^{\frac{1}{2}+\delta_1}\right)^2\left(n^{\max(\delta_1,\delta_2)-\gamma}\right)^p.$$

By taking p large enough, we get : $\sum_{n\geq 1}\mathbb{P}(A_n \setminus U_n) < +\infty$. □

3.1 Estimates on U_n

In this section, we suppose that we are in U_n. We will estimate :

$$I_n^{(1)}(\omega) := \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \left(\mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}}\xi_y N_{n-1}(y)}\middle|(S_p)_p\right](\omega)\right) e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\,dt.$$

Lemma 9. *There exists a real number $\delta > 0$ such that :* $\sup_{n\geq 1} n^{\delta}\sup_{\omega\in U_n} n^{\frac{1}{2}+\delta_3} I_n^{(1)}(\omega) < +\infty$.

To prove this lemma, we will use the following formula :

$$n^{\frac{1}{2}+\delta_3} I_n^{(1)}(\omega) = n^{\delta_2}\int_{\{|u|\leq 1\}} \left(\mathbb{E}\left[e^{iun^{-\frac{1}{2}-\delta_3+\delta_2}\sum_{y\in\mathbb{Z}}\xi_y N_{n-1}(y)}\middle|(S_p)_p\right](\omega)\right) e^{-\frac{u^2 n^{2\delta_2}}{2}}\,du.$$

The main idea is to prove that, in this formula, we can replace the term :

$$B_n(u)(\omega) := \mathbb{E}\left[e^{iun^{-\frac{1}{2}-\delta_3+\delta_2}\sum_{y\in\mathbb{Z}}\xi_y N_{n-1}(y)}\middle|(S_p)_p\right](\omega)$$

by the term : $A_n(u)(\omega) := e^{-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_{y,z}\mathbb{E}[\xi_y\xi_z](N_{n-1}(y)(\omega))^2}$. More precisely let us prove that we have :

Lemma 10. *There exists a real number $\delta_0 > 0$ such that we have :*

$$\sup_{n\geq 1} n^{\delta_0}\sup_{\omega\in U_n} n^{\delta_2}\int_{|u|\leq 1}|B_n(u)(\omega)-A_n(u)(\omega)|\,e^{-\frac{u^2 n^{2\delta_2}}{2}}\,du < +\infty. \tag{1}$$

After proving 10, we will prove that lemma 9 is a consequence of it. We will use the following notation : $\sigma_\xi^2 := \sum_{m\in\mathbb{Z}}\mathbb{E}[\xi_0\xi_m]$.

3.1.1 Proof of lemma 10

Our proof uses a method introduced by Jan (cf. [12], [13]). This method also gives a result of convergence in distribution for $\left(n^{-3/4}\sum_{k=0}^{n-1}\xi_{S_n}\right)_{n\geq 1}$ (see [17, 18]). Let n be an integer such that $n^\beta \geq 2$. Let us fix $\omega \in U_n$ and $u \in [-1;1]$. Let us recall that $0 < \beta < \frac{\delta_3}{2}-\delta_2$ et let us define : $L_n := \left\lfloor \frac{2\left\lfloor n^{\frac{1}{2}+\delta_1}\right\rfloor+1}{\lfloor n^\beta\rfloor}\right\rfloor$ (we have : $L_n \leq 4n^{\frac{1}{2}+\delta_1-\beta}$) and, for all integer $k = 0, ..., L_n$: $\alpha_{(k)} := -\left\lfloor n^{\frac{1}{2}+\delta_1}\right\rfloor + k\lfloor n^\beta\rfloor$ and $\alpha_{(L_n+1)} := \left\lfloor n^{\frac{1}{2}+\delta_1}\right\rfloor+1$; $b_k := e^{iun^{-\frac{1}{2}-\delta_3+\delta_2}\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\xi_y N_{n-1}(y)}$ and $a_k := e^{-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\sigma_\xi^2(N_{n-1}(y))^2}$. We have to estimate : $n^{\delta_2}\left|\mathbb{E}\left[\prod_{k=0}^{L_n} b_k\middle|(S_p)_p\right](\omega) - \prod_{k=0}^{L_n} a_k(\omega)\right|$. Hence it is enough to estimate :

$$n^{\delta_2}\sum_{k=0}^{L_n}\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-1} b_m\right)(b_k-a_k)\left(\prod_{m'=k+1}^{L_n} a_{m'}\right)\middle|(S_p)_p\right](\omega)\right|.$$

- We explain how we can restrict our study to the sum over the k such that $(r+1)^4 \leq k \leq L_n - 1$. Let $k \in \{0, ..., L_n\}$. We have :

$$\mathbb{E}\left[\left(\sum_{\ell=\alpha+1}^{\alpha+\theta} \xi_\ell N_{n-1}(\ell)\right)^2 |(S_p)_p\right](\omega) \leq \sum_{\ell=\alpha+1}^{\alpha+\theta}\sum_{m=\alpha+1}^{\alpha+\theta} |\mathbb{E}[\xi_\ell \xi_m]|\ N_{n-1}(\ell)(\omega)N_{n-1}(m)(\omega)$$
$$\leq \theta \sum_{m\in\mathbb{Z}} |\mathbb{E}[\xi_0\xi_m]| n^{1+2\delta_2}.$$

Hence we have :
$\mathbb{E}\left[|b_k - 1||(S_p)_p\right](\omega) \leq n^{-\frac{1}{2}-\delta_3+\delta_2}\left(\mathbb{E}\left[\left|\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \xi_y N_{n-1}(y)\right| |(S_p)_p\right](\omega)\right)$

$$\leq n^{-\frac{1}{2}-\delta_3+\delta_2} n^{\frac{\beta}{2}} \sqrt{\sum_{m\in\mathbb{Z}} |\mathbb{E}[\xi_0\xi_m]| n^{\frac{1}{2}+\delta_2}} \leq n^{-\frac{3}{4}\delta_3+\frac{3}{2}\delta_2}\sqrt{\sum_{m\in\mathbb{Z}} |\mathbb{E}[\xi_0\xi_m]|},$$

since we have $\beta < \frac{\delta_3}{2} - \delta_2$. Moreover we have :

$$|a_k(\omega) - 1| \leq \frac{\sigma_\xi^2 \sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1} (N_{n-1}(y)(\omega))^2}{2n^{1+2\delta_3-2\delta_2}} \leq \frac{\sigma_\xi^2 n^{1+2\delta_2}}{2n^{1+2\delta_3-2\delta_2}} n^\beta \sigma_\xi^2 n^{1+2\delta_2} \leq \frac{n^{-\frac{3}{2}\delta_3+3\delta_2}\sigma_\xi^2}{2}.$$

From which, we get :

$$n^{\delta_2} \sum_{k=0}^{(r+1)^4-1} \mathbb{E}\left[|b_k - a_k||(S_p)_p\right](\omega) + \mathbb{E}\left[|b_{L_n} - a_{L_n}||(S_p)_p\right](\omega) \leq c_0\left(n^{-\frac{3}{4}\delta_3+\frac{5}{2}\delta_2} + n^{-\frac{3}{2}\delta_3+4\delta_2}\right), \quad (2)$$

with $c_0 := ((r+1)^4+1)\sqrt{\sum_{m\in\mathbb{Z}} |\mathbb{E}[\xi_0\xi_m]|} + \frac{1}{2}\sigma_\xi^2$. Let us recall that $\frac{5}{3}\delta_2 < \frac{1}{2}\delta_3$.
Hence, it remains to estimate :

$$n^{\delta_2} \sum_{k=(r+1)^4}^{L_n-1} \left|\mathbb{E}\left[\left(\prod_{m=0}^{k-1} b_m\right)(b_k - a_k) \prod_{m'=k+1}^{L_n} a_{m'} |(S_p)_p\right]\right|. \quad (3)$$

- Let us introduce some holes in the indices m in order to use our decorrelation hypothesis. Let us control the following quantity :

$$\tilde{B}_n := n^{\delta_2} \sum_{k=(r+1)^4}^{L_n-1} \left|\mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^4} b_m\right) \prod_{j=1}^{3}\left(\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^j} b_m\right) - 1\right) \prod_{m''=k-r}^{k-1} b_{m''}(b_k - a_k) \prod_{m'=k+1}^{L_n} a_{m'} \middle| (S_p)_p\right]\right.$$

We have :

$$\tilde{B}_n(\omega) \leq n^{\delta_2} \sum_{k=(r+1)^4}^{L_n-1} \prod_{j=1}^{3} \left\|\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^j} b_m\right) - 1\right\|_{L^\infty(U_n)} \|b_k - a_k\|_{L^\infty(U_n)}.$$

On U_n, we have : $|b_k - 1| \leq n^{-\frac{1}{2}-\delta_3+\delta_2} n^\beta n^{\frac{1}{2}+\delta_2} \leq n^{-\delta_3+2\delta_2+\beta}$.
Analogously, we get : $\left|\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^j} b_m\right) - 1\right| \leq r(r+1)^j n^{-\delta_3+2\delta_2+\beta}$. On the other hand, we have : $|a_k - 1| \leq \frac{1}{2} n^{-2\delta_3+4\delta_2+\beta}\sigma_\xi^2$. Therefore, since we have $\beta < \frac{\delta_3}{2} - \delta_2$, we get :

$$\tilde{B}_n \leq 4n^{\delta_2} n^{\frac{1}{2}+\delta_1-\beta} r^3 (r+1)^6 \left(1 + \frac{1}{2}\sigma_\xi^2\right)\left(n^{-\delta_3+2\delta_2+\beta}\right)^4 = O\left(n^{\frac{1}{2}-\frac{5}{2}\delta_3+6\delta_2+\delta_1}\right).$$

The control of the quantity $\tilde{B}_n$ comes from the fact that $\frac{5}{2}\delta_3 > \frac{1}{2} + 6\delta_2 + \delta_1$.

It remains to estimate : $n^{\delta_2}\sum_{k=(r+1)^4+1}^{L_n-1}\sum_{1\leq j_0<j_1\leq j_2\leq 4}C_{n,k,j_0,j_1,j_2}$, where C_{n,k,j_0,j_1,j_2} is the following quantity :

$$\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^4}b_m\right)\left(\prod_{m=k-(r+1)^{j_2}+1}^{k-(r+1)^{j_1}}b_m\right)\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1}b_m\right)(b_k-a_k)\prod_{m'=k+1}^{L_n}a_{m'}\middle|(S_p)_p\right]\right|,$$

with the convention : $\prod_{m=\alpha}^{\beta}b_m=1$ if $\beta<\alpha$. Let j_0,j_1,j_2 be fixed. We have : $C_{n,k,j_0,j_1,j_2}\leq D_{n,k,j_0,j_1,j_2}+E_{n,k,j_0,j_1,j_2}$, with :

$$D_{n,k,j_0,j_1,j_2}:=\left|Cov_{|(S_p)_p}\left(\Delta_{n,k,j_1,j_2},\Gamma_{n,k,j_0}\right)\prod_{m'=k+1}^{L_n}a_{m'}\right|$$

$$\text{and}\quad E_{n,k,j_0,j_1,j_2}:=\left|\mathbb{E}\left[\Delta_{n,k,j_1,j_2}|\,(S_p)_p\right]\mathbb{E}\left[\Gamma_{n,k,j_0}|\,(S_p)_p\right]\prod_{m'=k+1}^{L_n}a_{m'}\right|.$$

with $\Delta_{n,k,j_1,j_2}:=\prod_{m=0}^{k-(r+1)^4}b_m\prod_{m'=k-(r+1)^{j_2}+1}^{k-(r+1)^{j_1}}b_{m'}$ and $\Gamma_{n,k,j_0}:=\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1}b_m\right)(b_k-a_k)$.

- <u>Control of the covariance terms (thanks to our decorrelation hypothesis).</u> Let j_0,j_1,j_2 be fixed. Let $k=(r+1)^4,...,L_n-1$. We have :

$$\begin{aligned}D_{n,k,j_0,j_1,j_2}\quad\leq\quad&\left|Cov_{|(S_p)_p}\left(\Delta_{n,k,j_1,j_2},\prod_{m=k-(r+1)^{j_0}+1}^{k}b_m\right)\prod_{m'=k+1}^{L_n}a_{m'}\right|+\\&+\left|Cov_{|(S_p)_p}\left(\Delta_{n,k,j_1,j_2},\prod_{m=k-(r+1)^{j_0}+1}^{k-1}b_m\right)\prod_{m'=k}^{L_n}a_{m'}\right|.\end{aligned}$$

But we have : $\prod_{m=\theta_1+1}^{\theta_1+\theta_2}b_m=e^{iun^{-\frac{1}{2}-\delta_3+\delta_2}\sum_{\ell=\alpha(\theta_1)}^{\alpha(\theta_1+\theta_2+1)^{-1}}\xi_\ell N_{n-1}(\ell)}$. Therefore, according to point 2 of the hypothesis of our theorem, we have :

$$D_{n,k,j_0,j_1,j_2}\leq 2C\left(1+n^{-\frac{1}{2}-\delta_3+\delta_2}\sum_{\ell\in\mathbb{Z}}N_{n-1}(\ell)\right)\varphi_{p,s}$$

with $p:=\lfloor n^\beta\rfloor((r+1)^{j_1}-(r+1)^{j_0})$ and $s:=\lfloor n^\beta\rfloor(r+1)^{j_0}-1$. Let us notice that we have : $p\geq rs$. Since $\sum_{\ell\in\mathbb{Z}}N_{n-1}(\ell)=n$, we have :

$$\begin{aligned}n^{\delta_2}\sum_{k=(r+1)^4}^{L_n-1}D_{n,k,j_0,j_1,j_2}\quad&\leq\quad 4C\left(n^{1-\delta_3+\delta_1-\beta+2\delta_2}\right)n^{-6\beta}\sup_{s\geq n^\beta}s^6\varphi_{rs,s}\\&\leq\quad 4C\left(n^{1-\frac{9}{8}+\delta_1+(\frac{27}{2}+16)\delta_2}\right)\sup_{s\geq n^\beta}s^6\varphi_{rs,s},\end{aligned}$$

since $\beta>\frac{\delta_3}{2}-2\delta_2$ and $\delta_3>\frac{1}{4}-3\delta_2$. We end this point by noticing that $\delta_1+(\frac{27}{2}+16)\delta_2<\frac{1}{8}$.

- <u>Control of the term with the product of the expectations.</u> Let j_0,j_1,j_2 be fixed. Let $k=(r+1)^4,...,L_n-1$. We can notice that E_{n,k,j_0,j_1,j_2} is bounded by the following quantity :

$$F_{n,k,j_0}:=\left|\mathbb{E}\left[\prod_{m=k-(r+1)^{j_0}+1}^{k}b_m-\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1}b_m\right)a_k\middle|(S_p)_p\right]\right|.$$

We approximate the terms with exponential using Taylor expansions.

– First we explain that, in F_{n,k,j_0}, we can replace

$$\prod_{m=k-(r+1)^{j_0}+1}^{k} b_m = \exp\left(iun^{-\frac{1}{2}-\delta_3+\delta_2} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) \right)$$

by the formula given by the second order Taylor expansion of the exponential function :

$$1 + iun^{-\frac{1}{2}-\delta_3+\delta_2} \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}} \left(\sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell) \right)^2 . \quad (4)$$

Indeed, the induced error is less than : $\frac{1}{6}n^{-\frac{3}{2}-3\delta_3+3\delta_2}\mathbb{E}\left[\left|\sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell)\right|^3 |(S_p)_p\right]$.
Moreover, we have :

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} \xi_\ell N_{n-1}(\ell)\right|^4 |(S_p)_p\right] &\leq \sum_{y_1,y_2,y_3,y_4=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k+1)}-1} |\mathbb{E}[\xi_{y_1}\xi_{y_2}\xi_{y_3}\xi_{y_4}]| \left(n^{\frac{1}{2}+\delta_2}\right)^4 \\
&\leq c_0' n^{2+4\delta_2}(r+1)^6 n^{2\beta},
\end{aligned}$$

according to the hypothesis of our theorem. Hence, taking the sum over $k = (r+1)^4, ..., L_n - 1$ and multiplying by n^{δ_2}, this substitution induces a total error bounded by :

$$\frac{(c_0')^{3/4}}{6} n^{\delta_2+\frac{1}{2}+\delta_1-\beta} n^{-\frac{3}{2}-3\delta_3+3\delta_2} n^{\frac{3}{2}+3\delta_2}(r+1)^{\frac{9}{2}} n^{\frac{3}{2}\beta}$$

and so by : $\frac{(c_0')^{3/4}}{6} n^{7\delta_2+\frac{1}{2}+\delta_1-3\delta_3+\frac{1}{2}\beta}(r+1)^{\frac{9}{2}}$. Since $\beta < \frac{\delta_3}{2} - \delta_2$, $\delta_3 > \frac{1}{4} - 3\delta_2$ and $\delta_1 + (\frac{27}{2} + 16)\delta_2 < \frac{1}{8}$, we have : $7\delta_2 + \frac{1}{2} + \delta_1 - 3\delta_3 + \frac{1}{2}\beta \leq -\frac{1}{16}$.

– Let us introduce $Y_k := \sum_{\ell=\alpha_{(k-(r+1)^{j_0}+1)}}^{\alpha_{(k)}-1} \xi_\ell N_{n-1}(\ell)$ and $Z_k := \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \sigma_\xi^2 N_{n-1}(\ell)^2$. We explain that, in F_{n,k,j_0}, we can replace $\left(\prod_{m=k-(r+1)^{j_0}+1}^{k-1} b_m\right) a_k = e^{\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k}$ by the formula given by the Taylor expansion of the exponential function at the second order :

$$1 + \frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k + \frac{1}{2}\left(\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k\right)^2 . \quad (5)$$

Indeed the modulus of the error between these two quantities is less than :

$$\frac{1}{6}\mathbb{E}\left[\left|\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k - \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k\right|^3 |(S_p)_p\right] \leq \frac{4}{3}\mathbb{E}\left[\left|\frac{1}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k\right|^3 + \left|\frac{1}{2n^{1+2\delta_3-2\delta_2}}Z_k\right|^3 |(S_p)_p\right].$$

We control the first term as in the preceding point. Moreover, we have :

$$\left|n^{-1-2\delta_3+2\delta_2}Z_k\right|^3 \leq n^{-3-6\delta_3+6\delta_2}\left(\sigma_\xi^2\right)^3 n^{3\beta} n^{3+6\delta_2} \leq n^{-6\delta_3+12\delta_2+3\beta}\left(\sigma_\xi^2\right)^3 .$$

Hence, taking the sum over $k = (r+1)^4, ..., L_n - 1$ and multiplying by n^{δ_2}, we get a quantity bounded by : $2n^{\frac{1}{2}+\delta_1-6\delta_3+13\delta_2+2\beta}\left(\sigma_\xi^2\right)^3$ and we have : $\frac{1}{2} + \delta_1 - 6\delta_3 + 13\delta_2 + 2\beta < 0$.

– Now, we show that in formula (5), we can omit the term with $(Z_k)^2$. Indeed, we have :

$$\begin{aligned}
n^{\delta_2}\sum_{(r+1)^4}^{L_n-1}\left(n^{-1-2\delta_3+2\delta_2}Z_k\right)^2 &\leq 2n^{\delta_2+\frac{1}{2}+\delta_1-\beta-2-4\delta_3+4\delta_2} n^{2\beta}(\sigma_\xi^2)^2 n^{2+4\delta_2} \\
&\leq 2n^{-\frac{1}{5}-\frac{2}{5}\delta_1-\frac{2}{5}\delta_2}(\sigma_\xi^2)^2
\end{aligned}$$

since $\beta < \frac{\delta_3}{2} - \delta_2$ and $\frac{5}{2}\delta_3 > \frac{1}{2} + 6\delta_2 + \delta_1$.

– Hence, it remains to estimate the following quantity called G_{n,k,j_0} :

$$\left|\mathbb{E}\left[\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}(Y_k+W_k)-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left(Y_k+W_k\right)^2-\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k+\right.\right.$$

$$\left.\left.+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}(Y_k)^2+\frac{iu}{n^{\frac{1}{2}+\delta_3-\delta_2}}Y_k\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k\right|(S_p)_p\right]\Bigg|,$$

with $W_k := \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\xi_\ell N_{n-1}(\ell)$. We get :

$$\begin{aligned}G_{n,k,j_0} &= \left|\mathbb{E}\left[-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left(Y_k+W_k\right)^2+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}Z_k+\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}(Y_k)^2\right|(S_p)_p\right]\right| \\ &= \frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left|\mathbb{E}\left[\left(W_k\right)^2+2W_kY_k-Z_k\right|(S_p)_p\right]\right|.\end{aligned}$$

Let us notice that we have :

$$Z_k := \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2]N_{n-1}(\ell)^2+2\sum_{m\leq\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)^2\right).$$

– Let us show that, in the last expression of G_{n,k,j_0}, we can replace Z_k by :

$$\tilde{Z}_k := \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2]N_{n-1}(\ell)^2+2\sum_{m\leq\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)N_{n-1}(m)\right).$$

Indeed, by definition of U_n, we have :

$$\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\mathbb{E}\left[\left|Z_k-\tilde{Z}_k\right|\,\Big|(S_p)_p\right]\leq$$

$$\begin{aligned}&\leq \frac{1}{n^{1+2\delta_3-2\delta_2}}\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\sum_{m\leq\ell-1}|\mathbb{E}[\xi_\ell\xi_m]|N_{n-1}(\ell)|N_{n-1}(m)-N_{n-1}(\ell)| \\ &\leq n^{-\frac{1}{4}-2\delta_3+3\delta_2+\beta+\frac{\gamma}{2}}\sum_{m\geq1}\sqrt{m}|\mathbb{E}[\xi_0\xi_m]|.\end{aligned}$$

Hence, taking the sum over $k=(r+1)^4,...,L_n-1$ and multiplying by n^{δ_2}, we get a quantity bounded by : $4n^{\frac{1}{4}+\delta_1-2\delta_3+4\delta_2+\frac{\gamma}{2}}\sum_{m\geq1}\sqrt{m}|\mathbb{E}[\xi_0\xi_m]|$. But, since $\delta_3>\frac{1}{4}-3\delta_2$ and $\gamma<\frac{1}{2}-22\max(\delta_1,\delta_2)$, we have : $\frac{1}{4}+\delta_1-2\delta_3+4\delta_2+\frac{\gamma}{2}<0$

– Hence we have to estimate :

$$\tilde{G}_{n,k,j_0}=\frac{u^2}{2n^{1+2\delta_3-2\delta_2}}\left|\mathbb{E}\left[\left(W_k\right)^2+2W_kY_k-\tilde{Z}_k\right|(S_p)_p\right]\right|.$$

We have :

$$\mathbb{E}\left[\left(W_k\right)^2\right|(S_p)_p\right]=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2](N_{n-1}(\ell))^2+2\sum_{m=\alpha_{(k)}}^{\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)N_{n-1}(m)\right).$$

Hence we have :

$$\mathbb{E}\left[\left(W_k\right)^2+2W_kY_k\right|(S_p)_p\right]=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}[(\xi_\ell)^2](N_{n-1}(\ell))^2+2\sum_{m=\alpha_{(k-(r+1)^{j_0}+1)}}^{\ell-1}\mathbb{E}[\xi_\ell\xi_m]N_{n-1}(\ell)N_{n-1}(m)\right).$$

We get :

$$\tilde{G}_{n,k,j_0} = \frac{u^2}{n^{1+2\delta_3-2\delta_2}} \left| \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \sum_{m \leq \alpha_{(k-(r+1)^{j_0}+1)}-1} \mathbb{E}[\xi_\ell \xi_m] N_{n-1}(\ell) N_{n-1}(m) \right|$$

$$\leq \frac{u^2}{n^{1+2\delta_3-2\delta_2}} n^\beta \sum_{m \geq \frac{(r+1)n^\beta}{2}} |\mathbb{E}[\xi_0 \xi_m]| n^{1+2\delta_2} \leq n^{-2\delta_3+4\delta_2+\beta} \sum_{m \geq \frac{(r+1)n^\beta}{2}} |\mathbb{E}[\xi_0 \xi_m]|.$$

Hence, taking the sum over $k = (r+1)^4, ..., L_n - 1$ of these quantities and multiplying by n^{δ_2}, we get a quantity bounded by :

$$4n^{\frac{1}{2}+\delta_1-2\delta_3+5\delta_2} \sum_{m \geq \frac{(r+1)n^\beta}{2}} |\mathbb{E}[\xi_0 \xi_m]| \leq 4n^{\delta_1+11\delta_2} \sum_{m \geq \frac{(r+1)n^\beta}{2}} |\mathbb{E}[\xi_0 \xi_m]|,$$

since $\delta_3 > \frac{1}{4} - 3\delta_2$. To conclude it suffices to notice that : $n^{\delta_1+11\delta_2} \sum_{m \geq \frac{(r+1)n^\beta}{2}} |\mathbb{E}[\xi_0 \xi_m]| = O(n^{-\varepsilon})$.

3.1.2 Proof of lemma 9

Let us consider $n \geq 2$. According to lemma 10, it suffices to prove that there exists a real number $\delta' > 0$ such that we have :

$$\sup_{n \geq 1} n^{\delta'} \sup_{\omega \in U_n} n^{\delta_2} \int_{|u| \leq 1} \exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}} \sum_{y,z} \mathbb{E}[\xi_y \xi_z] (N_{n-1}(y)(\omega))^2 \right) e^{-\frac{u^2 n^{2\delta_2}}{2}} \, du < +\infty.$$

Let us take $\omega \in U_n$. We have :

$$\exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}} \sum_{y,z} \mathbb{E}[\xi_y \xi_z] (N_{n-1}(y)(\omega))^2 \right) = \exp\left(-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}} \sigma_\xi^2 \sum_y (N_{n-1}(y)(\omega))^2 \right).$$

Let us define : $p_n := Card\{y \in \mathbb{Z} \ : \ N_{n-1}(y) \geq \frac{n^{\frac{1}{2}-\delta_4}}{3}\}$. We have :

$$\begin{aligned} n &= \sum_{y=-\lfloor n^{\frac{1}{2}+\delta_1} \rfloor}^{\lfloor n^{\frac{1}{2}+\delta_1} \rfloor} N_{n-1}(y) \leq p_n n^{\frac{1}{2}+\delta_2} + \frac{n^{\frac{1}{2}-\delta_4}}{3} \left(3n^{\frac{1}{2}+\delta_1} - p_n \right) \\ &\leq p_n n^{\frac{1}{2}+\delta_2} \left(1 - \frac{n^{-(\delta_2+\delta_4)}}{3} \right) + n^{1+\delta_1-\delta_4}. \end{aligned}$$

Since $\delta_1 < \delta_4$, we have : $p_n \geq n^{-\frac{1}{2}-\delta_2} \left(n - n^{1-(\delta_4-\delta_1)} \right) \geq n^{\frac{1}{2}-\delta_2} \left(1 - n^{-(\delta_4-\delta_1)} \right) \geq c_0 n^{\frac{1}{2}-\delta_2}$, with $c_0 := 1 - 2^{-(\delta_4-\delta_1)}$. Hence we have : $\sum_{y \in \mathbb{Z}} (N_{n-1}(y)(\omega))^2 \geq p_n \left(\frac{n^{\frac{1}{2}-\delta_4}}{3} \right)^2 \geq \frac{c_0 n^{\frac{3}{2}-\delta_2-2\delta_4}}{9}$ and

$$\begin{aligned} e^{-\frac{u^2 \sum_y \sigma_\xi^2 (N_{n-1}(y)(\omega))^2}{2n^{1+2\delta_3-2\delta_2}}} &\leq e^{-\frac{u^2 \sigma_\xi^2 c_0 n^{\frac{3}{2}-\delta_2-2\delta_4}}{18 n^{1+2\delta_3-2\delta_2}}} \\ &\leq e^{-\frac{u^2}{18} \sigma_\xi^2 c_0 n^{\frac{1}{2}+\delta_2-2\delta_3-2\delta_4}}. \end{aligned}$$

Therefore, we have :

$$n^{\delta_2} \int_{|u| \leq 1} e^{-\frac{u^2}{2n^{1+2\delta_3-2\delta_2}} \sum_{y,z} \mathbb{E}[\xi_y \xi_z] (N_{n-1}(y)(\omega))^2} e^{-\frac{u^2 n^{2\delta_2}}{2}} \, du \leq$$

$$\leq \quad n^{\delta_2} \int_{|u|\leq 1} e^{-\frac{u^2}{18}\sigma_\xi^2 c_0 n^{\frac{1}{2}+\delta_2-2\delta_3-2\delta_4}}\, du$$
$$\leq \quad n^{-\frac{1}{4}+\delta_4+\frac{1}{2}\delta_2+\delta_3} \int_{\mathbb{R}} e^{-\frac{v^2}{18}\sigma_\xi^2 c_0}\, dv.$$

This ends the proof since $\delta_4+\delta_3+\frac{1}{2}\delta_2<\frac{1}{4}$. □

4 About the model of Guillotin-Plantard and Le Ny

In this section, we prove that the hypothesis $\int_M \frac{1}{\sqrt{f_0(1-f_0)}}\, d\nu < +\infty$ of Guillotin-Plantard and Le Ny in [10] can be replaced by the existence of $p \geq 1$ such that $\int_M \frac{1}{(f_0(1-f_0))^p}\, d\nu < +\infty$, for some $p>0$. In this situation, there is no need to introduce the set U_n; we take $U_n = A_n$. If we take $\delta_1>0$, $\delta_2>0$ and $\delta_3>0$, all the points (of the sketch of the proof of section 3) except the point 3(b)(ii) come in the same way without the need of the hypothesis $\int_M \frac{1}{\sqrt{f_0(1-f_0)}}\, d\nu < +\infty$. It remains to estimate :

$$\sup_{\omega\in A_n} n^{\frac{1}{2}+\delta_3} I_n^{(1)}(\omega) := n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{it\sum_{y\in\mathbb{Z}}\xi_y N_{n-1}(y)} \middle| (S_p)_p \right](\omega) e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt.$$

Let us take $\omega\in A_n$. We suppose $\delta_3>2\delta_2$ and $\delta_1<\delta_4<\frac{1}{4}-\delta_3-\frac{\delta_2}{2}$. The idea of Guillotin-Plantard and Le Ny is to write :

$$\begin{aligned}
n^{\frac{1}{2}+\delta_3}\left|I_n^{(1)}\right| &\leq n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[\prod_{y\in\mathbb{Z}} |\cos(tN_{n-1}(y)) + i(2f_y-1)\sin(tN_{n-1}(y))| \middle| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt \\
&\leq n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[\prod_{y\in\mathbb{Z}} \sqrt{1-4f_y(1-f_y)\sin^2(tN_{n-1}(y))} \middle| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt \\
&\leq n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[\prod_{y\in\mathbb{Z}} \sqrt{1-f_y(1-f_y)\frac{16}{\pi^2}(tN_{n-1}(y))^2} \middle| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt \\
&\leq n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[\prod_{y\in\mathbb{Z}} e^{-\frac{8}{\pi^2}f_y(1-f_y)t^2 N_{n-1}(y)^2} \middle| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt
\end{aligned}$$

since $|tN_{n-1}(y)| \leq n^{-\frac{1}{2}-\delta_3+\delta_2} n^{\frac{1}{2}+\delta_2} = n^{2\delta_2-\delta_3}$. Hence, if n is large enough, then $|tN_{n-1}(y)|$ will be uniformly less than $\frac{\pi}{2}$ and $|\sin(tN_{n-1}(y))| \geq \frac{2}{\pi}|tN_{n-1}(y)|$. We also use the fact that, for positive u, we have : $1-u\leq e^{-u}$. According to the Hölder inequality with $\sum_y \frac{N_{n-1}(y)^2}{\sum_k N_{n-1}(k)^2}=1$, we have :

$$n^{\frac{1}{2}+\delta_3}\left|I_n^{(1)}\right| \leq n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{-\frac{8}{\pi^2}f_0(1-f_0)t^2\sum_k N_{n-1}(k)^2} \middle| (S_p)_p \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt.$$

Now, we use the fact that, since $\delta_4>\delta_1$, there exists a constant c such that we have :

$$\forall \omega'\in A_n,\quad \sum_{y\in\mathbb{Z}} (N_{n-1}(y))^2(\omega') \geq c n^{\frac{3}{2}-\delta_2-2\delta_4}.$$

This has been proved in the previous section entitled 'proof of lemma 9'. Hence, under the hypothesis $\int_M \frac{1}{\sqrt{f_0(1-f_0)}}\, d\nu < +\infty$ of Guillotin-Plantard and Le Ny, we have :

$$\begin{aligned}
n^{\frac{1}{2}+\delta_3}\left|I_n^{(1)}(\omega)\right| &\leq n^{\frac{1}{2}+\delta_3} \int_{\{|t|\leq n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{-\frac{8c}{\pi^2}f_0(1-f_0)t^2 n^{\frac{3}{2}-\delta_2-2\delta_4}} \right] e^{-\frac{t^2 n^{1+2\delta_3}}{2}}\, dt \\
&\leq n^{-\frac{1}{4}+\delta_3+\frac{\delta_2}{2}+\delta_4} \int_{\mathbb{R}} \mathbb{E}\left[\frac{1}{\sqrt{f_0(1-f_0)}} \right] e^{-\frac{8}{\pi^2}v^2}\, dv
\end{aligned}$$

with the change of variable $v = t\sqrt{f_0(1-f_0)n^{\frac{3}{2}-\delta_2-2\delta_4}}$. This gives the result of Guillotin-Plantard and Le Ny since $-\frac{1}{4}+\delta_3+\frac{\delta_2}{2}+\delta_4<0$. We adapt this argument to our hypothesis. Now let us replace the hypothesis $\int_M \frac{1}{\sqrt{f_0(1-f_0)}}\,d\nu<+\infty$ by $\int_M \frac{1}{[f_0(1-f_0)]^p}\,d\nu<+\infty$ for some $p>0$. Let us take $\delta_3>2\delta_2$ and $\delta_1<\delta_4<\frac{1}{4}-\delta_3-\frac{\delta_2}{2}-\frac{\delta_2}{p}$. We have :

$$\begin{aligned} n^{\frac{1}{2}+\delta_3}\int_{\{|t|\le n^{-\frac{3}{4}+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}}\}} \mathbb{E}\left[e^{-\frac{8}{\pi^2}f_0(1-f_0)t^2n^{\frac{3}{2}-\delta_2-2\delta_4}}\right]e^{-\frac{t^2n^{1+2\delta_3}}{2}}\,dt &\le 2n^{\frac{1}{2}+\delta_3}n^{-\frac{3}{4}+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}} \\ &\le 2n^{-\frac{1}{4}+\delta_3+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}}. \end{aligned}$$

On the other hand, let $c_p=\sup_{u>0}u^pe^{-u}$, we have :

$$n^{\frac{1}{2}+\delta_3}\int_{\{n^{-\frac{3}{4}+\frac{\delta_2}{2}+\delta_4+\frac{\delta_2}{p}}<|t|<n^{-\frac{1}{2}-\delta_3+\delta_2}\}} \mathbb{E}\left[e^{-\frac{8}{\pi^2}f_0(1-f_0)t^2n^{\frac{3}{2}-\delta_2-2\delta_4}}\right]e^{-\frac{t^2n^{1+2\delta_3}}{2}}\,dt \le$$

$$\le 2n^{\frac{1}{2}+\delta_3}n^{-\frac{1}{2}-\delta_3+\delta_2}\int_M e^{-\frac{8}{\pi^2}f_0(1-f_0)n^{\frac{2\delta_2}{p}}}\,d\nu \le n^{-\delta_2}c_p\left(\frac{\pi^2}{8}\right)^p\int_M[f_0(1-f_0)]^{-p}\,d\nu.$$

A Proof of proposition 4

In cases (a) and (b), $(\xi_k)_k$ is a stationary sequence of bounded centered random variables

A.1 Proof of *(a)*

We have : $\sum_{p\ge0}\sqrt{1+p}|\mathbb{E}[\xi_0\xi_p]|=\sum_{p\ge0}\sqrt{1+p}|\mathbb{E}_\nu[f.f\circ T^p]|$ which is less than : $c_0\|f\|_\infty\left(\|f\|_\infty+K_f^{(1)}+K_f^{(2)}\right)\sum_{p\ge0}\sqrt{}$ and hence is finite. Let us consider an integer $N\ge1$. We have :

$$\frac{1}{N^2}\sum_{k_1,k_2,k_3,k_4=0,\dots,N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]|\le\frac{24}{N^2}\sum_{0\le k_1\le k_2\le k_3\le k_4\le N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]|\,.$$

Let us consider the set $E_N^{(1)}$ of (k_1,k_2,k_3,k_4) such that $0\le k_1\le k_2\le k_3\le k_4\le N-1$ and $k_4-k_3\ge N^{\frac{1}{3}}$ We have :

$$\begin{aligned} \sum_{(k_1,k_2,k_3,k_4)\in E_N^{(1)}}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]| &= \sum_{(k_1,k_2,k_3,k_4)\in E_N^{(1)}}\left|Cov_\nu\left(f\circ T^{k_1-k_3}f\circ T^{k_2-k_3}f,f\circ T^{k_4-k_3}\right)\right| \\ &\le c_0N^4\left(\|f\|_\infty^4+\|f\|_\infty^3(K_f^{(2)}+3c_0K_f^{(1)})\right)\varphi_{\lceil N^{\frac{1}{3}}\rceil} \\ &\le c_0N^2\left(\|f\|_\infty^4+\|f\|_\infty^3(K_f^{(2)}+3c_0K_f^{(1)})\right)\sup_{n\ge1}n^6\varphi_n. \end{aligned}$$

Let us consider the set $E_N^{(2)}$ of (k_1,k_2,k_3,k_4) such that $0\le k_1\le k_2\le k_3\le k_4\le N-1$ and $k_4-k_3<N^{\frac{1}{3}}$ and $k_3-k_2\ge rN^{\frac{1}{3}}$. We have :

$$\begin{aligned} \sum_{(k_1,k_2,k_3,k_4)\in E_N^{(2)}}|Cov\left(\xi_{k_1}\xi_{k_2},\xi_{k_3}\xi_{k_4}\right)| &= \sum_{(k_1,k_2,k_3,k_4)\in E_N^{(2)}}\left|Cov_\nu\left(f\circ T^{k_1-k_2}f,(f.f\circ T^{k_4-k_3})\circ T^{k_3-k_2}\right)\right| \\ &\le 2^6c_0N^2\left(\|f\|_\infty^4+2c_0\|f\|_\infty^3(K_f^{(2)}+K_f^{(1)})\right)\sup_{n\ge1}n^6(1+\kappa_n)\varphi_{rn}. \end{aligned}$$

Moreover, we have :

$$\sum_{(k_1,k_2,k_3,k_4)\in E_N^{(2)}} |\mathbb{E}[\xi_{k_1}\xi_{k_2}]\mathbb{E}[\xi_{k_3}\xi_{k_4}]| \leq \left(\sum_{0\leq k_1\leq k_2\leq N-1} |\mathbb{E}[\xi_{k_1}\xi_{k_2}]|\right)^2 \leq \left(N\sum_{k\geq 0}\left|\mathbb{E}_\nu[f.f\circ T^k]\right|\right)^2$$
$$\leq N^2\left(c_0\left(\|f\|_\infty^2+\|f\|_\infty(K_f^{(1)}+K_f^{(2)})\right)\sum_{k\geq 0}\varphi_k\right)^2.$$

Let us consider the set $E_N^{(3)}$ of (k_1,k_2,k_3,k_4) such that $0\leq k_1\leq k_2\leq k_3\leq k_4\leq N-1$ and $k_4-k_3<N^{\frac{1}{3}}$ and $k_3-k_2<rN^{\frac{1}{3}}$ and $k_2-k_1\geq r(1+r)N^{\frac{1}{3}}$. By the same method, we get :

$$\sum_{(k_1,k_2,k_3,k_4)\in E_N^{(3)}} |\mathbb{E}\left[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}\right]| \leq N^2\frac{c_0 2^6}{(1+r)^6}\left(\|f\|_\infty^4+3c_0\|f\|_\infty^3(K_f^{(2)}+K_f^{(1)})\right)\sup_{n\geq 1} n^6(1+\kappa_n)\varphi_{rn}.$$

Since the number of (k_1,k_2,k_3,k_4) such that $0\leq k_1\leq k_2\leq k_3\leq k_4\leq N-1$ and that do not belong to $E_N^{(1)}\cup E_N^{(2)}\cup E_N^{(3)}$ is bounded by $N^2 2(r+1)^3$, we get :

$$\sup_{N\geq 1}\frac{1}{N^2}\sum_{k_1,k_2,k_3,k_4=0,\ldots,N-1}|\mathbb{E}[\xi_{k_1}\xi_{k_2}\xi_{k_3}\xi_{k_4}]|<+\infty.$$

Now, let us prove the point 2 of the hypothesis of theorem 1. Let n_1, n_2, n_3 and n_4 be four integers such that $0\leq n_1\leq n_2\leq n_3\leq n_4$. Let us consider any real numbers $\alpha_{n_1},..,\alpha_{n_2}$ and $\beta_{n_3},...,\beta_{n_4}$. We have :

$$\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\xi_k},e^{i\sum_{k=n_3}^{n_4}\beta_k\xi_k}\right)\right| = \left|Cov_\nu\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k f\circ T^{-(n_2-k)}},\left(e^{i\sum_{k=n_3}^{n_4}\beta_k f\circ T^{k-n_3}}\right)\circ T^{n_3-n_2}\right)\right|$$
$$\leq c_0\left(1+K^{(1)}_{\exp\left(i\sum_{k=n_1}^{n_2}\alpha_k f\circ T^{-(n_2-k)}\right)}+K^{(2)}_{\exp\left(i\sum_{k=n_3}^{n_4}\beta_k f\circ T^{k-n_3}\right)}\right)\varphi_{n_3-n_2}$$
$$\leq c_0\left(1+\sum_{k=n_1}^{n_2}c_0c_1|\alpha_k|+\sum_{k=n_3}^{n_4}c_0c_1|\beta_k|(1+\kappa_{n_4-n_3})\right)\varphi_{n_3-n_2}.$$

This gives the point 2 of the hypothesis of theorem 1 with $\varphi_{p,s}:=(1+\kappa_s)\varphi_p$.

A.2 Proof of *(b)*

Let us define the function $g=2f-1$. This function is ν-centered. More generally, for any integer $m\geq 1$, let us define : $g_{2m}=1$ and $g_{2m+1}=g$. We observe that, conditionally to $\omega\in M$, the expectation of $(\xi_k(\omega,\cdot))^m$ is equal to $g_m\circ T^k(\omega)$. Using the Fubini theorem and starting by integrating over $[0;1]^{\mathbb{Z}}$, we observe that, for any integers $p\geq 1$, we have : $\mathbb{E}[\xi_0\xi_p]=\mathbb{E}_\nu[g.g\circ T^p]$ and that, for any integers k_1,k_2,k_3,k_4, we have : $\mathbb{E}\left[{\xi_{k_1}}^{n_1}{\xi_{k_2}}^{n_2}{\xi_{k_3}}^{n_3}{\xi_{k_4}}^{n_4}\right]=\mathbb{E}_\nu\left[\prod_{j=1}^4 g_{n_j}\circ T^{k_j}\right]$. Hence, we can prove the point 1 of theorem 1 as we did for *(a)*.

Now, let us prove the point 2 of the hypothesis of theorem 1. We observe that, conditionally to $\omega\in M$, the $\xi_k(\omega,\cdot)$ are independent and that the expectation of $\exp(iu\xi_k(\omega,\cdot))$ is $h_u\circ T^k(\omega)$ with $(h_u:=e^{-iu}+2i\sin(u)f\circ T^k$. The modulus of this function is bounded by 1 and we have : $\max\left(K_{h_u}^{(1)},K_{h_u}^{(2)}\right)\leq 2c_1|u|$. Let n_1, n_2, n_3 and n_4 be four integers such that $0\leq n_1\leq n_2<n_3\leq n_4$. Let us consider any real numbers $\alpha_{n_1},..,\alpha_{n_2}$ and $\beta_{n_3},...,\beta_{n_4}$. We have :

$$\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\xi_k},e^{i\sum_{k=n_3}^{n_4}\beta_k\xi_k}\right)\right| = \left|Cov_\nu\left(\prod_{k=n_1}^{n_2}h_{\alpha_k}\circ T^k,\prod_{k=n_3}^{n_4}h_{\beta_k}\circ T^k\right)\right|$$
$$\leq c_0\left(1+2c_0c_1\left(\sum_{k=n_1}^{n_2}|\alpha_k|+\sum_{k=n_3}^{n_4}|\beta_k|\right)\right)(1+\kappa_{n_4-n_3})\varphi_{n_3-n_2}.$$

B Proof of theorem 2 : α-mixing condition

Let us define $(M, \mathcal{F}) = (\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R})^{\otimes \mathbb{Z}})$. Let $T : M \to M$ be such that $T((\omega_k)_{k\in\mathbb{Z}}) = (\omega_{k+1})_{k\in\mathbb{Z}}$. Let ν be the image probability measure on $(M, \mathcal{F})$ of $\Pi : \Omega \to \mathbb{R}^{\mathbb{Z}}$ with $\Pi(\omega) = (\xi_k(\omega))_{k\in\mathbb{Z}}$. The process $(\xi_k)_{k\in\mathbb{Z}}$ (with respect to $\mathbb{P}$) has the same distribution as $(f \circ T^k)_{k\in\mathbb{Z}}$ (with respect to ν) with $f : M \to \mathbb{R}$ given by $f((\omega_k)_{k\in\mathbb{Z}}) = \omega_0$. According to [11], lemma 1.2, $(M, \mathcal{F}, \nu, T)$ is strongly mixing (in the sense of our definition 3) with the following choice of $K_\cdot^{(1)}$ and of $K_\cdot^{(2)}$. If g is $\sigma(f \circ T^k,\ k \leq 0)$-measurable, we have $K_g^{(1)} := 0$; otherwise we have $K_g^{(1)} := \infty$. If h is $\sigma(f \circ T^k,\ k \geq 0)$-measurable, we have $K_h^{(2)} := 0$; otherwise we have $K_h^{(2)} := \infty$. We conclude with proposition 4.

C Proof of example 2.1

C.1 Case 1

Let $\eta > 0$. Let us denote by $\Gamma^{(s,e)}$ the set of stable-central manifolds and by Γ^u the set of unstable manifolds. In [16], each $\gamma^u \in \Gamma^u$ is endowed with some metric d^u and each $\gamma^{(s,e)} \in \Gamma^{(s,e)}$ is endowed with some metric $d^{(s,e)}$ such that there exist $\tilde{c}_0 > 0$, $\delta_0 \in]0;1[$ and $\beta > 0$ such that, for any integer $n \geq 0$, for any $\gamma^u \in \Gamma^u$ and any $\gamma^{(s,e)} \in \Gamma^{(s,e)}$, we have :

- For any $y, z \in \gamma^u$, $d^u(y,z) \geq d(y,z)$ and for any $y', z' \in \gamma^{(s,e)}$, $d^{(s,e)}(y',z') \geq d(y',z')$.
- For any $y, z \in \gamma^u$, there exists $\gamma^u_{(n)} \in \Gamma^u$ such that $T^{-n}(y)$ and $T^{-n}(z)$ belong to $\gamma^u_{(n)}$ and we have : $d^u(T^{-n}(y), T^{-n}(z)) \leq \tilde{c}_0(\delta_0)^n d^u(y,z)$.
- For any $y, z \in \gamma^{(s,e)}$, there exists $\gamma^{(s,e)}_{(n)} \in \Gamma^{(s,e)}$ such that $T^n(y)$ and $T^n(z)$ belong to $\gamma^{(s,e)}_{(n)}$ and we have : $d^{(s,e)}(T^n(y), T^n(z)) \leq \tilde{c}_0(1 + n^\beta) d^{(s,e)}(y,z)$.

We take :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{y,z \in \gamma^u : y \neq z} \frac{|f(y) - f(z)|}{(d^u(y,z))^\eta} \quad \text{and} \quad K_f^{(2)} := \sup_{\gamma^{(s,e)} \in \Gamma^{(s,e)}} \sup_{y,z \in \gamma^{(s,e)} : y \neq z} \frac{|f(y) - f(z)|}{(d^{(s,e)}(y,z))^\eta}.$$

For these examples, the result follows from [16] (cf. lemme 1.3.1 in [16]).

C.2 Case 2 : Sinai billiard

Since the early work of Sinai [19], this billiard system has been studied by many authors ([1, 2, 3, 4, 8] and others). Let us recall that a point of M is a couple (q, v) corresponds to a reflected unit speed vector v at the position q on some obstacle O_i and is parametrised by $(i, r\varphi)$ where i is the index of the obstacle O_i, r the curvilinear of x on it and φ the measure of the angle (taken in $[-\pi/2; \pi/2]$) made by v with the unit normal vector $\vec{n}(q)$ to O_i at q directed to the outside of the obstacle. We endow M with a metric d such that : $d((i,r,\varphi),(i,r',\varphi')) = |r - r'| + |\varphi - \varphi'|$. Let us denote by R_0 the set of points in M corresponding to a reflected vectors tangent to the obstacles, i.e. such that $\varphi = \pm\pi/2$. The transformation T^n defines a C^1-diffeomorphism from $M \setminus \bigcup_{k=0}^n T^{-k}(R_0)$ onto $M \setminus \bigcup_{k=0}^n T^k(R_0)$. Let us consider the set $\mathcal{C}_m$ of connected components of $M \setminus \bigcup_{k=-m}^m T^k(R_0)$. For all $k = -m, ..., m$, T^k is C^1 on each C belonging to $\mathcal{C}_m$. We will use the notations of Chernov in [6]. Let us consider the set Γ^s of homogeneous stable curves and the set Γ^u of homogeneous unstable curves and the two separation times $s_+(\cdot,\cdot)$ (in the future) and $s_-(\cdot,\cdot)$ (in the past) considered in [6]. We recall that there exist two constants $c_1 > 0$ and $\delta_1 \in]0;1[$ such that, for any nonnegative integer n, for any y and z in M, we have :

- If y and z belong to the same homogeneous unstable curve, then $s_+(x,y) \in \mathbb{Z}_+$, moreover $T^{-n}(y)$ and $T^{-n}(z)$ belong to a same homogeneous unstable curve and we have : $d(T^{-n}(y), T^{-n}(z)) \leq c_1 {\delta_1}^n$ and $s_+(T^{-n}(x), T^{-n}(y)) \geq n + s_+(x,y)$.

- If y and z belong to the same homogeneous stable curve, then $s_-(x,y) \in \mathbb{Z}_+$, moreover $T^n(y)$ and $T^n(z)$ belong to a same homogeneous stable curve and we have : $d(T^n(y), T^n(z)) \leq c_1 {\delta_1}^n$ and $s_-(T^n(x), T^n(y)) \geq n + s_-(x,y)$.

With these notations, according to [6] (theorem 4.3 in [6] and the remark after theorem 4.3 in [6]), this system is strongly mixing with :

$$K_f^{(1)} := \sup_{\gamma^u \in \Gamma^u} \sup_{\substack{y,z \in \gamma^u; y \neq z; \\ s_+(y,z) \geq m+1}} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_+(y,z)}} \text{ and } K_f^{(2)} := \sup_{\gamma^s \in \Gamma^s} \sup_{\substack{y,z \in \gamma^s; y \neq z; \\ s_-(y,z) \geq m+1}} \frac{|f(y) - f(z)|}{(\delta_1)^{\eta s_-(y,z)}}.$$

D Proof of conclusion (B) of theorem 5

We will use b and δ of proposition 2.1. First let us notice that there exists $c'_A > 0$ such that, for every $\varepsilon \in]0;1[$, there exists a Lipschitz continuous function f_ε such that : $\|\mathbf{1}_A - f_\varepsilon\|_{L^1(\nu)} \leq c_A \varepsilon^\zeta$, $\|f_\varepsilon\|_\infty \leq 1$ and $C_{f_\varepsilon}^{(1)} \leq \frac{c'_A}{\varepsilon}$. It suffices to take $f_\varepsilon = \max\left(0, 1 - \frac{d(\cdot, A)}{\varepsilon}\right)$.

- Let us prove that : $\sum_{p \geq 0} \sqrt{1+p} |\mathbb{E}[\xi_0 \xi_p]| < +\infty$. This quantity can be rewritten :

$$4 \sum_{p \geq 0} \sqrt{1+p} |Cov_\nu(\mathbf{1}_A, \mathbf{1}_A \circ T^p)|$$

  and is less than : $4 \sum_{p \geq 0} \sqrt{1+p} |Cov_\nu(f_{p^{-2/\zeta}}, f_{p^{-2/\zeta}} \circ T^p) + 2c_A p^{-2}|$. Moreover, we have :

$$|Cov_\nu(f_{p^{-2/\zeta}}, f_{p^{-2/\zeta}} \circ T^p)| \leq c_0 \left(1 + K^{(1)}_{f_{p^{-2/\zeta}}} + K^{(2)}_{f_{p^{-2/\zeta}}}\right) \alpha^p \leq c_0 \left(1 + 2c'_A p^{2/\zeta}\right) \alpha^p.$$

- Let us prove that :

$$\sup_{N \geq 1} N^{-2} \sum_{k_1,k_2,k_3,k_4 = 0,\ldots,N-1} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]| < +\infty.$$

  We use the notations $E_N^{(1)}$, $E_N^{(2)}$ and $E_N^{(3)}$ and the calculations done in section A.1.

  - To estimate $\sum_{(k_1,k_2,k_3,k_4) \in E_N^{(1)} \cup E_N^{(3)}} |\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]|$, we replace each ξ_k by $g_N \circ T^k$, with $g_N := 2\left(f_{N^{\frac{-2}{\zeta}}} - \mathbb{E}_\nu[f_{N^{\frac{-2}{\zeta}}}]\right)$. We have : $\|\xi_k - g_N \circ T^k\|_{L^1(\nu)} \leq 4c_A N^{-2}$. This substitution makes a total error in $O(N^2)$. Moreover, according to the calculations of section A.1, we have :

$$\sum_{(k_1,k_2,k_3,k_4) \in E_N^{(1)} \cup E_N^{(3)}} \left|\mathbb{E}[g_N \circ T^{k_1} g_N \circ T^{k_2} g_N \circ T^{k_3} g_N \circ T^{k_4}]\right| \leq CN^4 (1 + C_{g_N}^{(1)})(1 + N^{\beta/3}) \alpha^{N^{\frac{1}{3}}} = O(N^2).$$

  - With the same technique, we get : $\sum_{(k_1,k_2,k_3,k_4) \in E_N^{(2)}} |Cov_\nu(\xi_{k_1} \xi_{k_2}, \xi_{k_3} \xi_{k_4})| = O(N^2)$. Moreover, as in section A.1, we have : $\sum_{(k_1,k_2,k_3,k_4) \in E_N^{(2)}} |\mathbb{E}_\nu[\xi_{k_1} \xi_{k_2}] \mathbb{E}_\nu[\xi_{k_3} \xi_{k_4})]| \leq N^2 \left(\sum_{k \geq 0} |\mathbb{E}[\xi_0 \xi_k]|\right)^2$ and we have already proved that : $\sum_{k \geq 0} |\mathbb{E}[\xi_0 \xi_k]| < +\infty$.
  - The sum of $|\mathbb{E}[\xi_{k_1} \xi_{k_2} \xi_{k_3} \xi_{k_4}]|$ over the $k = (k_1, k_2, k_3, k_4)$ such that $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4$ but that do not belong to $E_N^{(1)} \cup E_N^{(2)} \cup E_N^{(3)}$ is controlled as in section A.1.

- Let us prove point 2 of hypothesis of theorem 1. By replacing each ξ_k by $\hat{\xi}_k^{(n_3 - n_2)} := h_{n_3 - n_2} \circ T^k$, with $h_N := 2\left(f_{N^{\frac{-7}{\zeta}}} - \mathbb{E}_\nu[f_{N^{\frac{-7}{\zeta}}}]\right)$, we make a total error in $\left(1 + \sum_{k=n_1}^{n_2} |\alpha_k| + \sum_{k=n_3}^{n_4} |\beta_k|\right)(n_3 - n_2)^{-7}$. Moreover, according to the calculations done in section A.1, we have :

$$\left|Cov\left(e^{i\sum_{k=n_1}^{n_2}\alpha_k\hat{\xi}_k^{n_3-n_2}}, e^{i\sum_{k=n_3}^{n_4}\beta_k\hat{\xi}_k^{n_3-n_2}}\right)\right| \leq$$

$$\begin{aligned}
&\leq c_0\left(1+(\sum_{k=n_1}^{n_2}|\alpha_k|+\sum_{k=n_3}^{n_4}|\beta_k|)c_0C^{(1)}_{h_{n_3-n_2}}\right)(1+(n_4-n_3)^{\beta})\alpha^{n_3-n_2}\\
&\leq C\left(1+\sum_{k=n_1}^{n_2}|\alpha_k|+\sum_{k=n_3}^{n_4}|\beta_k|\right)(1+(n_4-n_3)^{\beta})(n_3-n_2)^{\frac{7}{\zeta}}\alpha^{n_3-n_2}.
\end{aligned}$$

This gives the point 2 of the hypothesis of theorem 1 with $\varphi_{p,s} = p^{-7} + (1+s^{\beta})p^{7/\zeta}\delta^p$.

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