Rates in the strong invariance principle for ergodic automorphisms of the torus

Jérôme Dedecker [a], Florence Merlevède [b] and Françoise Pène [c] [1]

[a] Université Paris Descartes, Sorbonne Paris Cité, Laboratoire MAP5 and CNRS UMR 8145. Email: jerome.dedecker@parisdescartes.fr

[b] Université Paris Est, LAMA and CNRS UMR 8050.
E-mail: florence.merlevede@univ-mlv.fr

[c] Université de Brest, Laboratoire de Mathématiques de Bretagne Atlantique UMR CNRS 6205. E-mail: francoise.pene@univ-brest.fr

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Abstract

Let T be an ergodic automorphism of the d-dimensional torus $\mathbb{T}^d$. In the spirit of Le Borgne [10], we give conditions on the Fourier coefficients of a function f from $\mathbb{T}^d$ to $\mathbb{R}$ under which the partial sums $f\circ T+f\circ T^2+\cdots+f\circ T^n$ satisfies a strong invariance principle. Next, reinforcing the condition on the Fourier coefficients in a natural way, we obtain explicit rates of convergence in the strong invariance principle, up to $n^{1/4}\log n$.

1 Introduction

We endow the d-dimensional torus $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$ with the Lebesgue measure $\bar{\lambda}$, and we denote by $\mathbb{E}(\cdot)$ the expectation with respect to $\bar{\lambda}$. As usual, the $\mathbb{L}^p$ norm of a f from $\mathbb{T}^d$ to $\mathbb{R}$ is denoted by $\|f\|_p=(\mathbb{E}(|f|^p))^{1/p}$.

For $d\geq 2$, let T be an ergodic automorphism of $\mathbb{T}^d$, and let f be a function from $\mathbb{T}^d$ to $\mathbb{R}$ such that $\mathbb{E}(f^2)<\infty$ and $\mathbb{E}(f)=0$. In [10], Le Borgne has proved that if the Fourier coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ of f are such that, for $\theta>2$ and every integer $b>1$,

$$\sum_{|\mathbf{k}|\geq b}|c_{\mathbf{k}}|^2\leq R\log^{-\theta}(b)\,,\qquad \text{where } |\mathbf{k}|=\max_{1\leq i\leq d}|k_i|\,, \tag{1.1}$$

then the partial sums process

$$\Big\{\sum_{i=1}^{[nt]}f\circ T^i, t\in[0,1]\Big\} \tag{1.2}$$

properly normalized, satisfies both the weak and strong invariance principles. More precisely, Le Borgne has introduced in [10] an appropriate σ-field $\mathcal{F}_0$ such that $\mathcal{F}_0\subseteq T^{-1}(\mathcal{F}_0)$, for which the quantities $\|\mathbb{E}(f\circ T^k|\mathcal{F}_0)\|_2$ and $\|f\circ T^{-k}-\mathbb{E}(f\circ T^{-k}|\mathcal{F}_0)\|_2$ can be controlled for any positive integer k. The weak and strong invariance principles follow then, by applying Gordin's result (see [8]) and Heyde's result (see [9]) respectively.

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In Theorem 2.1 of this paper, we show that the weak and strong invariance principles still hold for functions f satisfying (1.1) with $\theta > 1$ only, and we give a multivariate version of these results. For the weak invariance principle, this follows from an improvement of Gordin's criterion, which was already known in the univariate case (see [7]). For the strong invariance principle, this will follow from a new criterion for stationary sequences, presented in Theorem 4.1 of the appendix. Note that the condition (1.1) with $\theta > 1$ is satisfied if, for a positive constant A,

$$|c_{\mathbf{k}}|^2 \leq A \prod_{i=1}^{d} \frac{1}{(1+|k_i|)\log^{1+\alpha}(2+|k_i|)} \quad \text{for some } \alpha > 1, \tag{1.3}$$

improving on the condition $\alpha > 2$ given by Leonov in 1969 (see [11], Remark 1). Note that Leonov has also given a condition in terms of the modulus of continuity of f in $\mathbb{L}^2$.

The strong invariance principle means that, enlarging $\mathbb{T}^d$ if necessary, there exists a sequence of independent identically distributed (iid) Gaussian random variables Z_i such that

$$\sup_{1\leq k\leq n} \Big| \sum_{i=1}^{k} f\circ T^i - \sum_{i=1}^{k} Z_i \Big| = o\big(n^{1/2}(\log\log n)^{1/2}\big) \text{ almost surely, as } n\to\infty. \tag{1.4}$$

It is also possible to exhibit rates of convergence in (1.4), provided that we reinforce the assumption (1.1). This has been done recently, thanks to a general result giving rates of convergence in the strong invariance principle for partial sums of stationary sequences. More precisely, let $p \in]2,4]$ and $q = p/(p-1)$. We have proved in Theorem 2.1 of [6] that if there exists $R > 0$ such that for every integer $b > 1$,

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q \leq R\log^{-\theta}(b) \quad \text{for some } \theta > \frac{p^2-2}{p(p-1)}\,, \tag{1.5}$$

and

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^2 \leq R\log^{-\beta}(b) \quad \text{for some } \beta > \frac{3p-4}{p}\,, \tag{1.6}$$

then the strong approximation (1.4) holds true with an error of order $o\big(n^{1/p}(\log n)^{(t+1)/2}\big)$, for $t > 2/p$. A condition on the ℓ^q norm of $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ seems appropriate in this context, since this ℓ^q-norm dominates the $\mathbb{L}^p$ norm of f, which is required to be finite to get the rate $o(n^{1/p})$ in the iid situation.

If we assume that the Fourier coefficients of f are such that,

$$|c_{\mathbf{k}}|^q \leq A \prod_{i=1}^{d} \frac{1}{(1+|k_i|)\log^{1+\alpha}(2+|k_i|)}\,, \tag{1.7}$$

then the conditions (1.5) and (1.6) are both satisfied provided that $\alpha > (p^2-2)/(p^2-p)$. Now, considering (1.3), one can wonder if $\alpha > 1$ in (1.7) is enough to get an approximation error of order $o(n^{1/p}L(n))$ in (1.4), where $L(n)$ is a slowly varying function. The main result of this paper, Theorem 2.2 below, shows that the answer is positive.

2 Invariance principles for ergodic automorphisms of the torus

Let us first recall some probabilistic notations. A measurable function $f : \mathbb{T}^d \to \mathbb{R}^m$ (with coordinates $f_1, ..., f_m$) is said to be centered if every f_i is integrable and centered. Such a function f is said to be square integrable if every f_i is square integrable. Now, for every centered and square integrable

functions $f, g : \mathbb{T}^d \to \mathbb{R}^m$ (with $f = (f_1, ..., f_m)$ and $g = (g_1, ..., g_m)$), we define the covariance matrix $\mathrm{Cov}(f, g)$ of f and g and the variance matrix $\mathrm{Var}(f)$ by

$$\mathrm{Cov}(f, g) = (\mathbb{E}(f_i g_j))_{i,j=1,\dots,m}\,, \quad \text{and} \quad \mathrm{Var}(f) = \mathrm{Cov}(f, f)\,.$$

Let us now recall some facts about ergodic automorphisms of $\mathbb{T}^d$. A group automorphism T of $\mathbb{T}^d$ is the quotient map of a linear map $\tilde{T} : \mathbb{R}^d \to \mathbb{R}^d$ given by $\tilde{T}(x) = S.x$ (. being the matrix product), where S is a $d \times d$-matrix with integer entries and with determinant ± 1. Any automorphism T of $\mathbb{T}^d$ preserves the Lebesgue measure $\bar{\lambda}$. Therefore $(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), T, \bar{\lambda})$ is a probability dynamical system (where $\mathcal{B}(\mathbb{T}^d)$ stands for the Borel σ-algebra of $\mathbb{T}^d$).

This dynamical system is ergodic if and only if no root of the unity is an eigenvalue of the matrix S associated to T. In this case, we say that T is an ergodic automorphism of $\mathbb{T}^d$.

An automorphism T of $\mathbb{T}^d$ is said to be hyperbolic if the matrix S associated to T admits no eigenvalue of modulus one. With the preceding characterization of ergodic automorphisms of $\mathbb{T}^d$, it is clear that every hyperbolic automorphism of $\mathbb{T}^d$ is ergodic. Ergodic automorphisms of $\mathbb{T}^d$ are partially hyperbolic but not necessarily hyperbolic (an example of a non-hyperbolic ergodic automorphism of $\mathbb{T}^d$ can be found in [10]).

In the next Theorem, we give weak and strong invariance principles for the partial sum process (1.2) of $\mathbb{R}^m$-valued functions.

Theorem 2.1. *Let T be an ergodic automorphism of $\mathbb{T}^d$. For any $j \in \{1, \dots, m\}$, let $f_j : \mathbb{T}^d \to \mathbb{R}$ be a centered function and assume that its Fourier coefficients $(c_{\mathbf{k},j})_{\mathbf{k}\in\mathbb{Z}^d}$ satisfy the following condition: there exists a positive constant R such that for every integer $b > 1$,*

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k},j}|^2 \leq R \log^{-\theta}(b) \quad \textit{for some } \theta > 1\,. \tag{2.1}$$

Let $f = (f_1, \dots, f_m) : \mathbb{T}^d \to \mathbb{R}^m$. Then the series $\Sigma = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(f, f \circ T^k)$ converges, and

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Var}\Big(\sum_{i=1}^{n} f \circ T^i\Big) = \Sigma\,. \tag{2.2}$$

In addition,

1. *The process $\{n^{-1/2} \sum_{i=1}^{[nt]} f \circ T^i, t \in [0, 1]\}$ converges in $D([0, 1], \mathbb{R}^m)$ equipped with the uniform topology to a Wiener process $\{W(t), t \in [0, 1]\}$ with variance matrix $\mathrm{Var}(W(1)) = \Sigma$.*
2. *Enlarging $\mathbb{T}^d$ if necessary, there exists a sequence $(Z_i)_{i\geq 1}$ of iid $\mathbb{R}^m$-valued Gaussian random variables with zero mean and variance matrix $\mathrm{Var}(Z_1) = \Sigma$ such that*

$$\sup_{1\leq k\leq n} \Big| \sum_{i=1}^{k} f \circ T^i - \sum_{i=1}^{k} Z_i \Big| = o\big(n^{1/2} (\log\log n)^{1/2}\big) \textit{ almost surely, as } n \to \infty.$$

When $m = 1$, it is also possible to exhibit rates of convergence in (1.4) provided that we reinforce Condition (2.1).

Theorem 2.2. *Let T be an ergodic automorphism of $\mathbb{T}^d$. Let $p \in]2, 4]$ and $q := p/(p - 1)$. Let $f : \mathbb{T}^d \to \mathbb{R}$ be a centered function with Fourier coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ satisfying the following conditions: there exists a positive constant R such that for every integer $b > 1$,*

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q \leq R \log^{-\theta}(b) \quad \textit{for some } \theta > 1\,, \tag{2.3}$$

and

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^2 \leq R\, b^{-\zeta} \quad \textit{for some } \zeta > 0\,. \tag{2.4}$$

Then the series

$$\sigma^2 = \sum_{k\in\mathbb{Z}} \mathbb{E}(f.f\circ T^k) \tag{2.5}$$

converges absolutely and, enlarging $\mathbb{T}^d$ *if necessary, there exists a sequence* $(Z_i)_{i\geq 1}$ *of iid Gaussian random variables with zero mean and variance* σ^2 *such that*

$$\sup_{1\leq k\leq n} \Big| \sum_{i=1}^k f\circ T^i - \sum_{i=1}^k Z_i \Big| = o\big(n^{1/p}\log n\big) \textit{ almost surely, as } n\to\infty. \tag{2.6}$$

Observe that if (1.7) holds with $\alpha > 1$ then (2.3) and (2.4) are both satisfied, so that the strong approximation (2.6) holds. However Theorem 2.1 in [6] and Theorem 2.2 above have different ranges of applicability. Indeed, let $\gamma > 1$, and define $c_{\mathbf{k}} = \ell^{-\gamma/q}$ if $\mathbf{k} = (2^\ell, 0, \ldots, 0)$, $c_{\mathbf{k}} = -\ell^{-\gamma/q}$ if $\mathbf{k} = (-2^\ell, 0, \ldots, 0)$ for $\ell \in \mathbb{N}$, and $c_{\mathbf{k}} = 0$ otherwise. Let now b and r be positive integers such that $2^{r-1} < b \leq 2^r$. Since

$$\sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q = 2\sum_{\ell\geq r} \frac{1}{\ell^\gamma}\,,$$

it follows that $\lambda_1(\log b)^{1-\gamma} \leq \sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^q \leq \lambda_2(\log b)^{1-\gamma}$ (where λ_1 and λ_2 are two positive constants). Similarly $\lambda_1(\log b)^{1-2\gamma/q} \leq \sum_{|\mathbf{k}|\geq b} |c_{\mathbf{k}}|^2 \leq \lambda_2(\log b)^{1-2\gamma/q}$. In this situation, the conditions (1.5) and (1.6) are both satisfied provided that $\gamma > 1 + (p^2-2)/(p^2-p)$ whereas condition (2.4) fails.

To prove Theorem 2.2, we shall still use martingale approximations as done in [6], but with the following modifications: Condition (2.4) allows us to consider a non stationary sequence $X_\ell^* = f_\ell \circ T^\ell$, where the functions f_ℓ are defined through a truncated series of the Fourier coefficients of f. For the partial sums associated to this non stationary sequence, the approximation error by a non stationary martingale can be suitably handled with the help of Condition (2.3).

3 Proofs of Theorems 2.1 and 2.2

As in [6], we consider the filtration as defined in [12, 10] that enables to suitably approximate the partial sums $\sum_{i=1}^n f\circ T^i$ by a martingale. To be more precise, given a finite partition $\mathcal{P}$ of $\mathbb{T}^d$, we define the measurable partition $\mathcal{P}_0^\infty$ by :

$$\forall \bar{x}\in\mathbb{T}^d,\quad \mathcal{P}_0^\infty(\bar{x}) := \bigcap_{k\geq 0} T^k\mathcal{P}(T^{-k}(\bar{x}))$$

and, for every integer n, the σ-algebra $\mathcal{F}_n$ generated by

$$\forall \bar{x}\in\mathbb{T}^d,\quad \mathcal{P}_{-n}^\infty(\bar{x}) := \bigcap_{k\geq -n} T^k\mathcal{P}(T^{-k}(\bar{x})) = T^{-n}(\mathcal{P}_0^\infty(T^n(\bar{x}))\,.$$

These definitions coincide with the ones of [10] applied to the ergodic toral automorphism T^{-1}. We obviously have $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} = T^{-1}\mathcal{F}_n$. Note that the sequence $(f\circ T^i)_{i\geq 1}$ is non adapted to $(\mathcal{F}_i)_{i\geq 1}$.

In what follows, we use the notation $\mathbb{E}_n(f) = \mathbb{E}(f|\mathcal{F}_n)$.

3.1 Proof of Theorem 2.1

According to Theorem 4.1 and Remark 4.3 given in Appendix, it suffices to verify that condition (4.35) is satisfied. Therefore, it suffices to verify that for any $j \in \{1, \dots, m\}$,

$$\sum_{n\geq 3} \frac{\log n}{n^{1/2}(\log\log n)^{1/2}} \|\mathbb{E}_0(f_j \circ T^n)\|_2 < \infty \text{ and } \sum_{n\geq 3} \frac{\log n}{n^{1/2}(\log\log n)^{1/2}} \|f_j - \mathbb{E}_n(f_j)\|_2 < \infty \,. \tag{3.1}$$

But, according to the proof of Propositions 4.2 and 4.3 of [6] (see also [10]), for any f_j satisfying (2.1),

$$\|\mathbb{E}_{-n}(f_j)\|_2 + \|f - \mathbb{E}_n(f_j)\|_2 \ll n^{-\theta/2} \,.$$

Since $\theta > 1$, (3.1) is satisfied. □

3.2 Proof of Theorem 2.2

Let $f : \mathbb{T}^d \to \mathbb{R}$ be a centered function with Fourier coefficients $(c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$. For every nonnegative integer m, we write

$$f_m := \sum_{|k|\leq m} c_k e^{2i\pi\langle k,\cdot\rangle} \,. \tag{3.2}$$

Notice that if f satisfies (2.4), then

$$\|f - f_m\|_2 \leq R m^{-\zeta/2} \,, \tag{3.3}$$

and if f satisfies (2.3), then

$$\|f - f_m\|_p \leq R(\log(m))^{-\theta(p-1)/p} \,. \tag{3.4}$$

According to the proofs of Propositions 4.2, 4.3 and 4.4 of [6], there exist $c \geq 1$ and $\gamma, \lambda \in (0,1)$ such that, setting $b(n) := [\gamma^{-n}]$, we have

$$\sup_{m\leq b(n)} (\|\mathbb{E}_{-n}(f_m)\|_p + \|f_m - \mathbb{E}_n(f_m)\|_p) \ll \lambda^n \tag{3.5}$$

(according to (4.50), (4.51) and (4.53) of [6]), and

$$\sup_{N\geq cn} \sup_{m\leq b(n)} \sup_{\ell\in\{0,\dots,n\}} \|\mathbb{E}_{-N}(f_m f_m \circ T^\ell) - \mathbb{E}(f_m f_m \circ T^\ell)\|_{p/2} \ll \lambda^n \tag{3.6}$$

(according to (4.61) and (4.62) of [6]). Moreover, according to the proof of Propositions 4.2 and 4.3 of [6], we have, for any f satisfying (2.3),

$$\sup_{m\geq 1} (\|\mathbb{E}_{-n}(f_m)\|_p + \|f_m - \mathbb{E}_n(f_m)\|_p) \ll n^{-\theta(p-1)/p} \,, \tag{3.7}$$

and

$$\|\mathbb{E}_{-n}(f)\|_p + \|f - \mathbb{E}_n(f)\|_p \ll n^{-\theta(p-1)/p} \,. \tag{3.8}$$

In addition, according to the proof of Proposition 4.4 of [6], there exists a positive integer c, such that for any f satisfying (2.3) and (2.4),

$$\max_{1\leq k\leq n} \|\mathbb{E}_{-nc}(S_k^2(f)) - \mathbb{E}(S_k^2(f))\|_{p/2} \ll n^{2-2\theta(p-1)/p} \,. \tag{3.9}$$

For any f satisfying (2.4), using the arguments developed in the proofs of Propositions 4.2 and 4.3 of [6], we infer that there exists $\beta \in (0,1)$ such that

$$\sup_{m\geq 1} (\|\mathbb{E}_{-n}(f_m)\|_2 + \|f_m - \mathbb{E}_n(f_m)\|_2) \ll \beta^n \,, \tag{3.10}$$

and

$$\|\mathbb{E}_{-n}(f)\|_2 + \|f - \mathbb{E}_n(f)\|_2 \ll \beta^n \,. \tag{3.11}$$

Let us write $P_\ell(\cdot) = \mathbb{E}_\ell(\cdot) - \mathbb{E}_{\ell-1}(\cdot)$. Now, let α be a positive real such that $\alpha\zeta \geq 3 - 2/p$. We then define

$$d_1^* := \sum_{k\in\mathbb{Z}} P_1\left(f_1 \circ T^k\right) \ , \ X_1^* := f_1 \circ T \,,$$

and, for every $j \geq 0$ and every $\ell \in \{2^j + 1, ..., 2^{j+1}\}$,

$$d_\ell^* := \sum_{k\in\mathbb{Z}} P_\ell\left(f_{[2^{\alpha j}]} \circ T^k\right) \ , \ X_\ell^* := f_{[2^{\alpha j}]} \circ T^\ell .$$

For every positive integer n, we define

$$M_n^*(f) := \sum_{\ell=1}^n d_\ell^* \ \text{ and } \ S_n^*(f) := \sum_{\ell=1}^n X_\ell^* \,.$$

The conclusion of Theorem 2.2 comes from the three following lemmas.

Lemma 3.1. *We have* $|S_n(f) - S_n^*(f)| = o(n^{1/p}(\log n))$ *almost surely.*

Lemma 3.2. *We have* $|S_n^*(f) - M_n^*(f)| = o(n^{1/p}(\log n))$ *almost surely.*

Lemma 3.3. *The conclusion of Theorem 2.2 holds with* $M_n^*(f)$ *replacing* $S_n(f)$.

Proof of Lemma 3.1. For any nonnegative integer j, let

$$D_j := \sup_{1\leq k\leq 2^j} \Big| \sum_{\ell=2^j+1}^{k+2^j} (X_\ell - X_\ell^*) \Big| .$$

Let $N \in \mathbb{N}^*$ and let $k \in]1, 2^N]$. We first notice that $D_j \geq |\sum_{\ell=2^j+1}^{2^{j+1}} (X_\ell - X_\ell^*)|$, so if K is the integer such that $2^{K-1} < k \leq 2^K$, then

$$\left|S_k - S_k^*\right| \leq |X_1 - X_1^*| + \sum_{j=0}^{K-1} D_j \,.$$

Consequently, since $K \leq N$,

$$\max_{1\leq k\leq 2^N} |S_k - S_k^*| \leq |X_1 - X_1^*| + \sum_{j=0}^{N-1} D_j \,. \tag{3.12}$$

Therefore, by standard arguments, Lemma 3.1 will follow if we can prove that $D_j = o\left(j\, 2^{j/p}\right)$ almost surely. This will hold true as soon as

$$\sum_{j\geq 1} \frac{\|D_j\|_q^q}{2^{jq/p}\, j^q} < \infty \ \text{ for some } q \in [1, p] \,. \tag{3.13}$$

We shall verify (3.13) for $q = 2$. Notice that

$$\|D_j\|_2 = \Big\| \sup_{1\leq k\leq 2^j} \Big| \sum_{\ell=2^j+1}^{k+2^j} (f - f_{[2^{\alpha j}]}) \circ T^\ell \Big| \Big\|_2 \leq 2^j \|f - f_{[2^{\alpha j}]}\|_2 \,.$$

Hence, by using (2.4), $\|D_j\|_2^2 \ll 2^{2j}2^{-\zeta\alpha j}$, which together with the fact that $\alpha\zeta \geq 2-2/p$ implies (3.13) with $q=2$, and then Lemma 3.1. $\square$

Proof of Lemma 3.2. Without loss of generality, we assume that $\theta < (p^2-2)/(p(p-1))$. Following the beginning of the proof of Lemma 3.1, Lemma 3.2 will be proven if (3.13) holds with D_j defined by

$$D_j = \sup_{1\leq k\leq 2^j} \Big| \sum_{\ell=2^j+1}^{k+2^j} (X_\ell^* - d_\ell^*) \Big| . \tag{3.14}$$

With this aim, setting, for every $k \in \{1, ..., 2^j\}$,

$$R_{j,k} = \sum_{\ell=1}^{k} \Big(f_{[2^{\alpha j}]} \circ T^\ell - \sum_{m\in\mathbb{Z}} P_\ell \left(f_{[2^{\alpha j}]} \circ T^m \right) \Big) = \sum_{\ell=2^j+1}^{k+2^j} (X_\ell^* - d_\ell^*) \circ T^{-2^j} ,$$

we first observe that

$$\|D_j\|_p = \Big\| \sup_{1\leq k\leq 2^j} |R_{j,k}| \Big\|_p \ll 2^{j/p} \sum_{k=0}^{j} 2^{-k/p} \|R_{j,2^k}\|_p , \tag{3.15}$$

(where for the inequality we have used inequality (6) in [15]). Now, according to the proof of Proposition 5.1 in [6] with $X_\ell = f_{[2^{\alpha j}]} \circ T^\ell$ and using again stationarity, we get that for any integer $k \geq 0$ and any integer $N \geq 2^k$,

$$\begin{aligned} \max_{1\leq m\leq 2^k} \|R_{j,m}\|_p \ll & \sum_{\ell=1}^{N} \|\mathbb{E}_{-\ell}(f_{[2^{\alpha j}]})\|_p + \sum_{\ell=0}^{N-1} \|f_{[2^{\alpha j}]} - \mathbb{E}_\ell(f_{[2^{\alpha j}]})\|_p \\ & + \Big(\sum_{m=1}^{2^k} \Big\| \sum_{\ell\geq m+N} P_{-\ell}(f_{[2^{\alpha j}]}) \circ T^\ell \Big\|_p^2 \Big)^{1/2} + \Big(\sum_{m=1}^{2^k} \Big\| \sum_{\ell\geq m+N} P_\ell(f_{[2^{\alpha j}]}) \circ T^{-\ell} \Big\|_p^2 \Big)^{1/2} . \end{aligned} \tag{3.16}$$

Let us first consider the case where $[2^{\alpha j}] \leq b(2^k)$. Starting from (3.16) with $N = 2^k$ and using the fact that $\|P_{-\ell}(f_{[2^{\alpha j}]})\|_p \leq 2\|\mathbb{E}_{-\ell}(f_{[2^{\alpha j}]})\|_p$ and that $\|P_\ell(f_{[2^{\alpha j}]})\|_p \leq \|f_{[2^{\alpha j}]} - \mathbb{E}_{\ell-1}(f_{[2^{\alpha j}]})\|_p$, we get that

$$\begin{aligned} \max_{1\leq m\leq 2^k} \|R_{j,m}\|_p \ll & \sum_{\ell=1}^{2^k} \|\mathbb{E}_{-\ell}(f_{[2^{\alpha j}]})\|_p + \sum_{\ell=0}^{2^k-1} \|f_{[2^{\alpha j}]} - \mathbb{E}_\ell(f_{[2^{\alpha j}]})\|_p \\ & + 2^{k/2} \sum_{\ell\geq 2^k+1} \|\mathbb{E}_{-\ell}(f_{[2^{\alpha j}]})\|_p + 2^{k/2} \sum_{\ell\geq 2^k} \|f_{[2^{\alpha j}]} - \mathbb{E}_\ell(f_{[2^{\alpha j}]})\|_p . \end{aligned}$$

Therefore, taking into account the upper bound (3.7) for the two first terms in the right hand side, and the upper bound (3.5) to handle the two last terms (since $[2^{\alpha j}] \leq b(2^k)$), we derive that

$$\max_{1\leq m\leq 2^k} \|R_{j,m}\|_p \ll 2^{k(1-\theta(p-1)/p)} \tag{3.17}$$

(recall that $\theta(p-1)/p < 1$). On the other hand, starting from (3.16) with $N = 2[2^{kp/2}]$ and using Lemma 5.1 in [6], we get that

$$\begin{aligned} \max_{1\leq m\leq 2^k} \|R_{j,m}\|_p \ll & \sum_{\ell=1}^{2[2^{kp/2}]} \|\mathbb{E}_{-\ell}(f_{[2^{\alpha j}]})\|_p + \sum_{\ell=0}^{2[2^{kp/2}]} \|f_{[2^{\alpha j}]} - \mathbb{E}_\ell(f_{[2^{\alpha j}]})\|_p \\ & + 2^{k/2} \sum_{\ell\geq [2^{kp/2}]} \frac{\|\mathbb{E}_{-\ell}(f_{[2^{\alpha j}]})\|_p}{\ell^{1/p}} + 2^{k/2} \sum_{\ell\geq [2^{kp/2}]} \frac{\|f_{[2^{\alpha j}]} - \mathbb{E}_\ell(f_{[2^{\alpha j}]})\|_p}{\ell^{1/p}} . \end{aligned}$$

Therefore, it follows from (3.7) that

$$\max_{1\le m\le 2^k} \|R_{j,m}\|_p \ll 2^{(kp(1-\theta(p-1)/p))/2}\,. \tag{3.18}$$

Let

$$C = [\alpha^{-1}(\log(\gamma^{-1}))/(\log 2)] \text{ and } j_0 = (\log 2)^{-1}(\log j - \log C)\,. \tag{3.19}$$

Clearly, if $j_0 \le k$ then $[2^{\alpha j}] \le b(2^k)$. Therefore using the upper bound (3.18) when $k < j_0$ and the upper bound (3.17) when $k \ge j_0$, we get that for any positive integer j

$$\sum_{k=0}^{j} 2^{-k/p}\|R_{j,2^k}\|_p \ll \frac{j^{p/2}}{j^{1/p}j^{\theta(p-1)/2}}\,,$$

since $\theta < (p^2-2)/(p(p-1))$. Now, since $\theta > 1$, it follows that

$$\sum_{j\ge 1} j^{-p}\Big(\sum_{k=0}^{j} 2^{-k/p}\|R_{j,2^k}\|_p\Big)^p < \infty\,.$$

From (3.15), this implies that (3.13) holds with D_j defined by (3.14) and $q = p$. This ends the proof of lemma 3.2. □

Proof of Lemma 3.3. Let $M_n = \sum_{\ell=1}^n d_\ell$ where $d_\ell = d_0 \circ T^\ell$ with $d_0 = \sum_{i\in\mathbb{Z}} P_0(f\circ T^i)$. Notice that the upper bound (3.8) and the fact that $\theta > 1$ imply in particular that

$$\sum_{n>1} n^{-1/p}\|\mathbb{E}_{-n}(f)\|_p < \infty \text{ and } \sum_{n>1} n^{-1/p}\|f - \mathbb{E}_n(f)\|_p < \infty\,,$$

and then that $\sum_{k\in\mathbb{Z}} \|P_0(X_k)\|_p < \infty$ (use for instance Lemma 5.1 in [6] to see this). Therefore $\sum_{k\in\mathbb{Z}} \|P_0(X_k)\|_2 < \infty$. Using (4.2) of Lemma 4.1, we get that

$$\|S_n(f) - M_n\|_2 = o(\sqrt{n})\,. \tag{3.20}$$

From (2.2) of Theorem 2.1, we know that $n^{-1}\|S_n(f)\|_2^2$ converges to σ^2. It follows from (3.20) that $\sigma^2 = n^{-1}\mathbb{E}(M_n^2) = \mathbb{E}(d_0^2)$.

We shall prove now that

$$\|M_n^* - M_n\|_2 = O\big(n^{1/p}\big)\,. \tag{3.21}$$

Let N be the positive integer such that $2^{N-1} < n \le 2^N$. Since $M_n^* - M_n$, is a martingale, we have that

$$\|M_n^* - M_n\|_2^2 = \sum_{\ell=1}^{n} \mathbb{E}((d_\ell^* - d_\ell)^2) \le \mathbb{E}((d_1^* - d_1)^2) + \sum_{j=0}^{N-1}\sum_{\ell=2^j+1}^{2^{j+1}} \mathbb{E}((d_\ell^* - d_\ell)^2)\,. \tag{3.22}$$

By stationarity, for any $\ell \in [2^j+1, 2^{j+1}] \cap \mathbb{N}$ we get that

$$\begin{aligned}\|d_\ell^* - d_\ell\|_2 &= \|\sum_{i\in\mathbb{Z}} P_0((f - f_{[2^{\alpha j}]})\circ T^i)\|_2 \\ &\le 2^{j+3}\|f - f_{[2^{\alpha j}]}\|_2 + \sum_{i\ge 2^{j+1}} \|P_{-i}(f - f_{[2^{\alpha j}]})\|_2 + \sum_{i\ge 2^{j+1}} \|P_i(f - f_{[2^{\alpha j}]})\|_2\,.\end{aligned}$$

According to (3.3)

$$\|f - f_{[2^{\alpha j}]}\|_2 \le R2^{-\zeta\alpha j/2}\,. \tag{3.23}$$

On the other hand, by Lemma 5.1 in [6],

$$\sum_{i\geq 2^{j+1}} \|P_{-i}(f-f_{[2^{\alpha j}]})\|_2 \ll \sum_{k\geq 2^j} k^{-1/2}\|\mathbb{E}_{-k}(f-f_{[2^{\alpha j}]})\|_2$$

and

$$\sum_{i\geq 2^{j+1}} \|P_{i}(f-f_{[2^{\alpha j}]})\|_2 \ll \sum_{k\geq 2^j} k^{-1/2}\|f-f_{[2^{\alpha j}]}+\mathbb{E}_{k}(f-f_{[2^{\alpha j}]})\|_2\,.$$

Using the estimate (3.10) and (3.11), it follows that

$$\sum_{i\geq 2^{j+1}} \big(\|P_{-i}(f-f_{[2^{\alpha j}]})\|_2+\|P_{-i}(f-f_{[2^{\alpha j}]})\|_2\big) = O(\beta^{2^j})\,. \tag{3.24}$$

Combining the upper bounds (3.23) and (3.24) with the fact that $\alpha\zeta \geq 3-2/p$, it follows that

$$\|d_\ell^*-d_\ell\|_2 \ll 2^{-j(p-2)/(2p)}\,.$$

Using this estimate in (3.22), we obtain that $\|M_n^*-M_n\|_2^2 \ll n^{2/p}$, proving (3.21).

Now, let us recall Theorem 2.1 in [13] (used with $a_n = n^{2/p}(\log n)$): if there exists a finite constant K such that

$$\sup_{k\geq 1}\|d_k^*\|_p \leq K\,, \tag{3.25}$$

and if

$$\sum_{i=1}^n \big(\mathbb{E}((d_i^*)^2|\mathcal{F}_{i-1})-\mathbb{E}((d_i^*)^2)\big) = o\big(n^{2/p}(\log n)\big) \quad a.s.\,, \tag{3.26}$$

then, since $\mathbb{E}((M_n^*)^2)\sim n\sigma^2$, enlarging $\mathbb{T}^d$ if necessary, there exists a sequence $(Z_\ell^*)_{\ell\geq 1}$ of independent Gaussian random variables with zero mean and variance $\mathbb{E}(Z_\ell^*)^2 = \mathbb{E}(d_\ell^*)^2 = (\sigma_\ell^*)^2$ such that

$$\sup_{1\leq k\leq n}\Big|M_k^*-\sum_{\ell=1}^k Z_\ell^*\Big| = o\big(n^{1/p}(\log n)\big) \text{ almost surely, as } n\to\infty. \tag{3.27}$$

Let $(\delta_k)_{k\geq 1}$ be a sequence of iid Gaussian random variables with mean zero and variance σ^2, independent of the sequence $(Z_\ell^*)_{\ell\geq 1}$. We now construct a sequence $(Z_\ell)_{\ell\geq 1}$ as follows. If $\sigma_\ell^*=0$, then $Z_\ell=\delta_\ell$, else $Z_\ell = (\sigma/\sigma_\ell^*)Z_\ell^*$. By construction, the Z_ℓ's are iid Gaussian random variables with mean zero and variance σ^2. Let $G_\ell = Z_\ell - Z_\ell^*$ and note that $(G_\ell)_{\ell\geq 1}$ is a sequence of independent Gaussian random variables with mean zero and variances $\mathrm{Var}(G_\ell) = (\sigma-\sigma_\ell^*)^2$. Notice now that

$$v_n^2 = \mathrm{Var}\Big(\sum_{i=1}^n G_i\Big) = \sum_{i=1}^n \big(\|d_i\|_2-\|d_i^*\|_2\big)^2 \leq \|M_n-M_n^*\|_2^2\,.$$

From the basic inequality

$$\mathbb{P}\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k G_i\Big| > x\Big) \leq 2\exp\Big(-\frac{x^2}{2v_n^2}\Big)\,,$$

and the fact that by (3.21), $v_n^2 \ll n^{2/p}$, it follows that for any $\varepsilon>0$,

$$\sum_{n>1} n^{-1}\mathbb{P}\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^k G_i\Big| > \varepsilon n^{1/p}(\log n)\Big) < \infty\,,$$

showing that $\max_{1\le k\le n}|\sum_{i=1}^k G_i| = o(n^{1/p}(\log n))$ almost surely. Therefore starting from (3.27), we conclude that if (3.25) and (3.26) hold then Lemma 3.3 does; namely, enlarging $\mathbb{T}^d$ if necessary, there exists a sequence $(Z_\ell)_{\ell\ge1}$ of iid Gaussian random variables with zero mean and variance σ^2 such that

$$\sup_{1\le k\le n}\Big|M_k^* - \sum_{\ell=1}^k Z_\ell\Big| = o(n^{1/p}(\log n)) \quad \text{almost surely, as } n\to\infty.$$

It remains to show that (3.25) and (3.26) are satisfied. We start with (3.25). Notice that $\|d_1^*\|_p \le \sum_{k\in\mathbb{Z}}\|P_{-k}(f_1)\|_p$ and that, for every $j\ge0$ and every $\ell\in\{2^j+1,...,2^{j+1}\}$,

$$\|d_\ell^*\|_p \le \sum_{k\in\mathbb{Z}}\|P_{-k}(f_{[2^{\alpha j}]})\|_p\,.$$

By Lemma 5.1 in [6],

$$\sum_{k\in\mathbb{Z}}\|P_{-k}(g)\|_p \ll \sum_{k\ge1}k^{-1/p}\|\mathbb{E}_{-k}(g)\|_p + \sum_{k\ge1}k^{-1/p}\|g-\mathbb{E}_k(g)\|_p\,,$$

with constants non depending on g. Hence, using the estimate (3.7) and the fact that $\theta>1$, we get that, for every $j\ge0$ and every $\ell\in\{2^j+1,...,2^{j+1}\}$, there exists a constant K non depending on j such that $\|d_\ell^*\|_p\le K$. This ends the proof of (3.25).

To prove (3.26), we proceed as follows. Following the beginning of the proof of Lemma 3.1, we infer that (3.26) will be proven if we can show that

$$D_j := \sup_{1\le\ell\le2^j}\Big|\sum_{i=2^j+1}^{2^j+\ell}(\mathbb{E}((d_i^*)^2|\mathcal{F}_{i-1})-\mathbb{E}((d_i^*)^2))\Big| = o\big(j\,2^{2j/p}\big)\quad a.s.\,. \tag{3.28}$$

This will hold true as soon as

$$\sum_{j\ge1}\frac{\|D_j\|_{p/2}^{p/2}}{2^j\,j^{p/2}}<\infty\,. \tag{3.29}$$

For any j fixed and any $i\in\mathbb{Z}$, let $d_{j,i}=\sum_{k\in\mathbb{Z}}P_i\left(f_{[2^{\alpha j}]}\circ T^k\right)$. By stationarity

$$\|D_j\|_{p/2} := \Big\|\sup_{1\le\ell\le2^j}\Big|\sum_{i=1}^{\ell}(\mathbb{E}(d_{j,i}^2|\mathcal{F}_{i-1})-\mathbb{E}(d_{j,i}^2))\Big|\Big\|_{p/2}\,.$$

Observe now that, for any j fixed, $(d_{j,i})_{i\in\mathbb{Z}}$ is a stationary sequence of martingale differences in $\mathbb{L}^p$. Let

$$M_{j,n} := \sum_{i=1}^n d_{j,i}\,.$$

Applying Theorem 3 in [16] (since $1<p/2\le2$) and using the martingale property of the sequence $(M_{j,n})_{n\ge1}$, we get that

$$\mathbb{E}\Big(\sup_{1\le\ell\le2^j}\Big|\sum_{i=1}^{\ell}(\mathbb{E}(d_{j,i}^2|\mathcal{F}_{i-1})-\mathbb{E}(d_{j,i}^2))\Big|^{p/2}\Big) \ll 2^j\|d_{j,1}^2\|_{p/2}^{p/2}+2^j\left(\sum_{k=0}^{j-1}\frac{\|\mathbb{E}(M_{j,2^k}^2|\mathcal{F}_0)-\mathbb{E}(M_{j,2^k}^2)\|_{p/2}}{2^{2k/p}}\right)^{p/2}.$$

Using the fact that $\|d_{j,1}^2\|_{p/2}=\|d_{j,1}\|_p^2\le K$ where K does not depend on j, the convergence (3.29) will be then proven if we can show that

$$\sum_{j\ge1}\frac{1}{j^{p/2}}\Big(\sum_{k=0}^{j-1}2^{-2k/p}\|\mathbb{E}_0(M_{j,2^k}^2)-\mathbb{E}(M_{j,2^k}^2)\|_{p/2}\Big)^{p/2}<\infty\,.$$

According to the arguments developed in the proof of Theorems 3.1 and 3.2 in [6] (see (3.19) and (3.20) of [6]), since $(M_{j,k})_{k\geq 1}$ is a sequence of martingales, we infer that this last convergence will be satisfied as soon as there exists a positive integer c such that

$$\sum_{j\geq 1}\frac{1}{j^{p/2}}\Big(\sum_{k=0}^{j}2^{-2k/p}\big\|\mathbb{E}_{-c2^k}(M_{j,2^k}^2)-\mathbb{E}(M_{j,2^k}^2)\big\|_{p/2}\Big)^{p/2}<\infty\,. \tag{3.30}$$

We shall prove in what follows that this convergence holds as soon as c is chosen in such a way that (3.9) holds true.

For any positive integer n, let

$$S_{j,n}=\sum_{\ell=1}^{n}f_{[2^{\alpha j}]}\circ T^{\ell}\ \text{ and }\ R_{j,n}=S_{j,n}-M_{j,n}\,.$$

We first write that

$$\begin{aligned}\|\mathbb{E}_{-c2^k}(M_{j,2^k}^2)-\mathbb{E}(M_{j,2^k}^2)\|_{p/2}&\leq\|\mathbb{E}_{-c2^k}(S_{j,2^k}^2)-\mathbb{E}(S_{j,2^k}^2)\|_{p/2}\\&\quad+4\|\mathbb{E}_{-c2^k}(S_{j,2^k}R_{j,2^k})\|_{p/2}+2\|R_{j,2^k}\|_p^2\,.\end{aligned} \tag{3.31}$$

Let j_0 be defined as in (3.19). Using the upper bound (3.18) when $k<j_0$ and the upper bound (3.17) when $k\geq j_0$, we get that for any positive integer j,

$$\sum_{k=0}^{j}2^{-2k/p}\max_{1\leq m\leq 2^k}\|R_{j,m}\|_p^2\ll\frac{j^p}{j^{2/p}j^{\theta(p-1)}}\,,$$

since we can assume without loss of generality that $\theta<(p^2-1)/(p(p-1))$. Now, since $\theta>1$, it follows that

$$\sum_{j\geq 1}\frac{1}{j^{p/2}}\Big(\sum_{k=0}^{j}2^{-2k/p}\max_{1\leq m\leq 2^k}\|R_{j,m}\|_p^2\Big)^{p/2}<\infty\,. \tag{3.32}$$

On an other hand, c being chosen such that (3.9) holds true, the upper bound in (3.9) together with the fact that $\theta>1$, implies that

$$\sum_{k=0}^{j}2^{-2k/p}\max_{1\leq m\leq 2^k}\|\mathbb{E}_{-c2^k}(S_{j,m}^2)-\mathbb{E}(S_{j,m}^2)\big\|_{p/2}\ll\sum_{k=0}^{j}2^{-2k/p}2^{2k(1-\theta(p-1)/p)}=O(1)\,.$$

Therefore,

$$\sum_{j\geq 1}\frac{1}{j^{p/2}}\Big(\sum_{k=0}^{j}2^{-2k/p}\max_{1\leq m\leq 2^k}\|\mathbb{E}_{-c2^k}(S_{j,m}^2)-\mathbb{E}(S_{j,m}^2)\big\|_{p/2}\Big)^{p/2}<\infty\,. \tag{3.33}$$

Starting from (3.31), and taking into account (3.32) and (3.33), we then infer that (3.30) will hold true if we can show that

$$\sum_{j\geq 1}\frac{1}{j^{p/2}}\Big(\sum_{k=0}^{j}2^{-2k/p}\|\mathbb{E}_{-c2^k}(S_{j,2^k}R_{j,2^k})\big\|_{p/2}\Big)^{p/2}<\infty\,. \tag{3.34}$$

With this aim, we use Inequality (3.24) in [6] (taking $n=2^k$, $u_n=[2^{k/2}]$, $r=c2^k$). Therefore,

$$\begin{aligned}\|\mathbb{E}_{-c2^k}(S_{j,2^k}R_{j,2^k})\big\|_{p/2}&\ll 2^{k/4}\big(\|\mathbb{E}_0(S_{j,2^k})\|_2+\|S_{j,2^k}-\mathbb{E}_{2^k}(S_{j,2^k})\|_2\big)\\&\quad+\max_{m=\{2^k,2^k-[2^{k/2}]\}}\|R_{j,m}\|_p^2+2^{k/2}\big(\|\mathbb{E}_{-[2^{k/2}]}(S_{j,2^k})\|_2+\|S_{j,2^k}-\mathbb{E}_{2^k+[2^{k/2}]}(S_{j,2^k})\|_2\big)\\&\quad+\max_{m=\{2^k,[2^{k/2}]\}}\|\mathbb{E}_{-c2^k}(S_{j,m}^2)-\mathbb{E}(S_{j,m}^2)\|_{p/2}+2^k\sum_{|\ell|\geq 2^k}\|P_0(f_{[2^{j\alpha}]}\circ T^{\ell})\|_2\,.\end{aligned} \tag{3.35}$$

By stationarity,

$$\|\mathbb{E}_0(S_{j,2^k})\|_2 + \|S_{j,2^k} - \mathbb{E}_{2^k}(S_{j,2^k})\|_2 \le \sum_{\ell=1}^{2^k} \|\mathbb{E}_{-\ell}(f_{[2^{j\alpha}]})\|_2 + \sum_{\ell=0}^{2^k-1} \|f_{[2^{j\alpha}]} - \mathbb{E}_\ell(f_{[2^{j\alpha}]})\|_2 \,.$$

Hence by using (3.10),

$$2^{k/4}\big(\|\mathbb{E}_0(S_{j,2^k})\|_2 + \|S_{j,2^k} - \mathbb{E}_{2^k}(S_{j,2^k})\|_2\big) \ll 2^{k/4} \,. \tag{3.36}$$

Using again (3.10) and the stationarity, we get that

$$\begin{aligned}&\|\mathbb{E}_{-[2^{k/2}]}(S_{j,2^k})\|_2 + \|S_{j,2^k} - \mathbb{E}_{2^k+[2^{k/2}]}(S_{j,2^k})\|_2 \\ &\qquad \le \sum_{\ell=1}^{2^k} \|\mathbb{E}_{-([2^{k/2}]+\ell)}(f_{[2^{j\alpha}]})\|_2 + \sum_{\ell=0}^{2^k-1} \|f_{[2^{j\alpha}]} - \mathbb{E}_{[2^{k/2}]+\ell}(f_{[2^{j\alpha}]})\|_2 \ll \beta^{2^{k/2}} \,.\end{aligned}$$

Therefore

$$2^{k/2}\big(\|\mathbb{E}_{-[2^{k/2}]}(S_{j,2^k})\|_2 + \|S_{j,2^k} - \mathbb{E}_{2^k+[2^{k/2}]}(S_{j,2^k})\|_2\big) = O(1) \,. \tag{3.37}$$

On an other hand, using again (3.10) and the stationarity,

$$\sum_{|\ell|\ge 2^k} \|P_0(f_{[2^{j\alpha}]} \circ T^\ell)\|_2 \le \sum_{\ell \ge 2^k} \big(\|\mathbb{E}_{-\ell}(f_{[2^{j\alpha}]})\|_2 + \|f_{[2^{j\alpha}]} - \mathbb{E}_{\ell-1}(f_{[2^{j\alpha}]})\|_2\big) \ll \beta^{2^k} \,,$$

which implies that

$$2^k \sum_{|\ell|\ge 2^k} \|P_0(f_{[2^{j\alpha}]} \circ T^\ell)\|_2 = O(1) \,. \tag{3.38}$$

Starting from (3.35) and taking into account the convergence (3.32) and (3.33), and the upper bounds (3.36), (3.37) and (3.38), we then derive that (3.34) holds. This ends the proof of (3.26) and therefore of Lemma 3.3. □

4 Appendix

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. Let us denote by $|\cdot|_m$ the euclidean norm on $\mathbb{R}^m$ and by $<\cdot,\cdot>_m$ the associated scalar product. For a σ-algebra $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, we define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let $\mathcal{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \mathcal{F}_k$ and $\mathcal{F}_\infty = \bigvee_{k\in\mathbb{Z}} \mathcal{F}_k$. For a random variable X with values in $\mathbb{R}^m$, we denote by $\|X\|_{p,m} = (\mathbb{E}(|X|_m^p))^{1/p}$ its norm in $\mathbb{L}^p(\mathbb{R}^m)$.

In what follows X_0 is a random variable with values in $\mathbb{R}^m$, and we define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. We shall use the notations $\mathbb{E}_k(X) = \mathbb{E}(X|\mathcal{F}_k)$, $\mathbb{E}_\infty(X) = \mathbb{E}(X|\mathcal{F}_\infty)$, $\mathbb{E}_{-\infty}(X) = \mathbb{E}(X|\mathcal{F}_{-\infty})$, and $P_k(X) = \mathbb{E}_k(X) - \mathbb{E}_{k-1}(X)$.

The aim of this section is to collect some results about invariance principles for stationary sequences that are non necessarily adapted to the underlying filtration. We start with a martingale approximation result. The estimate (4.2) of Proposition 4.1 below is a generalization of Item 2 of Theorem 1 in [15] to the multidimensional case and to the case where the variables are non necessarily adapted to the filtration under consideration. The convergence (4.3) is new. Notice that the proof of the next lemma is based on algebraic computations and on Burkholder's inequality. Burkholder's inequality being also valid in Hilbert spaces (see [2]), the approximation lemma below is then also valid for variables taking values in a separable Hilbert space, $\mathcal{H}$, by replacing the norm $|\cdot|_m$ by the norm on $\mathcal{H}$, let say $|\cdot|_{\mathcal{H}}$.

Proposition 4.1. *Let $p \in [1,\infty[$ and $p' = \min(2,p)$. Assume that $\mathbb{E}_{-\infty}(X_0) = 0$ almost surely, that $\mathbb{E}_{\infty}(X_0) = X_0$ almost surely, and that*

$$\sum_{i\in\mathbb{Z}} \|P_0(X_i)\|_{p,m} < \infty\,. \tag{4.1}$$

Let $d_0 = \sum_{i\in\mathbb{Z}} P_0(X_i)$, $M_n := \sum_{i=1}^n d_0 \circ T^i$ and $R_n := S_n - M_n$. For any positive integer n,

$$\|R_n\|_{p,m}^{p'} \ll \sum_{k=1}^n \Big(\sum_{|\ell|\geq k} \|P_\ell(X_0)\|_{p,m} \Big)^{p'}\,. \tag{4.2}$$

In addition,

$$\Big\| \max_{1\leq k\leq n} |R_k|_m \Big\|_p = o(n^{1/p'}) \quad \text{as } n\to\infty. \tag{4.3}$$

Remark 4.1. *The constant appearing in (4.2) depends only on p and not on $(\Omega, \mathcal{A}, \mathbb{P}, T, X_0, \mathcal{F}_0)$.*

Proof of Proposition 4.1. It will be useful to note that $P_i(X_j) = P_{i-j}(X_0)\circ T^j = P_0(X_{j-i})\circ T^i$ almost surely. The following decomposition is valid:

$$\begin{aligned} R_n &= \sum_{k=1}^n \Big(X_k - \sum_{j=1}^n P_j(X_k) \Big) - \sum_{k=1}^n \sum_{j\geq n+1} P_k(X_j) - \sum_{k=1}^n \sum_{j=0}^\infty P_k(X_{-j}) \\ &= \mathbb{E}_0(S_n) - \sum_{k=1}^n \sum_{j\geq n+1} P_k(X_j) + S_n - \mathbb{E}_n(S_n) - \sum_{k=1}^n \sum_{j=0}^\infty P_k(X_{-j})\,. \end{aligned} \tag{4.4}$$

Applying Burkholder's inequality for multivariate martingales, and using the stationarity, we obtain that there exists a positive constant c_p such that, for any positive integer n,

$$\Big\| \sum_{k=1}^n \sum_{j\geq n+1} P_k(X_j) \Big\|_{p,m}^{p'} \leq c_p \sum_{k=1}^n \Big\| \sum_{j\geq n+1} P_k(X_j) \Big\|_{p,m}^{p'} \leq c_p \sum_{k=1}^n \Big(\sum_{j\geq k} \|P_0(X_j)\|_{p,m} \Big)^{p'}\,, \tag{4.5}$$

and

$$\Big\| \sum_{k=1}^n \sum_{j\geq 0} P_k(X_{-j}) \Big\|_{p,m}^{p'} \leq c_p \sum_{k=1}^n \Big\| \sum_{j\geq 0} P_k(X_{-j}) \Big\|_{p,m}^{p'} = c_p \sum_{k=1}^n \Big(\sum_{j\geq k} \|P_0(X_{-j})\|_{p,m} \Big)^{p'}\,. \tag{4.6}$$

On an other hand, since $\mathbb{E}_{-\infty}(X_0) = 0$ almost surely, we have $\mathbb{E}_0(S_n) = \sum_{k\geq 0} P_{-k}(S_n)$ almost surely. Hence by Burkholder's inequality for multivariate martingales together with stationarity, there exists a positive constant c_p depending only on p such that

$$\begin{aligned} \|\mathbb{E}_0(S_n)\|_{p,m}^{p'} &\leq c_p \sum_{k\geq 0} \|P_{-k}(S_n)\|_{p,m}^{p'} \leq c_p \sum_{k\geq 0} \Big(\sum_{\ell=1}^n \|P_{-k}(X_\ell)\|_{p,m} \Big)^{p'} \\ &\leq c_p \sum_{k=0}^{n-1} \Big(\sum_{\ell\geq k+1} \|P_{-\ell}(X_0)\|_{p,m} \Big)^{p'} + c_p \sum_{k\geq n} \Big(\sum_{\ell=k+1}^{k+n} \|P_{-\ell}(X_0)\|_{p,m} \Big)^{p'} \\ &\leq c_p \sum_{k=0}^{n-1} \Big(\sum_{\ell\geq k+1} \|P_{-\ell}(X_0)\|_{p,m} \Big)^{p'} + c_p \Big(\sum_{i\geq n+1} \|P_{-i}(X_0)\|_{p,m} \Big)^{p'-1} \sum_{k\geq n} \sum_{\ell=k+1}^{k+n} \|P_{-\ell}(X_0)\|_{p,m} \\ &\leq c_p \sum_{k=0}^{n-1} \Big(\sum_{\ell\geq k+1} \|P_{-\ell}(X_0)\|_{p,m} \Big)^{p'} + c_p n \Big(\sum_{\ell\geq n+1} \|P_{-\ell}(X_0)\|_{p,m} \Big)^{p'}\,. \end{aligned}$$

Therefore

$$\|\mathbb{E}_0(S_n)\|_{p,m}^{p'} \le 2c_p \sum_{k=1}^{n} \Big(\sum_{\ell \ge k} \|P_{-\ell}(X_0)\|_{p,m} \Big)^{p'} . \tag{4.7}$$

We handle now the quantity $\|S_n - \mathbb{E}_n(S_n)\|_{p,m}^{p'}$. Since $\mathbb{E}_\infty(X_0) = X_0$ almost surely, we first write that $S_n - \mathbb{E}_n(S_n) = \sum_{k \ge n+1} P_k(S_n)$. Hence, applying Burkholder's inequality for multivariate martingales and using the stationarity, we infer that there exists a positive constant c_p depending only on p such that

$$\begin{aligned}
\|S_n - \mathbb{E}_n(S_n)\|_{p,m}^{p'} &\le c_p \sum_{k \ge n+1} \|P_k(S_n)\|_{p,m}^{p'} \le c_p \sum_{k \ge n+1} \Big(\sum_{\ell=1}^{n} \|P_k(X_\ell)\|_{p,m} \Big)^{p'} \\
&\le c_p \sum_{k=n+1}^{2n} \Big(\sum_{\ell=k-n}^{k-1} \|P_\ell(X_0)\|_{p,m} \Big)^{p'} + c_p \sum_{k \ge 2n+1} \Big(\sum_{\ell=k-n}^{k-1} \|P_\ell(X_0)\|_{p,m} \Big)^{p'} \\
&\le c_p \sum_{k=1}^{n} \Big(\sum_{\ell \ge k} \|P_\ell(X_0)\|_{p,m} \Big)^{p'} + c_p \Big(\sum_{i \ge n+1} \|P_i(X_0)\|_{p,m} \Big)^{p'-1} \sum_{k \ge 2n+1} \sum_{\ell=k-n}^{k-1} \|P_\ell(X_0)\|_{p,m} \\
&\le c_p \sum_{k=1}^{n} \Big(\sum_{\ell \ge k} \|P_\ell(X_0)\|_{p,m} \Big)^{p'} + c_p n \Big(\sum_{\ell \ge n+1} \|P_\ell(X_0)\|_{p,m} \Big)^{p'} .
\end{aligned}$$

Therefore

$$\|S_n - \mathbb{E}_n(S_n)\|_{p,m}^{p'} \le 2c_p \sum_{k=1}^{n} \Big(\sum_{\ell \ge k} \|P_\ell(X_0)\|_{p,m} \Big)^{p'} . \tag{4.8}$$

Starting from (4.4) and taking into account the upper bounds (4.5), (4.6), (4.7) and (4.8), the inequality (4.2) follows.

We turn now to the proof of (4.3). Let r be some fixed positive integer. Since $M_k = \sum_{i=1}^{k} d_0 \circ T^i$ and $d_0 \circ T^i = \sum_{j \in \mathbb{Z}} P_i(X_j)$, the following decomposition holds:

$$R_k = \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} \sum_{\ell=-r-i}^{r+i} P_i(X_\ell) - \sum_{i=1}^{k} \sum_{\ell \ge r+i+1} P_i(X_\ell) - \sum_{i=1}^{k} \sum_{\ell \ge r+i+1} P_i(X_{-\ell}) . \tag{4.9}$$

Applying Burkholder's inequality for multivariate martingales and using the stationarity, we infer that there exists a positive constant c_p depending only on p such that, for any positive integer n,

$$\Big\| \max_{r \le k \le n} \Big| \sum_{i=1}^{k} \sum_{\ell \ge r+i+1} P_i(X_\ell) \Big|_m \Big\|_p^{p'} \le c_p \sum_{i=1}^{n} \Big\| \sum_{\ell \ge r+i+1} P_i(X_\ell) \Big\|_{p,m}^{p'} \le c_p n \Big(\sum_{j \ge r+1} \|P_0(X_j)\|_{p,m} \Big)^{p'} , \tag{4.10}$$

since $P_i(X_\ell) = P_0(X_{\ell-i}) \circ T^i$. Similarly

$$\Big\| \max_{r \le k \le n} \Big| \sum_{i=1}^{k} \sum_{\ell \ge r+i+1} P_i(X_{-\ell}) \Big|_m \Big\|_p^{p'} \le c_p n \Big(\sum_{j \ge r+1} \|P_0(X_{-j})\|_{p,m} \Big)^{p'} . \tag{4.11}$$

We write now that

$$\sum_{i=1}^{k} X_i - \sum_{i=1}^{k} \sum_{\ell=-r-i}^{r+i} P_i(X_\ell) = \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} \sum_{\ell=1}^{r+i} P_i(X_\ell) - \sum_{i=1}^{k} \sum_{\ell=0}^{r+i} P_i(X_{-\ell}) . \tag{4.12}$$

The following decomposition holds

$$\sum_{i=1}^{k}\sum_{\ell=1}^{r+i} P_i(X_\ell) = \sum_{i=1}^{k-r}\sum_{\ell=1}^{r+i} P_i(X_\ell) + \sum_{i=k-r+1}^{k}\sum_{\ell=1}^{r+i} P_i(X_\ell)$$
$$= \sum_{\ell=1}^{k}\sum_{i=1}^{k-r} \mathbf{1}_{i\geq \ell-r} P_i(X_\ell) + \sum_{\ell=1}^{k+r}\sum_{i=k-r+1}^{k} \mathbf{1}_{i\geq \ell-r} P_i(X_\ell)\,. \tag{4.13}$$

Now,

$$\sum_{\ell=1}^{k}\sum_{i=1}^{k-r} \mathbf{1}_{i\geq \ell-r} P_i(X_\ell) = \sum_{\ell=1}^{r}\sum_{i=1}^{k-r} P_i(X_\ell) + \sum_{\ell=r+1}^{k}\sum_{i=\ell-r}^{k-r} P_i(X_\ell)$$
$$= \mathbb{E}_{k-r}(S_r) - \mathbb{E}_0(S_r) + \mathbb{E}_{k-r}(S_k - S_r) - \sum_{\ell=r+1}^{k} \mathbb{E}_{\ell-r-1}(X_\ell)$$
$$= \mathbb{E}_{k-r}(S_k) - \mathbb{E}_0(S_r) - \sum_{\ell=r+1}^{k} \mathbb{E}_{\ell-r-1}(X_\ell)\,, \tag{4.14}$$

and

$$\sum_{\ell=1}^{k+r}\sum_{i=k-r+1}^{k} \mathbf{1}_{i\geq \ell-r} P_i(X_\ell) = \sum_{\ell=1}^{k}\sum_{i=k-r+1}^{k} P_i(X_\ell) + \sum_{\ell=k+1}^{k+r}\sum_{i=\ell-r}^{k} P_i(X_\ell)$$
$$= \mathbb{E}_k(S_k) - \mathbb{E}_{k-r}(S_k) + \mathbb{E}_k(S_{k+r} - S_k) - \sum_{\ell=k+1}^{k+r} \mathbb{E}_{\ell-r-1}(X_\ell)$$
$$= \mathbb{E}_k(S_{k+r}) - \mathbb{E}_{k-r}(S_k) - \sum_{\ell=k+1}^{k+r} \mathbb{E}_{\ell-r-1}(X_\ell)\,. \tag{4.15}$$

Therefore starting from (4.12), and considering the decompositions (4.13), (4.14) and (4.15), we get that

$$\sum_{i=1}^{k} X_i - \sum_{i=1}^{k}\sum_{\ell=-r-i}^{r+i} P_i(X_\ell) = S_k - \mathbb{E}_k(S_{k+r}) + \mathbb{E}_0(S_r) + \sum_{\ell=r+1}^{k+r} \mathbb{E}_{\ell-r-1}(X_\ell) - \sum_{i=1}^{k}\sum_{\ell=0}^{r+i} P_i(X_{-\ell})\,. \tag{4.16}$$

The decomposition (4.9) together with the upper bounds (4.10), (4.11) and (4.16) imply that

$$\Big\| \max_{r\leq k\leq n} |R_k|_m \Big\|_p^{p'} \ll n\Big(\sum_{|j|\geq r+1} \|P_0(X_j)\|_{p,m}\Big)^{p'} + \Big\| \max_{r\leq k\leq n} \big|S_k - \mathbb{E}_k(S_{k+r})\big|_m \Big\|_p^{p'} + \|\mathbb{E}_0(S_r)\|_{p,m}^{p'}$$
$$+ \Big\| \max_{r\leq k\leq n} \Big| \sum_{\ell=r+1}^{k+r} \mathbb{E}_{\ell-r-1}(X_\ell)\Big|_m \Big\|_p^{p'} + \Big\| \max_{r\leq k\leq n} \Big| \sum_{i=1}^{k}\sum_{\ell=0}^{r+i} P_i(X_{-\ell})\Big| \Big\|_p^{p'}\,. \tag{4.17}$$

Applying Burkholder's inequality for multivariate martingales and using the stationarity, there exists a positive constant c_p depending only on p such that, for any positive integer n,

$$\Big\| \max_{r\leq k\leq n} \Big| \sum_{i=1}^{k}\sum_{\ell=0}^{r+i} P_i(X_{-\ell})\Big| \Big\|_p^{p'} \leq c_p \sum_{i=1}^{n} \Big\| \sum_{\ell=0}^{r+i} P_i(X_{-\ell})\Big\|_{p,m}^{p'} \leq c_p \sum_{i=1}^{n} \Big(\sum_{j\geq i} \|P_0(X_{-j})\|_{p,m}\Big)^{p'}\,. \tag{4.18}$$

To handle the fourth term in the right-hand side of (4.17) we proceed as follows. Since $\mathbb{E}_{-\infty}(X_\ell) = 0$ almost surely, we first write that

$$\mathbb{E}_{\ell-r-1}(X_\ell) = \sum_{j=r+1}^{\infty} P_{\ell-j}(X_\ell)\,.$$

Then

$$\max_{r\le k\le n}\Big|\sum_{\ell=r+1}^{k+r}\mathbb{E}_{\ell-r-1}(X_\ell)\Big|_m \le \sum_{j=r+1}^{\infty}\max_{r\le k\le n}\Big|\sum_{\ell=r+1}^{k+r}P_{\ell-j}(X_\ell)\Big|_m\,.$$

Let now

$$u_i = \|P_0(X_i)\|_{p,m}\,,\ C_r = \sum_{i\ge r+1} u_i \ \text{ and } \ \alpha_i = C_r^{-1}u_i\,.$$

By using the facts that for any $p\ge 1$, $x\mapsto x^p$ is convex and that $\alpha_i\ge 0$ with $\sum_{i\ge r+1}\alpha_i = 1$ and writing $\sum_j a_j = \sum_j \alpha_j(a_j/\alpha_j)$, we obtain that

$$\Big\|\max_{r\le k\le n}\Big|\sum_{\ell=r+1}^{k+r}\mathbb{E}_{\ell-r-1}(X_\ell)\Big|_m\Big\|_p^p \le \sum_{j=r+1}^{\infty}\alpha_j^{1-p}\mathbb{E}\Big(\max_{r\le k\le n}\Big|\sum_{\ell=r+1}^{k+r}P_{\ell-j}(X_\ell)\Big|_m^p\Big)\,.$$

Applying Burkholder's inequality for multivariate martingales and using the stationarity, we infer that there exists a positive constant c_p depending only on p such that, for any positive integer n,

$$\Big\|\max_{r\le k\le n}\Big|\sum_{\ell=r+1}^{k+r}P_{\ell-j}(X_\ell)\Big|_m\Big\|_p^{p'} \le c_p\sum_{\ell=r+1}^{n+r}\|P_{\ell-j}(X_\ell)\|_{p,m}^{p'} \le c_p\,n\,\|P_0(X_j)\|_{p,m}^{p'}\,.$$

So, overall

$$\Big\|\max_{r\le k\le n}\Big|\sum_{\ell=r+1}^{k+r}\mathbb{E}_{\ell-r-1}(X_\ell)\Big|_m\Big\|_p^p \le (c_pn)^{p/p'}\sum_{j=r+1}^{\infty}\alpha_j^{1-p}u_j^p = (c_pn)^{p/p'}\Big(\sum_{j=r+1}^{\infty}\|P_0(X_j)\|_p\Big)^p\,. \tag{4.19}$$

We handle now the second term in the right-hand side of (4.17). We first write that

$$S_k - \mathbb{E}_k(S_{k+r}) = S_k - S_{k-r} - \mathbb{E}_k(S_{k+r}-S_{k-r}) + S_{k-r} - \mathbb{E}_k(S_{k-r})\,. \tag{4.20}$$

Let

$$Y_r = \sum_{i=-(r-1)}^{0} X_i - \sum_{i=-(r-1)}^{r}\mathbb{E}_0(X_i)\,.$$

With this notation,

$$S_k - S_{k-r} - \mathbb{E}_k(S_{k+r}-S_{k-r}) = Y_r\circ T^k\,.$$

Hence, for any positive real A,

$$\begin{aligned}\Big\|\max_{r\le k\le n}\big|S_k - S_{k-r} - \mathbb{E}_k(S_{k+r}-S_{k-r})\big|_m\Big\|_p^p &\le 2^pA^p + 2^p\Big\|\max_{r\le k\le n}|Y_r\mathbf{1}_{|Y_r|_m>A}\circ T^k|_m\Big\|_p^p\\ &\le 2^pA^p + 2^p\,n\,\|Y_r\mathbf{1}_{|Y_r|_m>A}\|_{p,m}^p\,.\end{aligned}$$

Since $\|Y_r\|_{p,m}\le K_r$ where K_r is a constant depending on r, we get that

$$\lim_{A\to\infty}\limsup_{n\to\infty}\frac{1}{n}\Big\|\max_{r\le k\le n}\big|S_k - S_{k-r} - \mathbb{E}_k(S_{k+r}-S_{k-r})\big|_m\Big\|_p^{p'} = 0\,. \tag{4.21}$$

We deal now with the term $\| \max_{r\le k\le n} |S_{k-r} - \mathbb{E}_k(S_{k-r})|_m \|_p$. Since $\mathbb{E}_\infty(X_0) = X_0$ almost surely, we have that, almost surely

$$S_{k-r} - \mathbb{E}_k(S_{k-r}) = \sum_{\ell=1}^{k-r} \sum_{j=-\infty}^{\ell-k-1} P_{\ell-j}(X_\ell) = \sum_{j=-\infty}^{-k} \sum_{\ell=1}^{k-r} P_{\ell-j}(X_\ell) + \sum_{j=-k+1}^{-r-1} \sum_{\ell=k+j+1}^{k-r} P_{\ell-j}(X_\ell)\,.$$

Therefore

$$\begin{aligned}\max_{r\le k\le n} |S_{k-r} - \mathbb{E}_k(S_{k-r})|_m &\le \sum_{j=-\infty}^{-r} \max_{r\le k\le n} \Big| \sum_{\ell=1}^{k-r} P_{\ell-j}(X_\ell)\Big|_m + \sum_{j=-n+1}^{-r-1} \max_{r\le k\le n} \Big| \sum_{\ell=k+j+1}^{k-r} P_{\ell-j}(X_\ell)\Big|_m \\ &\le \sum_{j=-\infty}^{-r} \max_{r\le k\le n} \Big| \sum_{\ell=1}^{k-r} P_{\ell-j}(X_\ell)\Big|_m + \sum_{j=-n+1}^{-r-1} \max_{r\le k\le n} \Big| \sum_{\ell=r-n}^{k-r} P_{\ell-j}(X_\ell)\Big|_m \\ &\qquad + \sum_{j=-n+1}^{-r-1} \max_{r\le k\le n} \Big| \sum_{\ell=r-n}^{k+j} P_{\ell-j}(X_\ell)\Big|_m\,. \qquad (4.22)\end{aligned}$$

Let $u_i = \|P_0(X_i)\|_{p,m}$, $C_r = \sum_{i=-\infty}^{-r} u_i$ and $\alpha_i = C_r^{-1} u_i$. As before, using the facts that for any $p \ge 1$, $x \mapsto x^p$ is convex and that $\alpha_i \ge 0$ with $\sum_{i=-\infty}^{-r} \alpha_i = 1$, we obtain that

$$\Big\| \sum_{j=-\infty}^{-r} \max_{r\le k\le n} \Big| \sum_{\ell=1}^{k-r} P_{\ell-j}(X_\ell)\Big|_m \Big\|_p^p \le \sum_{j=-\infty}^{-r} \alpha_j^{1-p} \mathbb{E}\Big(\max_{r\le k\le n} \Big| \sum_{\ell=1}^{k-r} P_{\ell-j}(X_\ell)\Big|_m^p \Big)\,.$$

Applying Burkholder's inequality for multivariate martingales and using the stationarity, we infer that there exists a positive constant c_p depending only on p such that, for any positive integer n,

$$\Big\| \sum_{j=-\infty}^{-r} \max_{r\le k\le n} \Big| \sum_{\ell=1}^{k-r} P_{\ell-j}(X_\ell)\Big|_m \Big\|_p^p \le (c_p n)^{p/p'} \Big(\sum_{i\ge r} \|P_0(X_{-i})\|_p \Big)^p\,. \qquad (4.23)$$

With similar arguments, we derive that

$$\begin{aligned}\Big\| \sum_{j=-n+1}^{-r-1} \max_{r\le k\le n} \Big| \sum_{\ell=r-n}^{k-r} P_{\ell-j}(X_\ell)\Big|_m \Big\|_p^p + \Big\| \sum_{j=-n+1}^{-r-1} \max_{r\le k\le n} \Big| \sum_{\ell=r-n}^{k+j} P_{\ell-j}(X_\ell)\Big|_m \Big\|_p^p \\ \le 2(2c_p n)^{p/p'} \Big(\sum_{i=r+1}^{n-1} \|P_0(X_{-i})\|_p \Big)^p\,. \qquad (4.24)\end{aligned}$$

Starting from (4.22) and considering the upper bounds (4.23) and (4.24), we get that

$$\Big\| \max_{r\le k\le n} |S_{k-r} - \mathbb{E}_k(S_{k-r})|_m \Big\|_p \le n^{1/p'} \sum_{i\ge r} \|P_0(X_{-i})\|_p\,. \qquad (4.25)$$

From the decomposition (4.20) together with (4.21) and (4.25), it follows that

$$\Big\| \max_{r\le k\le n} |S_k - \mathbb{E}_k(S_{k+r})|_m \Big\|_p \le n^{1/p'} \sum_{i\ge r} \|P_0(X_{-i})\|_p + o(n^{1/p'})\,. \qquad (4.26)$$

Starting from (4.17) and considering (4.18), (4.19), (4.26) and the condition (4.1), we derive that

$$\Big\| \max_{r\le k\le n} |R_k|_m \Big\|_p \ll n^{1/p'} \sum_{|j|\ge r} \|P_0(X_j)\|_{p,m} + o(n^{1/p'}) + r^{1/p'}\,,$$

(with the decomposition of (4.18) in $\sum_{i=1}^{r} + \sum_{i=r+1}^{n}$) which, combined with (4.2) and Condition (4.1), implies that

$$\Big\| \max_{1\le k\le n} |R_k|_m \Big\|_p \le \Big\| \max_{1\le k\le r} |R_k|_m \Big\|_p + \Big\| \max_{r\le k\le n} |R_k|_m \Big\|_p \ll r^2 + n^{1/p'} \sum_{|j|\ge r} \|P_0(X_j)\|_{p,m} + o(n^{1/p'})\,.$$

Letting first n tend to infinity and next r tend to infinity, (4.3) follows. □

Starting from Proposition 4.1 one can prove the following theorem concerning the weak and strong invariance principles for non-adapted sequences.

Theorem 4.1. *Let X_0 be a zero mean random variable in $\mathbb{L}^2(\mathbb{R}^m)$ and $\mathcal{F}_0$ a σ-algebra satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$. For any $i \in \mathbb{Z}$, let $X_i = X_0 \circ T^i$ and $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let $S_n = X_1 + \cdots + X_n$. Assume that T is ergodic, that $\mathbb{E}_{-\infty}(X_0) = 0$ almost surely, and that $\mathbb{E}_{\infty}(X_0) = X_0$ almost surely.*

1. *Assume that*
$$\sum_{n\in\mathbb{Z}} \|P_0(X_n)\|_{2,m} < \infty\,. \tag{4.27}$$
Then $n^{-1}\mathrm{Var}(S_n)$ converges to
$$C = \sum_{k\in\mathbb{Z}} \mathrm{Cov}\big(X_0, X_k\big)\,. \tag{4.28}$$
In addition the process $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ converges in $D([0,1],\mathbb{R}^m)$ equipped with the uniform topology to a Wiener process $\{W(t), t \in [0,1]\}$ with variance matrix $\mathrm{Var}(W(1)) = C$.

2. *Assume that*
$$\sum_{n\ge 3} \frac{\log n(\|P_0(X_n)\|_{2,m} + \|P_0(X_{-n})\|_{2,m})}{(\log\log n)^{1/2}} < \infty\,. \tag{4.29}$$
Then, enlarging the probability space if necessary, there exists a sequence $(Z_i)_{i\ge1}$ of iid Gaussian random variables in $\mathbb{R}^m$ with zero mean and variance matrix $\mathrm{Var}(Z_i) = C$ given by (4.28), such that
$$\sup_{1\le k\le n} \Big| \sum_{i=1}^{k} X_0\circ T^i - \sum_{i=1}^{k} Z_i \Big|_m = o\big((n\log\log n)^{1/2}\big) \text{ almost surely, as } n\to\infty. \tag{4.30}$$

Remark 4.2. *The weak invariance principle (Item 1 of Theorem 4.1) still holds if T is not ergodic, but in that case the limiting distribution is a mixture of Brownian motion (this has been proved in [7] when $m = 1$). This weak invariance principle can be also extended to separable Hilbert spaces, with the appropriate covariance operator. In the adapted case (i.e. X_0 is $\mathcal{F}_0$-measurable), the non ergodic Hilbert-valued version of Item 1 has been proved in [4].*

Proof of Theorem 4.1. Let $S_{m,1} = \{x \in \mathbb{R}^m : |x|_m = 1\}$. For a matrix A from $\mathbb{R}^m$ to $\mathbb{R}^m$, let $|A|_m = \sup_{x\in S_{m,1}} |Ax|_m$. By stationarity $n^{-1}\mathrm{Var}(S_n) = n^{-1}\sum_{|k|<n}(n-|k|)\mathrm{Cov}(X_0, X_k)$. Hence $n^{-1}\mathrm{Cov}(S_n)$ converges to C provided that $\sum_{k\in\mathbb{Z}} |\mathrm{Cov}(X_0, X_k)|_m < \infty$. Since $\mathbb{E}_{-\infty}(X_k) = 0$ almost surely and $\mathbb{E}_{\infty}(X_k) = X_k$ almost surely, it follows that $X_k = \sum_{i\in\mathbb{Z}} P_i(X_k)$ almost surely. Moreover $\mathrm{Cov}(P_i(X_0), P_j(X_k)) = 0$ for $i \ne j$. Hence,

$$\mathrm{Cov}(X_0, X_k) = \sum_{i\in\mathbb{Z}} \mathrm{Cov}(P_i(X_0), P_i(X_k))\,,$$

and consequently $|\mathrm{Cov}(X_0, X_k)|_m \le \sum_{i\in\mathbb{Z}} \|P_i(X_0)\|_{2,m}\|P_i(X_k)\|_{2,m}$. By (4.27) it follows that

$$\sum_{k\in\mathbb{Z}} |\mathrm{Cov}(X_0, X_k)|_m \le \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} \|P_i(X_0)\|_{2,m}\|P_i(X_k)\|_{2,m} = \Big(\sum_{i\in\mathbb{Z}} \|P_0(X_i)\|_{2,m}\Big)^2 < \infty\,,$$

which proves the convergence of $n^{-1}\mathrm{Var}(S_n)$ to C.

Let now $d_0 := \sum_{j\in\mathbb{Z}} P_0(X_j)$. Since (4.27) is assumed, d_0 belongs to $\mathbb{L}^2(\mathbb{R}^m)$. In addition $\mathbb{E}(d_0|\mathcal{F}_{-1}) = 0$ almost surely. Let $d_i := d_0 \circ T^i$ for all $i \in \mathbb{Z}$. Then $(d_i)_{i\in\mathbb{Z}}$ is a stationary ergodic sequence of martingale differences in $\mathbb{L}^2(\mathbb{R}^m)$. Let

$$M_n := \sum_{i=1}^n d_i \ \text{ and } \ R_n := S_n - M_n\,.$$

Using (4.27), it follows from (4.3) of Lemma 4.1 that

$$\Big\| \max_{1\le k\le n} |R_k|_m \Big\|_2^2 = o(n)\,. \tag{4.31}$$

Since $n^{-1}\mathrm{Var}(S_n)$ converges to C, it follows that $\mathrm{Var}(d_0) = C$. Therefore, Item 1 of Theorem 4.1 follows from the weak invariance principle for partial sums of stationary multivariate martingale differences in $\mathbb{L}^2(\mathbb{R}^m)$ (see [4] for the non ergodic Hilbert-valued version) together with the maximal martingale approximation given in (4.31).

We turn now to the proof of Item 2. According to Theorem 3.1 in [1] (that is the generalization of the Strassen's invariance principle [14] for real martingales with ergodic increments to the multivariate case), enlarging the probability space if necessary, there exists a sequence $(Z_i)_{i\ge1}$ of iid Gaussian random variables in $\mathbb{R}^m$ with zero mean and covariance $\mathrm{Var}(Z_1) = C$ such that

$$\sup_{1\le k\le n} \Big| \sum_{i=1}^k d_0\circ T^i - \sum_{i=1}^k Z_i \Big|_m = o\big((n\log\log n)^{1/2}\big) \ \text{ almost surely, as } n\to\infty.$$

Therefore the strong approximation result (4.30) will follow if we can show that

$$|R_n|_m = o\big((n\log\log n)^{1/2}\big) \ \text{ almost surely, as } n\to\infty. \tag{4.32}$$

Since $R_n = \sum_{i=1}^n (f - d_0)\circ T^i$, (4.32) will follow by Theorem 4.7 in [3] if we can prove that

$$\sum_{n>3} \frac{\|R_n\|_2}{n^{3/2}(\log\log n)^{1/2}} < \infty\,.$$

Using (4.2) of Lemma 4.1, this last convergence will hold provided that

$$\sum_{n>3} \frac{\Big(\sum_{k=1}^n \Big(\sum_{|\ell|\ge k} \|P_\ell(X_0)\|_{2,m}\Big)^2\Big)^{1/2}}{n^{3/2}(\log\log n)^{1/2}} < \infty\,. \tag{4.33}$$

Notice that

$$\sum_{n>3} \frac{\Big(\sum_{k=1}^n \Big(\sum_{|\ell|\geq k} \|P_\ell(X_0)\|_{2,m}\Big)^2\Big)^{1/2}}{n^{3/2}(\log\log n)^{1/2}}$$

$$\ll \sum_{k>2} 2^{-k/2}(\log k)^{-1/2}\Big(\sum_{j=1}^{2^k}\Big(\sum_{\ell\geq j}(\|P_0(X_\ell)\|_{2,m}+\|P_0(X_{-\ell})\|_{2,m})\Big)^2\Big)^{1/2}$$

$$\ll \sum_{k\geq 0} 2^{-k/2}(\log k)^{-1/2}\Big(\sum_{j=0}^{k} 2^j\Big(\sum_{\ell\geq 2^j}(\|P_0(X_\ell)\|_{2,m}+\|P_0(X_{-\ell})\|_{2,m})\Big)^2\Big)^{1/2}.$$

Now, using the subadditivity of $x\mapsto x^{1/2}$, it follows that (4.33) will be satisfied as soon as

$$\sum_{k>2} 2^{-k/2}(\log k)^{-1/2}\sum_{j=0}^{k} 2^{j/2}\sum_{\ell\geq 2^j}(\|P_0(X_\ell)\|_{2,m}+\|P_0(X_{-\ell})\|_{2,m})<\infty\,,$$

which holds as soon as (4.29) does (changing the order of summation in $\sum_\ell\sum_j\sum_k$). This ends the proof of Item 2 of Theorem 4.1. □

For the sake of applications, we now give sufficient conditions for (4.27) and (4.29) to hold.

Remark 4.3. *The condition* (4.27) *is satisfied if we assume that*

$$\sum_{n\geq 1}\frac{1}{n^{1/2}}\|\mathbb{E}_0(X_n)\|_{2,m}<\infty \text{ and } \sum_{n\geq 1}\frac{1}{n^{1/2}}\|X_{-n}-\mathbb{E}_0(X_{-n})\|_{2,m}<\infty\,, \tag{4.34}$$

and the condition (4.29) *holds if we assume that*

$$\sum_{n\geq 3}\frac{\log n}{n^{1/2}(\log\log n)^{1/2}}\|\mathbb{E}_0(X_n)\|_{2,m}<\infty \text{ and } \sum_{n\geq 3}\frac{\log n}{n^{1/2}(\log\log n)^{1/2}}\|X_{-n}-\mathbb{E}_0(X_{-n})\|_{2,m}<\infty\,. \tag{4.35}$$

The proof of the remark above is omitted since it uses exactly the arguments developed to prove Remarks 3.3 and 3.6 in [5] (see Section 5.5 of [5]). Notice that the conditions (4.34) or (4.35) imply clearly that $\mathbb{E}_{-\infty}(X_0)=0$ almost surely and that $\mathbb{E}_\infty(X_0)=X_0$ almost surely.

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