Chern-Einstein metrics on symplectic manifolds

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Partially based on a joint work with Alice Gatti (LBNL)
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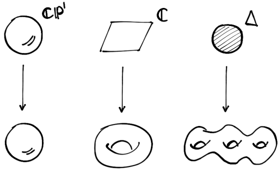
Motivation

As is well known, any compact Riemann surface is covered by either $\mathbf{CP}^1$, or $\mathbf{C}$, or the unit disk $\Delta \subset \mathbf{C}$, according to the sign of the curvature of a constant Gaussian curvature metric that may exist on it.



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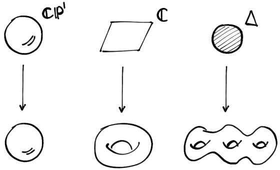

In complex dimension two, this results generalizes as the following

Theorem

If (M, J, ω) is a compact Kähler-Einstein surface then

$$c_1^2 - 3c_2 \leq 0.$$

Equality holds if and only if M is covered by either $\mathbf{CP}^2$, or $\mathbf{C}^2$, or the unit ball $B^2 \subset \mathbf{C}^2$.

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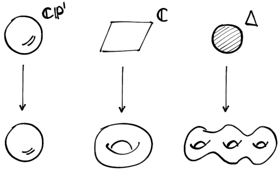

In higher dimension this results generalizes as the following

Theorem

If (M, J, ω) is a compact Kähler-Einstein n-fold then

$$\left(c_1^2 - \frac{2(n+1)}{n} c_2\right) \cup [\omega]^{n-2} \leq 0.$$

Equality holds if and only if M is covered by either $\mathbf{CP}^n$, or $\mathbf{C}^n$, or the unit ball $B^n \subset \mathbf{C}^n$.

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- For *any* Kähler metric ω on (M, J) one proves that

$$\left(c_1^2 - \frac{2(n+1)}{n}c_2\right) \cup [\omega]^{n-2} = \int_M \left(k_1\,|\rho - (s/n)\omega|^2 - k_2|B|^2\right)\omega^n,$$

  where $k_1, k_2 > 0$ are constants depending just on n, ρ is the Ricci form of ω, s is the scalar curvature and $B \in \Omega^2(\mathrm{End}(TM))$ vanishes if and only if ω has constant holomorphic sectional curvature (Chern-Weil theory).

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- (M, J, ω) has constant holomorphic sectional curvature if and only if it is isometrically covered by either $\mathbf{CP}^n$, or $\mathbf{C}^n$, or $B^n \subset \mathbf{C}^n$ equipped with their standard metrics, up to scaling (Uniformization Theorem).

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From now on consider a compact *symplectic* manifold (M, ω) and the set of all compatible almost complex structures on it.

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Does a compatible almost complex structure J having **special curvature properties** *constrain the topology of (M, ω)?*

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Let (M, ω) be a compact symplectic $2n$-fold. If J is a compatible almost complex structure satisfying $\rho = \lambda\omega$, then

$$\left(c_1^2 - \frac{2(n+1)}{n} c_2\right) \cup [\omega]^{n-2} \le \int_M \left(k_1 \lambda |N|^2 + k_2 |N|^4 + k_3 |\nabla N|^2\right) \omega^n$$

where $k_1, k_2, k_3 > 0$ are constants depending just on n, N is the Nijenhuis tensor of J, and ∇ is the Chern connection of J.

Geometry of compatible complex structures

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$$\rho(X, Y) = \mathrm{tr}(JR(X, Y)), \qquad d\rho = 0, \qquad c_1 = \frac{1}{4\pi}[\rho]$$

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5. $\theta \in \Omega^4(M)$ and Chern class $c_2 \in H^4(M)$

$$\theta = \frac{1}{2}\rho^2 + \mathrm{tr}(R \wedge R), \qquad d\theta = 0, \qquad c_2 = \frac{1}{16\pi^2}[\theta]$$

Special compatible almost complex structures

Definition

A compatible almost complex structure J on (M, ω) is **Chern-Einstein** if there is $\lambda \in \mathbf{R}$ such that

$$\rho = \lambda\omega.$$

(...sometimes called Hermitian-Einstein or special. Already considered in Apostolov-Drăghichi 2003)

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If J is Chen-Einstein, then $4\pi c_1 = \lambda[\omega]$. Therefore (M, ω) is

- symplectic general type if $\lambda < 0$,
- symplectic Calabi-Yau if $\lambda = 0$,
- symplectic Fano (or monotone) if $\lambda > 0$

Moreover, the Hermitian scalar curvature is constant $s = n\lambda$.

Kähler examples and their deformations

Example (After Moser, Aubin, Yau, Chen-Donaldson-Sun, Tian)

If (M, ω) satisfies $4\pi c_1 = \lambda[\omega]$ and admits a K-stable integrable J_0, then it also admits an **integrable Chern-Einstein** J.

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Example (Lejmi 2010)

If (M, J_0, ω) is a locally toric Kähler-Einstein complex surface then there exist **Chern-Einstein non-integrable** deformations J_ε of J_0.

These examples include: $\mathbf{CP}^2$, $\mathbf{CP}^1 \times \mathbf{CP}^1$, $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}^2}$, $\Gamma \backslash B^2$, $\Gamma \backslash (\Delta \times \Delta)$.

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All this examples are

- Chern-Ricci flat ($\rho = 0$),
- locally homogeneous ($M = \Gamma \backslash G/V$)
  - $V \subset G$ compact
  - $\Gamma \subset G$ discrete and torsion-free (lattice)
  - ω and J descends from homogeneous structures on G/V

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Let (M, ω) such a twistor space. By Theorem before one has

$$-8c_2 \cup [\omega] \leq \inf_{J \text{ s.t. } \rho=0} \left\{ \frac{1}{96\pi^2} \int_M \left(253|N|^4 + 96|\nabla N|^2\right) \frac{\omega^3}{6} \right\}$$

- $\|N\|_{L^4} + \|N\|_{W^{1,2}}$ cannot be arbitrarily small for a Chern-Ricci flat J.

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Theorem

For all $X, Y \in \mathfrak{m}$ one has $\rho(X, Y) = \operatorname{tr}\left(\operatorname{ad}_{H[X,Y]_{\mathfrak{g}}} - H \operatorname{ad}_{[X,Y]_{\mathfrak{g}}}\right)$.

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Corollary

If V has discrete center, then $(G/V, \omega)$ has $c_1 = 0$.

Homogeneous symplectic manifolds - Symplectic Lie groups

A **symplectic Lie group** (G, ω) is a Homogeneous symplectic manifold with trivial isotropy V. Equivalently:

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Theorem (Lauret-Will 2017)

If G is unimodular and 4-dimensional, then a Chern-Ricci flat J on (G, ω) exists whenever $G \neq \mathrm{Nil}^4$ ($[e_1, e_4] = -e_3$, $[e_3, e_4] = -e_2$).

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Corollary (after Geiges 1992)

Let M be the total space of a T^2-bundle over T^2 and let $c \in H^2(M)$ such that $c^2 \neq 0$. If $c(\text{fiber}) \neq 0$ then there exist ω symplectic such that $[\omega] = c$ and J Chern-Ricci flat.

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G and V have the same rank. This is a familiar situation in Hodge theory. On G/V is defined an *integrable* almost complex structure J' which, in general, is not compatible with ω.

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Definition (ADV, ADV-Gatti, Alekseevsky-Podestà)

There is a *canonical* homogeneous compatible almost complex structure J on $(G/V, \omega)$.

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Example

The twistor space of a real hyperbolic $2m$-fold $\Gamma\backslash H^{2m}$ is a symplectc manifold (M, ω_Γ) admitting J_Γ with $\rho_\Gamma = 2(m-2)\omega_\Gamma$.

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- (M, ω_Γ) is a symplectic Fano $m(m+1)$-fold if $m > 2$

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- M has dimension $p(p+2q+1)$

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Example

A weight-two period domain of type $(2p, q)$ is a symplectc manifold (M, ω_Γ) admitting J_Γ with $\rho_\Gamma = 2(p - q - 1)\omega_\Gamma$.

- $\Gamma \subset SO(2p, q)$ is a lattice
- $V = U(p) \times SO(q)$
- $M = \Gamma \backslash SO(2p, q)/V$
- $SO(2p, q)/V$ is an adjoint orbit with $\rho = 2(p - q - 1)\omega$
- M is not homotopy Kähler if $p \geq 2$, and $q \neq 2$ (Carlson-Toledo 1989)
- M has dimension $p(p + 2q + 1)$
- (M, ω_Γ) is symplectic GT, CY, or Fano according to the sign of $p - q - 1$

Homogeneous symplectic manifolds - (Co)adjoint orbits

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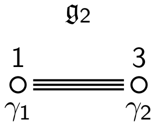
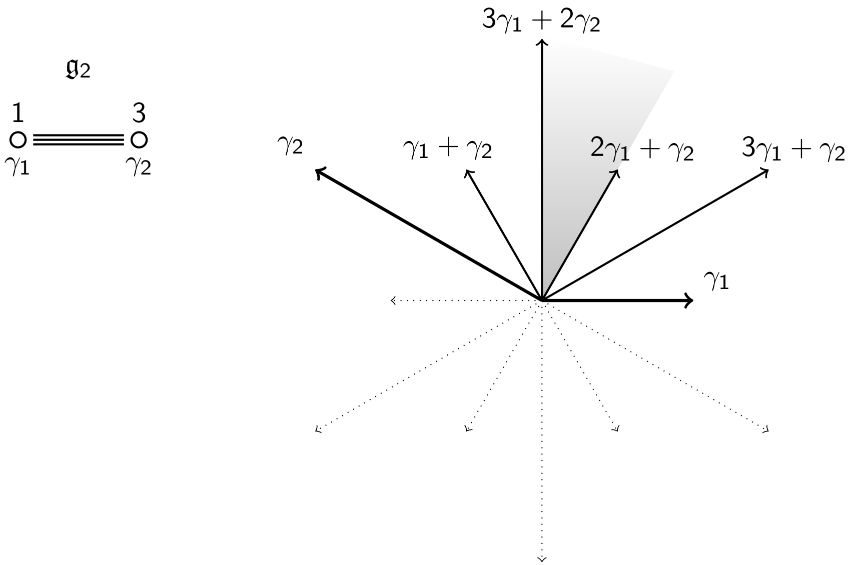
  Vogan diagrams demonstrated to be the appropriate combinatorial device for algorithmic listing. No hope to guessing the general pattern at the moment.

Homogeneous symplectic manifolds - (Co)adjoint orbits

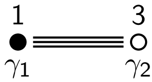


Homogeneous symplectic manifolds - (Co)adjoint orbits

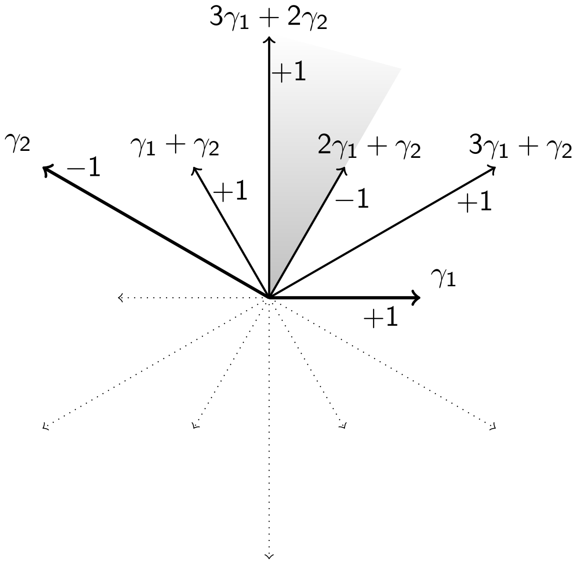


Homogeneous symplectic manifolds - (Co)adjoint orbits

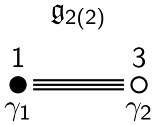
$\mathfrak{g}_{2(2)}$



$\varepsilon_{\gamma_1} = +1,\ \varepsilon_{\gamma_2} = -1$



Homogeneous symplectic manifolds - (Co)adjoint orbits



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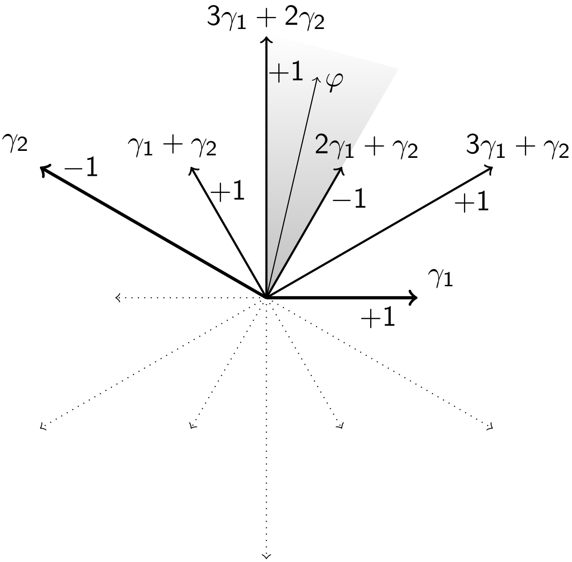


- $v \in \mathfrak{g} \rightsquigarrow \varphi \in \mathfrak{h}_{\mathbf{R}}^*$

  $\varphi(u) = -iB(v, u)$

- $\varphi' = -2\sum_{\alpha \in \Delta^+ \setminus \varphi^\perp} \varepsilon_\alpha \alpha$

- $\rho = \lambda\omega$ iff $\varphi' = \lambda\varphi$

Homogeneous symplectic manifolds - (Co)adjoint orbits

A_2	$\dim \mathfrak{g} = 8$. One non-compact simple real form with trivial automorphism: $\mathfrak{su}(1,2)$.	Fundamental dominant weights $\varphi_1 = \frac{2}{3}\gamma_1 + \frac{1}{3}\gamma_2$, $\varphi_2 = \frac{1}{3}\gamma_1 + \frac{2}{3}\gamma_2$						
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 1 ●—○ γ_1 γ_2	φ_1	no	GT	-12	4	4	$\mathfrak{su}(1,2)$	$\mathfrak{su}(2) \oplus \mathbf{R}$
1 1 ●—● γ_1 γ_2	$t_1\varphi_1 + t_2\varphi_2$ for all $t_1, t_2 > 0$	no	sCY	0	2	6	$\mathfrak{su}(1,2)$	$\mathbf{R} \oplus \mathbf{R}$

B_2	$\dim \mathfrak{g} = 10$. Two simple non-compact real forms: $\mathfrak{so}(4,1)$, $\mathfrak{so}(2,3)$.	Fundamental dominant weights $\varphi_1 = \gamma_1 + \frac{1}{2}\gamma_2$, $\varphi_2 = \gamma_1 + \gamma_2$						
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 2 ●═○ γ_1 γ_2	φ_1	no	sCY	0	4	6	$\mathfrak{so}(4,1)$	$\mathfrak{su}(2) \oplus \mathbf{R}$
1 2 ○═● γ_1 γ_2	φ_2	yes	GT	-18	4	6	$\mathfrak{so}(2,3)$	$\mathfrak{su}(2) \oplus \mathbf{R}$

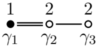
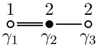
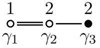
Homogeneous symplectic manifolds - (Co)adjoint orbits

G_2	$\dim \mathfrak{g} = 14$. One non-compact simple real form denoted by $\mathfrak{g}_{2(2)} = \mathrm{G}$.					Fundamental dominant weights $\varphi_1 = 2\gamma_1 + \gamma_2$, $\varphi_2 = 3\gamma_1 + 2\gamma_2$		
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 ● ≡ ○ 3 ; γ_1, γ_2	φ_1	yes	sGT	-30	4	10	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) \oplus \mathbf{R}$
1 ○ ≡ ● 3 ; γ_1, γ_2	φ_2	yes	sGT	-10	4	10	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) \oplus \mathbf{R}$

A_3	$\dim \mathfrak{g} = 15$. Two non-compact simple real forms with trivial automorphism: $\mathfrak{su}(1,3)$, $\mathfrak{su}(2,2)$.					Fundamental dominant weights $\varphi_1 = \frac{3}{4}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{4}\gamma_3$, $\varphi_2 = \frac{1}{2}\gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3$, $\varphi_3 = \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{3}{4}\gamma_3$		
Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 ● — 1 ○ — 1 ○ ; γ_1, γ_2, γ_3	φ_1	no	GT	-24	9	6	$\mathfrak{su}(1,3)$	$\mathfrak{su}(3) \oplus \mathbf{R}$
1 ○ — 1 ● — 1 ○ ; γ_1, γ_2, γ_3	φ_2	no	GT	-32	7	8	$\mathfrak{su}(2,2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
1 ● — 1 ○ — 1 ● ; γ_1, γ_2, γ_3	$\varphi_1 + \varphi_3$	yes	sGT	-10	5	10	$\mathfrak{su}(2,2)$	$\mathfrak{su}(2) \oplus \mathbf{R} \oplus \mathbf{R}$

Homogeneous symplectic manifolds - (Co)adjoint orbits

B_3	$\dim \mathfrak{g} = 21$. Three non-compact simple real forms: $\mathfrak{so}(6,1)$, $\mathfrak{so}(4,3)$, $\mathfrak{so}(2,5)$.	Fundamental dominant weights $\varphi_1 = \frac{3}{2}\gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3$ $\varphi_2 = 2\gamma_1 + 2\gamma_2 + \gamma_3$ $\varphi_3 = \gamma_1 + \gamma_2 + \gamma_3$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	sF	24	9	12	$\mathfrak{so}(6,1)$	$\mathfrak{su}(3) \oplus \mathbf{R}$
	φ_2	yes	sGT	-28	7	14	$\mathfrak{so}(4,3)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
	φ_3	yes	GT	-50	11	10	$\mathfrak{so}(2,5)$	$\mathfrak{so}(5) \oplus \mathbf{R}$

Homogeneous symplectic manifolds - (Co)adjoint orbits

C_3	$\dim \mathfrak{g} = 21$. Two non-compact simple real forms: $\mathfrak{sp}(1,2)$, $\mathfrak{sp}(3,\mathbf{R})$.	Fundamental dominant weights $\varphi_1 = \gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3$ $\varphi_2 = \gamma_1 + 2\gamma_2 + \gamma_3$ $\varphi_3 = \gamma_1 + 2\gamma_2 + \frac{3}{2}\gamma_3$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 1 2 ● — ○ ═ ○ γ_1 γ_2 γ_3	φ_1	no	sGT	-20	11	10	$\mathfrak{sp}(1,2)$	$\mathfrak{sp}(2) \oplus \mathbf{R}$
1 1 2 ○ — ● ═ ○ γ_1 γ_2 γ_3	φ_2	yes	sF	14	7	14	$\mathfrak{sp}(1,2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
1 1 2 ○ — ○ ═ ● γ_1 γ_2 γ_3	φ_3	no	GT	-48	9	12	$\mathfrak{sp}(3,\mathbf{R})$	$\mathfrak{su}(3) \oplus \mathbf{R}$
1 1 2 ● — ○ ═ ● γ_1 γ_2 γ_3	$\varphi_1 + \varphi_3$	no	sGT	-16	5	16	$\mathfrak{sp}(3,\mathbf{R})$	$\mathfrak{su}(2) \oplus \mathbf{R} \oplus \mathbf{R}$

Homogeneous symplectic manifolds - (Co)adjoint orbits

A_4	$\dim \mathfrak{g} = 24$. Two non-compact simple real forms with trivial automorphism: $\mathfrak{su}(1,4)$, $\mathfrak{su}(2,3)$.	Fundamental dominant weights $\varphi_1 = \frac{4}{5}\gamma_1 + \frac{3}{5}\gamma_2 + \frac{2}{5}\gamma_3 + \frac{1}{5}\gamma_4$ $\varphi_2 = \frac{3}{5}\gamma_1 + \frac{6}{5}\gamma_2 + \frac{4}{5}\gamma_3 + \frac{2}{5}\gamma_4$ $\varphi_3 = \frac{2}{5}\gamma_1 + \frac{4}{5}\gamma_2 + \frac{6}{5}\gamma_3 + \frac{3}{5}\gamma_4$ $\varphi_4 = \frac{1}{5}\gamma_1 + \frac{2}{5}\gamma_2 + \frac{3}{5}\gamma_3 + \frac{4}{5}\gamma_4$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 1 1 1 ●—○—○—○ γ_1 γ_2 γ_3 γ_4	φ_1	no	GT	-40	16	8	$\mathfrak{su}(1,4)$	$\mathfrak{su}(4) \oplus \mathbf{R}$
1 1 1 1 ○—●—○—○ γ_1 γ_2 γ_3 γ_4	φ_2	no	GT	-60	12	12	$\mathfrak{su}(2,3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
1 1 1 1 ●—○—○—● γ_1 γ_2 γ_3 γ_4	$\varphi_1 + \varphi_4$	yes	sGT	-28	10	14	$\mathfrak{su}(2,3)$	$\mathfrak{su}(3) \oplus \mathbf{R} \oplus \mathbf{R}$
1 1 1 1 ○—●—●—○ γ_1 γ_2 γ_3 γ_4	$\varphi_2 + \varphi_3$	no	sF	16	8	16	$\mathfrak{su}(1,4)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbf{R} \oplus \mathbf{R}$
1 1 1 1 ●—●—●—● γ_1 γ_2 γ_3 γ_4	$\sum_{i=1}^{4} t_i\varphi_i$ for all $t_i > 0$	no	sCY	0	4	20	$\mathfrak{su}(2,3)$	$\mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$

Homogeneous symplectic manifolds - (Co)adjoint orbits

B_4	$\dim \mathfrak{g} = 36$. Four non-compact simple real forms: $\mathfrak{so}(8,1)$, $\mathfrak{so}(6,3)$, $\mathfrak{so}(4,5)$, $\mathfrak{so}(2,7)$.	Fundamental dominant weights $\varphi_1 = 2\gamma_1 + \frac{3}{2}\gamma_2 + \gamma_3 + \frac{1}{2}\gamma_4$ $\varphi_2 = 3\gamma_1 + 3\gamma_2 + 2\gamma_3 + \gamma_4$ $\varphi_3 = 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4$ $\varphi_4 = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 2 2 2 ●═○—○—○ γ_1 γ_2 γ_3 γ_4	φ_1	no	sF	80	16	20	$\mathfrak{so}(8,1)$	$\mathfrak{su}(4) \oplus \mathbf{R}$
	$\varphi_1 + 2\varphi_4$	no	sF	52	10	26	$\mathfrak{so}(8,1)$	$\mathfrak{su}(3) \oplus \mathbf{R} \oplus \mathbf{R}$
1 2 2 2 ○═●—○—○ γ_1 γ_2 γ_3 γ_4	φ_2	no	sGT	−24	12	24	$\mathfrak{so}(6,3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
1 2 2 2 ○═○—●—○ γ_1 γ_2 γ_3 γ_4	φ_3	yes	sGT	−88	14	22	$\mathfrak{so}(4,5)$	$\mathfrak{so}(5) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
1 2 2 2 ○═○—○—● γ_1 γ_2 γ_3 γ_4	φ_4	yes	GT	−98	22	14	$\mathfrak{so}(2,7)$	$\mathfrak{so}(7) \oplus \mathbf{R}$

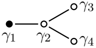
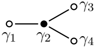
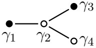
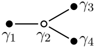
Homogeneous symplectic manifolds - (Co)adjoint orbits

C_4	$\dim \mathfrak{g} = 36$. Three non-compact simple real forms: $\mathfrak{sp}(1,3)$, $\mathfrak{sp}(2,2)$, $\mathfrak{sp}(4,\mathbf{R})$.	Fundamental dominant weights $\varphi_1 = \gamma_1 + \gamma_2 + \gamma_3 + \frac{1}{2}\gamma_4$ $\varphi_2 = \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4$ $\varphi_3 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + \frac{3}{2}\gamma_4$ $\varphi_4 = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 2\gamma_4$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 1 1 2 $\bullet - \circ - \circ = \circ$ γ_1 γ_2 γ_3 γ_4	φ_1	no	sGT	-56	22	14	$\mathfrak{sp}(1,3)$	$\mathfrak{sp}(3) \oplus \mathbf{R}$
1 1 1 2 $\circ - \bullet - \circ = \circ$ γ_1 γ_2 γ_3 γ_4	φ_2	yes	sGT	-22	14	22	$\mathfrak{sp}(2,2)$	$\mathfrak{su}(2) \oplus \mathfrak{sp}(2) \oplus \mathbf{R}$
1 1 1 2 $\circ - \circ - \bullet = \circ$ γ_1 γ_2 γ_3 γ_4	φ_3	no	sF	48	12	24	$\mathfrak{sp}(1,3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
	$3\varphi_1 + \varphi_3$	no	sF	28	8	28	$\mathfrak{sp}(1,3)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbf{R} \oplus \mathbf{R}$
1 1 1 2 $\circ - \circ - \circ = \bullet$ γ_1 γ_2 γ_3 γ_4	φ_4	no	GT	-100	16	20	$\mathfrak{sp}(4,\mathbf{R})$	$\mathfrak{su}(4) \oplus \mathbf{R}$

Homogeneous symplectic manifolds - (Co)adjoint orbits

D_4	$\dim \mathfrak{g} = 28$. Two non-compact simple real forms with trivial automorphism: $\mathfrak{so}(2,6)$, $\mathfrak{so}(4,4)$.	Fundamental dominant weights $\varphi_1 = \gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3 + \frac{1}{2}\gamma_4$ $\varphi_2 = \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4$ $\varphi_3 = \frac{1}{2}\gamma_1 + \gamma_2 + \gamma_3 + \frac{1}{2}\gamma_4$ $\varphi_4 = \frac{1}{2}\gamma_1 + \gamma_2 + \frac{1}{2}\gamma_3 + \gamma_4$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
	φ_1	no	GT	-72	16	12	$\mathfrak{so}(2,6)$	$\mathfrak{su}(4) \oplus \mathbf{R}$
	φ_2	yes	sGT	-54	10	18	$\mathfrak{so}(4,4)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
	$t_1\varphi_1 + t_2\varphi_3$ for all $t_1, t_2 > 0$	no	sCY	0	10	18	$\mathfrak{so}(2,6)$	$\mathfrak{su}(3) \oplus \mathbf{R} \oplus \mathbf{R}$
	$\varphi_1 + \varphi_3 + \varphi_4$	no	sGT	-22	6	22	$\mathfrak{so}(4,4)$	$\mathfrak{su}(2) \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$

Homogeneous symplectic manifolds - (Co)adjoint orbits

F_4	$\dim \mathfrak{g} = 52$. Two non-compact simple real forms denoted by $\mathfrak{f}_{4(4)} = \mathrm{F\,I}$, $\mathfrak{f}_{4(-20)} = \mathrm{F\,II}$.	Fundamental dominant weights $\varphi_1 = 2\gamma_1 + 3\gamma_2 + 2\gamma_3 + \gamma_4$ $\varphi_2 = 3\gamma_1 + 6\gamma_2 + 4\gamma_3 + 2\gamma_4$ $\varphi_3 = 4\gamma_1 + 8\gamma_2 + 6\gamma_3 + 3\gamma_4$ $\varphi_4 = 2\gamma_1 + 4\gamma_2 + 3\gamma_3 + 2\gamma_4$

Vogan diagram	φ	$\varphi \in \Delta$	Type	s	$\dim V$	$\dim G/V$	$\mathfrak{g}$	$\mathfrak{v}$
1 1 2 2 ●—○═○—○ γ_1 γ_2 γ_3 γ_4	φ_1	yes	sF	90	22	30	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(7) \oplus \mathbf{R}$
1 1 2 2 ○—●═○—○ γ_1 γ_2 γ_3 γ_4	φ_2	no	sF	120	12	40	$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
	$\varphi_1 + \varphi_2$	no	sF	84	10	42	$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(3) \oplus \mathbf{R} \oplus \mathbf{R}$
	$\varphi_2 + 3\varphi_4$	no	sF	44	8	44	$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathbf{R} \oplus \mathbf{R}$
1 1 2 2 ○—○═●—○ γ_1 γ_2 γ_3 γ_4	φ_3	no	sGT	-40	12	40	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathbf{R}$
1 1 2 2 ○—○═○—● γ_1 γ_2 γ_3 γ_4	φ_4	yes	sGT	-180	22	30	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3) \oplus \mathbf{R}$
1 1 2 2 ●—●═○—○ γ_1 γ_2 γ_3 γ_4	$\varphi_1 + \varphi_2$	no	sF	84	10	42	$\mathfrak{f}_{4(-20)}$	$\mathfrak{su}(3) \oplus \mathbf{R} \oplus \mathbf{R}$
1 1 2 2 ●—○═○—● γ_1 γ_2 γ_3 γ_4	$2\varphi_1 + \varphi_4$	no	sGT	-40	12	40	$\mathfrak{f}_{4(4)}$	$\mathfrak{so}(5) \oplus \mathbf{R} \oplus \mathbf{R}$

Thank you for your attention!