Deformation theory of nearly G_2 manifolds

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Special Geometries and Gauge Theory
would have been in Brest, France.
July 3, 2020

based on joint work with

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Brief Outline

- Introduction to G_2 and nearly G_2 structures.
- Dirac operators on nearly G_2 manifolds.
- Hodge theory on nearly G_2 manifolds.
- Infinitesimal deformations of nearly G_2 structures.
- Second-order deformations.
- Obstructions to higher-order deformations in Aloff–Wallach spaces.

Introduction to G_2 and nearly G_2 structures

Definition

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$$B_{ij} dx^1 \wedge \ldots \wedge dx^7 = \left(\frac{\partial}{\partial x^i} \lrcorner \varphi\right) \wedge \left(\frac{\partial}{\partial x^j} \lrcorner \varphi\right) \wedge \varphi$$

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From this, define the symmetric bilinear form g_{ij} by

$$g_{ij} = \frac{1}{6^{\frac{2}{9}}} \frac{B_{ij}}{\det(B)^{\frac{1}{9}}}$$

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- A G_2 structure on M exists if and only if M is *orientable* and *spinnable* which is equivalent to $w_1(M) = 0$ and $w_2(M) = 0$.

Introduction contd.

G_2 structure $\leftrightsquigarrow$ "non-degenerate" 3-form $\varphi \rightsquigarrow g_\varphi$ and orientation nonlinearly. and hence a Hodge star $*_\varphi$. Denote $*_\varphi\varphi = \psi$.

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Definition

A G_2 structure is a *nearly* G_2 structure if τ_0 is the only non-vanishing torsion form. Equivalently,

$$d\varphi = \tau_0 \psi, \qquad d\psi = 0. \tag{NG2}$$

In such a case, (M, φ) is a nearly G_2 manifold.

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If M is connected then differentiating (NG2) and using the fact that $\cdot \wedge \psi$ is an isomorphism from $\Omega^1(M)$ to $\Omega^5_1(M)$ implies $\tau_0 =$ constant.

The constant τ_0 can be altered by rescaling the metric and readjusting the orientation. Under our assumptions we fix $\tau_0 = 4$.

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To summarize,

(M^7, φ) is a nearly G_2 manifold if φ is a G_2-structure such that

$$d\varphi = 4\psi.$$

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Fact. Nearly G_2 manifolds are positive Einstein.

Dirac operators on nearly G_2 manifolds

Let $\not{S}(M)$ be the *rank 8* spinor bundle over M.

Definition

A spinor $\eta \in \Gamma(\not{S}(M))$ is called a *Killing* spinor if for any $X \in \Gamma(TM)$

$$\nabla_X \eta = -\frac{1}{2} X \cdot \eta$$

where " $\cdot$ " is the Clifford multiplication.

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Fact: nearly G_2 structures on $M \overset{\text{1-1}}{\longleftrightarrow}$ Killing spinors on $\$(M)$

If η is the real Killing spinor corresponding to φ then we have the decomposition

$$\$(M) = \underbrace{\Omega^0(M)\ \eta}_{\color{red}{Rank\ 1}} \oplus \underbrace{\Omega^1(M) \cdot \eta}_{\color{red}{Rank\ 7}} \cong \Omega^0(M) \oplus \Omega^1(M).$$

Dirac operators

Definition

The *Dirac operator* $\not{D}$ is a first order differential operator on $\$(M)$ defined as follows. Let $s = (f, X) \in \Gamma(\$(M))$. Then

$$\not{D}(f, X) = \left(\frac{7}{2}f + d^*X, df + dX \lrcorner \varphi - \frac{5}{2}X\right)$$

Moreover, $\not{D}^* = \not{D}$ and $\not{D}^2(f, X) = (\Delta f + d^*X + \frac{49}{4}f, \Delta_d X + dX \lrcorner \varphi + df + \frac{25}{4}X)$ and hence is elliptic.

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For our purposes we need to define a **modified Dirac operator**

$$\tilde{\not{D}}(f, X) = \left(2f - \frac{3}{7}d^*X, \frac{1}{2}df + 6X - dX \lrcorner \varphi\right)$$

Since $\$(M) \cong \Omega^0 \oplus \Omega^1 \cong \Omega^4_1 \oplus \Omega^4_7$ we can see this as an operator

$$\tilde{\not{D}} \colon \Omega^0 \oplus \Omega^1 \to \Omega^4_1 \oplus \Omega^4_7$$
$$(f, X) \mapsto \frac{1}{2}d(f\varphi) + \pi_{1\oplus 7}(d^*(X \wedge \psi))$$

Hodge theory on nearly G_2 manifolds

By explicit computation $\ker \tilde{\not{D}} = \mathcal{K} = \{$Killing vector fields on M$\}$

$$\Omega^4_{1\oplus 7} \cong \mathrm{Im}\tilde{\not{D}} \oplus \ker \tilde{\not{D}} \cong d\Omega^3_1 \oplus \pi_{1\oplus 7}(d^*\Omega^5_7) \oplus \{X \wedge \varphi, X \in \mathcal{K}\}$$

Thus we have the decomposition

$$\Omega^4 = d\Omega^3_1 \oplus d^*\Omega^5_7 \oplus \{X \wedge \varphi, X \in \mathcal{K}\} \oplus \Omega^4_{27}$$

$$\Omega^4_{exact} = d\Omega^3_1 \oplus \{X \wedge \varphi, X \in \mathcal{K}\} \oplus \Omega^4_{27.exact}$$

By explicit computation $\ker D = \mathcal{K} = \{\text{Killing vector fields on M}\}$

$$\Omega^4_{1\oplus 7} \cong \mathrm{Im} D \oplus \ker D \cong d\Omega^3_1 \oplus \pi_{1\oplus 7}(d^*\Omega^5_7) \oplus \{X \wedge \varphi \mid X \in \mathcal{K}\}$$

Thus we have the decomposition

$$\Omega^4 = d\Omega^3_1 \oplus d^*\Omega^5_7 \oplus \{X \wedge \varphi, X \in \mathcal{K}\} \oplus \Omega^4_{27} \qquad \text{not } L^2 \text{ orthogonal}$$

$$\Omega^4_{exact} = d\Omega^3_1 \oplus \{X \wedge \varphi \mid X \in \mathcal{K}\} \oplus \Omega^4_{27,exact} \qquad L^2 \text{ orthogonal}$$

Theorem (D.–Singhal '20)

Let (M, φ, ψ) be a complete nearly G_2 manifold not isometric to S^7. Then every harmonic 4-form lies in Ω^4_{27}. Equivalently every harmonic 3-form lies in Ω^3_{27}.

Idea of the proof

$\alpha \in \Omega^4$ harmonic

$$\alpha = \underbrace{X \wedge \varphi}_{\tilde{K}} + d(f\varphi) + d^*(\underbrace{Y}_{K^{\perp_{L^2}}} \wedge \varphi) + \underbrace{\alpha_0}_{\Omega^4_{27}}$$

$\Rightarrow$

$$0 = \langle \alpha, d(f\varphi)\rangle_{L^2} = \langle X \wedge \varphi, d(f\varphi)\rangle_{L^2} + \|d(f\varphi)\|^2_{L^2} + \underbrace{\langle \alpha_0, d(f\varphi)\rangle_{L^2}}_{=0}$$

$$= \langle \underbrace{d^*(X \wedge \varphi)}_{=4X \lrcorner \psi \text{ as } X \in \operatorname{Ker}(D)}, f\varphi\rangle + \|d(f\varphi)\|^2$$

$\therefore$ $\alpha = X \wedge \varphi + d^*(Y \wedge \varphi) + \alpha_0$

Proof contd.

$0 = d^*\alpha = d^*(x \wedge \varphi) + d^*\alpha_0$

$\Rightarrow \quad \|d^*\alpha_0\|^2 = \langle d^*(x\wedge\varphi), d^*(x\wedge\varphi)\rangle = 64\|x\|^2$

$\shortparallel$

$-\langle d^*\alpha_0, d^*(x\wedge\varphi)\rangle = -4\langle \alpha_0, d(x\lrcorner\psi)\rangle = 16\langle \alpha_0, x\wedge\varphi\rangle = 0$

$\therefore \quad \alpha = d^*(y\wedge\varphi) + \alpha_0$

Fact:- On NG_2 if $\alpha_0 \in \Omega^4_{27}$ w/ $d^*\alpha_0 = 0 \Rightarrow d\alpha_0 \in \Omega^5_{14}$.

$\therefore$

$0 = \langle \alpha, d^*(y\wedge\varphi)\rangle = \|d^*(y\wedge\varphi)\|^2 + \langle \alpha_0, d^*(y\wedge\varphi)\rangle$

$= 0$ by the fact.

$\alpha = \alpha_0.$ ∎

Cohomology contd.

We also have a partial result on the degree 2 (and 5) cohomology.

Proposition

Let $\beta \in \Omega^2$ *be a harmonic* 2*-form on* M. *Then*

$$\beta = \beta_7 + \beta_{14} = (X \lrcorner \varphi) + \beta_{14}.$$

If $\operatorname{curl} X = \lambda X$ *with* $\lambda \neq 12$, *then* $\beta \in \Omega^2_{14}$.

Deformation theory

Aim: Deform the nearly G_2 structure (φ, ψ) to a nearby nearly G_2 structure $(\tilde{\varphi}, \tilde{\psi})$.

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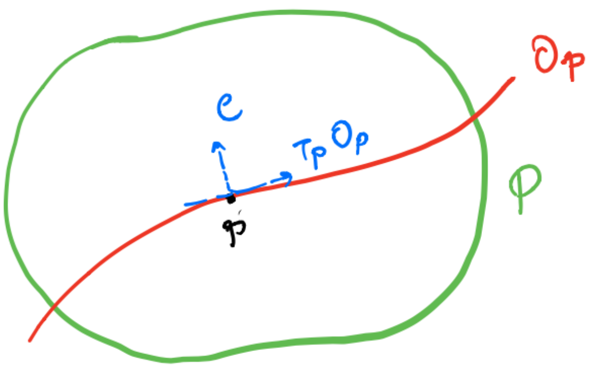
Infinitesimal deformations:
$\mathcal{P} :=$ space of G_2 structures on M
$\mathfrak{p} = (\varphi, \psi) \in \mathcal{P}$, nearly G_2 structure on M.
$\mathcal{O}_{\mathfrak{p}} :=$ Orbit of $\mathfrak{p}$ under $\mathrm{Diff}_0(M)$

$$T_{\mathfrak{p}}\mathcal{P} = T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}} \oplus \mathcal{C}$$



We are interested in finding the complement $\mathcal{C}$ of $T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}$ in $T_{\mathfrak{p}}\mathcal{P}$.

Infinitesimal deformations

Let $(\xi, \eta) \in T_{\mathfrak{p}}\mathcal{P}$. For some $(f, X, Y, \xi_0) \in \Omega^0(M) \times \mathcal{K} \times \mathcal{K}^{\perp_{L^2}} \times \Omega^3_{27}$

$$\eta = -4X \wedge \varphi + d(f\varphi) + d^*(Y \wedge \psi) + *\xi_0$$

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$$\eta = \underbrace{-4X \wedge \varphi}_{\mathcal{L}_X\psi \in T_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}}} + d(f\varphi) + d^*(Y \wedge \psi) + *\xi_0$$

thus modulo $\mathrm{Diff}_0(M)$

$$\eta = (4f - \frac{1}{7}d^*Y)\psi + (df + dY \lrcorner \varphi - 2Y) \wedge \varphi + *\xi_0$$

$$\xi = (3f - \frac{3}{28}d^*Y)\varphi - (df + dY \lrcorner \varphi - 2Y) \lrcorner \psi - \xi_0$$

Hence, $\mathcal{C} \cong \Omega^0(M) \times \mathcal{K}^{\perp_{L^2}} \times \Omega^4_{27}$.

Theorem (Alexandrov–Semmelmann'12, D.–Singhal.'20)

Let (M, φ, ψ) be a complete nearly G_2 manifold. Then the infinitesimal deformations of the nearly G_2 structure are in one to one correspondence with $(X, \xi_0) \in \mathcal{K} \times \Omega^3_{27}$ with

$$*d\xi_0 = -4\xi_0 \qquad \text{and} \qquad \Delta X = 12X$$

Idea of the proof.

If (ξ, η) nearly G_2 deformation $\Rightarrow \eta$ exact

$\therefore \quad \eta = d(f\varphi) + X \wedge \varphi + \eta_0$

$X \in \mathcal{K}$, $\quad \eta_0 \in \Omega^4_{27}$

$\Rightarrow \quad \xi = 3f\varphi - (X + df) \lrcorner \psi - *\eta_0$

and $\quad d\xi = 4\eta \implies \Delta f = 7f \rightsquigarrow f = 0$

by Obata's Thm.

Proof contd.

$X \in K \implies d^*X = 0$ and $dX = 2X \lrcorner \varphi$

$\implies \Delta X = d^*dX = d^*(2X \lrcorner \varphi) = 12X.$

Taking $*d\xi$ gives $-4\xi_0.$

$\square$

Second order deformations

Definition

Given a nearly G_2 structure (φ_0, ψ_0) and an infinitesimal deformation (ξ_1, η_1), a second order deformation of (φ_0, ψ_0) in the direction of (ξ_1, η_1) is a pair $(\xi_2, \eta_2) \in \Omega^3 \times \Omega^4$ such that

$$\varphi = \varphi_0 + \epsilon\xi_1 + \frac{\epsilon^2}{2}\xi_2, \qquad \psi = \psi_0 + \epsilon\eta_1 + \frac{\epsilon^2}{2}\eta_2$$

is a nearly G_2 structure up to terms of order $O(\epsilon^2)$. An infinitesimal deformation (ξ_1, η_1) is said to be *obstructed to second order* if there exists no second-order deformation in its direction.

In order to find the second order deformation we need to enlarge the space under consideration.

Second order deformations

Let $U \subset \Omega^4_{+,exact}$ be a small neighborhood of the 4-form ψ.
if $\|\eta\|_\varphi$ is small. $\tilde{\psi} = \psi + \eta \in \Omega^4_{+,exact}$

Proposition

The pair of positive forms $(\tilde{\varphi}, \tilde{\psi})$ defines a nearly G_2 structure if there exists a $Z \in \Gamma(TM)$ such that

$$d\tilde{\varphi} - 4\tilde{\psi} = d * d(Z \lrcorner \tilde{\psi}).$$

This condition is equivalent to the vanishing of the map

$$\Phi : U \times \Gamma(TM) \to \Omega^4_{exact}$$
$$(\tilde{\psi}, Z) \mapsto d * \tilde{\psi} - 4\tilde{\psi} - d * d(Z \lrcorner \tilde{\psi}).$$

Second order deformations

The obstructions on the first order deformation of the nearly G_2 structure (φ, ψ) are given by Im(DΦ) which is characterized by

Proposition

Let (φ, ψ) *be a nearly* G_2 *structure and* $(\xi, \eta) \in \Omega^3_{27} \times \Omega^4_{27,exact}$ *be their first order deformation in* $\mathcal{P}$*. Then* $\alpha \in \Omega^4_{exact}$ *lies in the image of* DΦ *if and only if*

$$\langle d^*\alpha - 4 * \alpha, \chi \rangle_{L^2} = 0$$

for all co-closed $\chi \in \Omega^3_{27}$ *such that* $\Delta\chi = 16\chi$*.*

Sketch of the proof.

$$\eta = \underbrace{\chi \wedge \varphi}_{\mathcal{K}} + d(f\varphi) + \eta_0 = d\left(-\tfrac{1}{4}\chi \lrcorner \psi + f\varphi\right) + \eta_0$$

$$\Xi = 3f\varphi - \left(df + \tfrac{1}{4}\chi\right) \lrcorner \psi - *\eta_0$$

$$\alpha = \underbrace{y \wedge \varphi}_{\mathcal{K}} + d(h\varphi) + \underbrace{\alpha_0}_{\Omega^4_{27}}$$

$$\alpha \in \mathrm{Im}(D\Phi) \implies$$

$$d\Xi - 4\eta - d*d(Z \lrcorner \psi) = \alpha = d\left(-\tfrac{1}{4} y \lrcorner \psi + h\varphi\right) + \alpha_0$$

$$\downarrow$$

$$f + \tfrac{1}{7} d^* Z = -h$$

$$df - 2Z + \mathrm{curl}\, Z = \tfrac{1}{4} y$$

$$-d*\eta_0 - 4\eta_0 = \alpha_0$$

(A)

Sketch of the proof.

Let $\alpha_0 = 0$. Then by IFT, a solution to the first pair in Ⓐ exists if

$$\tilde{D} : \Omega^0 \times \Omega^1 \longrightarrow \Omega^0 \times \Omega^1$$ is invertible.

$\tilde{D} \rightsquigarrow \tilde{\not{D}} \implies \ker(\tilde{D}) = \operatorname{coker}(\tilde{D}) \Rightarrow f = 0 = Z.$

If $\alpha_0 \neq 0$ satisfies the 3rd eq. in Ⓐ then

$$d^*\alpha_0 = -d^*d*\eta_0 - 4d^*\eta_0$$

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and $*\eta_0$ is co-closed $\rightarrow \langle d^*\alpha_0 - 4*\alpha_0, \xi\rangle_{L^2} = 0.$

Next show it for whole α. ▨

Second order deformations contd.

Consider the formal power series defining positive *exact* 4-form (up to order 2)

$$\psi_\epsilon = \psi_0 + \epsilon\eta_1 + \frac{\epsilon^2}{2}\eta_2 \qquad\qquad (\eta_i \in \Omega^4_{exact})$$

$$\varphi_\epsilon = \varphi_0 + \epsilon\xi_1 + \frac{\epsilon^2}{2}(\widehat{\eta_2} - Q_1(\eta_1))$$

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$(\widehat{\eta_2} - Q(\eta_1), \eta_2)$ is a second order deformation of the nearly G_2 structure (φ_0, ψ_0) if and only if

$$\langle *Q(\eta_1), d\chi - 4 * \chi\rangle_{L^2} = 0$$

for all $\chi \in \Omega^3_{27}$ such that $d^*\chi = 0, \Delta\chi = 16\chi$.

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Remarks:

- Foscolo [2017] proved similar results for nearly Kähler 6-manifolds.
- Infinitesimal deformations on Flag manifold $\mathbb{F}_3$ were found to be obstructed.
- He showed that the "*inner product*" does not always vanish.

Deformations of the Aloff–Wallach space.

- Aloff–Wallach space is the normal homogeneous space $\dfrac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$.
- $\mathrm{SU}(2)_d = \text{span}\left\{\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a\right) \mid a \in \mathrm{SU}(2)\right\}$ and

  $\mathrm{U}(1) = \text{span}\left\{\left(\begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-2it} \end{pmatrix}, 1\right) \mid t \in \mathbb{R}\right\}$

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Theorem (Alexandrov–Semmelmann' 12)

The space of infinitesimal deformations of the nearly G_2 structure on A–W space is isomorphic to 8-dimensional $\mathfrak{su}(3)$ as an $\mathrm{SU}(3)\times\mathrm{SU}(2)$ representation.

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Theorem (D.–Singhal'20)

The infinitesimal deformations of A–W space are all obstructed.

Idea of the proof.

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The infinitesimal deformations of A–W space are all obstructed.

Fix $\theta \in \mathfrak{su}(3) \rightsquigarrow$ inf. deformations $(\xi_\theta, \eta_\theta)$ of (φ_0, ψ_0). To prove obstructions, find $\theta' \in \mathfrak{su}(3)$ s.t

$$\langle Q(\eta_\theta), \xi_{\theta'} \rangle_{L^2} \neq 0$$

[θ' depends on θ]

Done this:- explicitly using the representation theory of the A–W space and $\mathfrak{su}(3)$.

Thank you for your attention