Smooth Loops and Loop Bundles

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Outline

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- It is known that some properties of G_2-structures on 7-manifolds can be expressed in terms of an octonion bundle. Smooth loop bundles are therefore a generalization of this construction.

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- In G_2-geometry one of the interesting questions is regarding properties of G_2-structures that correspond to the same Riemannian metric.
- It turns out that some of the properties of G_2-structures and octonion bundles are in fact quite generic and appear in the general framework of smooth loops.

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1. Product $(p, q) \mapsto pq$
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A *loop* is a quasigroup with an identity element 1. For any $q \in \mathbb{L}$, define left and right inverses

$$q^{\rho} = q\backslash 1 \text{ and } q^{\lambda} = 1/q.$$

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8. *Group*: clearly any associative loop is a group.

Pseudoautomorphisms

Definition

An invertible map $\alpha : \mathbb{L} \longrightarrow \mathbb{L}$ is a *right pseudoautomorphism* of $\mathbb{L}$ if there exists an element $A \in \mathbb{L}$ such that for any $p, q \in \mathbb{L}$

$$\alpha(p) \cdot \alpha(q) A = \alpha(pq) A. \tag{1}$$

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In a Moufang loop, the map Ad_q, given by $p \longmapsto qpq^{-1}$ is a right pseudoautomorphism with companion q^3.

Definition

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Lemma

The set $\Psi^R(\mathbb{L})$ of all pairs (α, A), where $\alpha \in \mathrm{PsAut}^R(\mathbb{L})$ and $A \in \mathbb{L}$ is its companion, is a group with identity element $(\mathrm{id}, 1)$ and the following group operations:

$$\textit{product:} \quad (\alpha_1, A_1)(\alpha_2, A_2) = (\alpha_1 \circ \alpha_2, \alpha_1(A_2) A_1) \tag{2a}$$

$$\textit{inverse:} \quad (\alpha, A)^{-1} = \left(\alpha^{-1}, \alpha^{-1}\left(A^{\lambda}\right)\right) = \left(\alpha^{-1}, \left(\alpha^{-1}(A)\right)^{\rho}\right). \tag{2b}$$

- $\Psi^{R}(\mathbb{L})$ has two actions on $\mathbb{L}$. Let $h=(\alpha,A)\in\Psi^{R}(\mathbb{L})$ and $p\in\mathbb{L}$,

$$\begin{aligned}\text{non-faithful } h'(p) &= \alpha(p)\\ \text{faithful } h(p) &= \alpha(p)A.\end{aligned}$$

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- Denote by $\mathring{\mathbb{L}}$ the set $\mathbb{L}$ equipped with the faithful action of $\Psi^R(\mathbb{L})$. In this case, $\mathrm{Aut}(\mathbb{L}) \cong \mathrm{Stab}_{\Psi^R(\mathbb{L})}(1)$. The set of companions $\mathcal{C}^R(\mathbb{L}) = \mathrm{Orb}_{\Psi^R(\mathbb{L})}(1)$. If $\Psi^R(\mathbb{L})$ acts transitively, $\mathbb{L}$ is known as a *G-loop*.

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- By (1), for $p, q \in \mathbb{L}$, we have

$$h(pq) = h'(p) \cdot h(q).$$

Example

Suppose $\mathbb{L} = S^3$ - the group of unit quaternions. We know that $\mathrm{Aut}\left(S^3\right) \cong SO\left(3\right)$. Now however, $\Psi^R\left(S^3\right)$ consists of all pairs $\left(\alpha, A\right) \in SO\left(3\right) \times S^3$ with the group structure defined by (2a),which is the semi-direct product

$$\Psi^R\left(S^3\right) \cong SO\left(3\right) \ltimes S^3 \cong Sp\left(1\right) Sp\left(1\right) \cong SO\left(4\right).$$

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Example

Suppose $\mathbb{L} = S^7$ - the Moufang loop of unit octonions. In this case, $\Psi^R\left(S^7\right) \cong Spin(7)$ and $\mathrm{PsAut}^R\left(S^7\right) \cong SO(7)$. The (right) nucleus is $\{\pm 1\}$, so the projection of a pair $(\alpha, A) \in \Psi^R\left(S^7\right)$ to $\alpha \in \mathrm{PsAut}^R\left(S^7\right)$ corresponds to the double cover $Spin(7) \longrightarrow SO(7)$.

Modified product

- Let $r \in \mathbb{L}$, and define the modified product $\circ_r$ on $\mathbb{L}$ via

$$p \circ_r q = (p \cdot qr)\,/r. \tag{3}$$

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Lemma

Let $h \in \Psi^R(\mathbb{L}, \cdot)$ and $p, q, r, x \in \mathbb{L}$, then

$$h'(p \circ_r q) = h'(p) \circ_{h(r)} h'(q). \tag{4}$$

and

$$p \circ_{rx} q = (p \circ_x (q \circ_x r))/_x r. \tag{5}$$

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- For any $r \in \mathbb{L}$, define the diffeomorphisms

$$\begin{aligned} L_r : \mathbb{L} &\longrightarrow \mathbb{L} & R_r : \mathbb{L} &\longrightarrow \mathbb{L} \\ q &\longmapsto rq & q &\longmapsto qr. \end{aligned}$$

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  Similarly, define $L_r^{(s)}$ and $R_r^{(s)}$ for products with respect to $\circ_s$.
- Given a tangent vector $\xi \in T_1\mathbb{L}$, define the vector field $\rho(\xi)$ given by

$$\rho(\xi)_q = (R_q)_* \xi \tag{6}$$

  at any $p \in \mathbb{L}$. If $\mathbb{L}$ is a Lie group, this is equivalent to the standard definition of a right-invariant vector field X such that $(R_q)_* X_p = X_{pq}$, however in the non-associative case, $R_q \circ R_p \neq R_{pq}$, so in that case, $\rho(\xi)$ is not right-invariant.

Exponential map

- If the loop $(\mathbb{L}, \circ_s)$ is monoassociative, it was shown by Kuz'min in 1971 that one can define an exponential map $\exp_s : T_1\mathbb{L} \longrightarrow \mathbb{L}$ as the solution of the equation (7) for some $\xi \in T_1\mathbb{L}$:

$$\begin{cases} \frac{dp_\xi(t)}{dt} = \left(R^{(s)}_{p_\xi(t)}\right)_* \xi \\ p_\xi(0) = 1. \end{cases} \tag{7}$$

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- Suppose $(\mathbb{L}, \cdot)$ is power-associative and moreover, power left-alternative, i.e. $x^k (x^l q) = x^{k+l} q$ for all $x, q \in \mathbb{L}$. Then, it can be shown that the exponential functions are equal for all $q \in \mathbb{L}$.

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- In general they will be different functions, but with the same derivative at $t = 0$.

Tangent algebra

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Definition

For any $\xi, \gamma \in T_1\mathbb{L}$, the p-bracket $[\cdot,\cdot]^{(p)}$ is defined as

$$[\xi, \gamma]^{(p)} = \frac{d}{dt}\left(\left(\mathrm{Ad}^{(p)}_{\exp_p(t\xi)}\right)_* \gamma\right)\bigg|_{t=0} = -\left(R_p^{-1}\right)_* [\rho(\xi), \rho(\gamma)]_p . \qquad (8)$$

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- The key difference with Lie algebras is that every $p \in \mathbb{L}$ defines a bracket. If p and q are in different orbits of $\Psi^R(\mathbb{L})$, then these algebras do not need to be isomorphic.

Maurer-Cartan form

- Given $p \in \mathbb{L}$ and and $\xi \in \mathfrak{l}$, define θ_p to be

$$\theta_p \left(\rho\left(\xi\right)_p \right) = \left(R_p^{-1}\right)_* \rho\left(\xi\right)_p = \xi. \tag{9}$$

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Theorem

Let $p \in \mathbb{L}$ and let $[\cdot,\cdot]^{(p)}$ be bracket on $\mathfrak{l}^{(p)}$. Then θ satisfies the following equation at p:

$$\left(d\theta\right)_p - \frac{1}{2}\left[\theta,\theta\right]^{(p)} = 0, \tag{10}$$

where $[\theta,\theta]^{(p)}$ is the bracket of $\mathbb{L}$-algebra-valued 1-forms such that for any $X, Y \in T_p\mathbb{L}$, $\frac{1}{2}\left[\theta,\theta\right]^{(p)}(X,Y) = \left[\theta(X),\theta(Y)\right]^{(p)}$.

- Define $b : \mathbb{L} \longrightarrow \mathfrak{l} \otimes \Lambda^2 \mathfrak{l}^*$ given by $p \mapsto [\cdot, \cdot]^{(p)}$. Then,

$$db|_p (\eta, \gamma) = [\eta, \gamma, \theta_p]^{(p)} - [\gamma, \eta, \theta_p]^{(p)}, \tag{11}$$

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where $[\cdot, \cdot, \cdot]^{(p)}$ is the $\mathbb{L}$-*algebra associator* on $\mathfrak{l}^{(p)}$ given by

$$\begin{aligned} [\eta, \gamma, \xi]^{(p)} &= \frac{d^3}{dt d\tau d\tau'} \exp(\tau\eta) \circ_p \left(\exp(\tau'\gamma) \circ_p \exp(t\xi)\right)\Big|_{t,\tau,\tau'=0} \\ &\quad - \frac{d^3}{dt d\tau d\tau'} \left(\exp(\tau\eta) \circ_p \exp(\tau'\gamma)\right) \circ_p \exp(t\xi)\Big|_{t,\tau,\tau'=0}. \end{aligned} \tag{12}$$

- In a left-alternative loop, the associator is skew in first two entries, but not in general so. Define $a_p(\eta, \gamma, \xi) = [\eta, \gamma, \xi]^{(p)} - [\gamma, \eta, \xi]^{(p)}$.

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$$\begin{aligned} [\eta,\gamma,\xi]^{(p)} &= \frac{d^3}{dt d\tau d\tau'} \exp(\tau\eta) \circ_p \left(\exp(\tau'\gamma) \circ_p \exp(t\xi)\right)\Big|_{t,\tau,\tau'=0} \\ &\quad - \frac{d^3}{dt d\tau d\tau'} \left(\exp(\tau\eta) \circ_p \exp(\tau'\gamma)\right) \circ_p \exp(t\xi)\Big|_{t,\tau,\tau'=0}. \end{aligned} \tag{12}$$

- In a left-alternative loop, the associator is skew in first two entries, but not in general so. Define $a_p(\eta,\gamma,\xi) = [\eta,\gamma,\xi]^{(p)} - [\gamma,\eta,\xi]^{(p)}$.
- In a Lie group, the exterior derivative of the Maurer-Cartan equation gives the Jacobi identity. In general, from (11) we obtain a generalization (known as the Akivis identity)

$$\mathrm{Jac}^{(p)}(\xi,\eta,\gamma) = a_p(\xi,\eta,\gamma) + a_p(\eta,\gamma,\xi) + a_p(\gamma,\xi,\eta). \tag{13}$$

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Definition

Define the map $\varphi : \mathring{\mathbb{L}} \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ such that for each $s \in \mathring{\mathbb{L}}$ and $\gamma \in \mathfrak{p}$,

$$\varphi_s(\gamma) = \frac{d}{dt}\left[\exp_{\mathfrak{p}}(t\gamma)(s)\right]/s\Big|_{t=0} \in \mathfrak{l}. \tag{14}$$

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- Suppose $\mathfrak{h}_s$ is the Lie algebra of $\mathrm{Aut}(\mathbb{L}, \circ_s)$ and $\mathfrak{q}^{(s)} = T_1\mathcal{C}^R(\mathbb{L}, \circ_s)$. Since $\mathcal{C}^R(\mathbb{L}, \circ_s) \cong \Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}, \circ_s)$, we have $\mathfrak{q}^{(s)} \cong \mathfrak{p}/\mathfrak{h}^{(s)}$ as linear representations of $\mathrm{Aut}(\mathbb{L}, \circ_s)$. We can then see that $\ker \varphi_s = \mathfrak{h}^{(s)}$ and the image of φ_s is precisely $\mathfrak{q}^{(s)}$.

- The action of $\mathfrak{p}$ on $\mathfrak{l}$ is given by $\gamma \cdot \xi = \frac{d^2}{dt d\tau} \exp_{\mathfrak{p}}(t\gamma)' (\exp_s \tau\xi)\Big|_{t,\tau=0}$.

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Suppose $\xi, \eta \in \mathfrak{p}$, then for any $s \in \mathbb{L}$, we have

$$\xi \cdot \varphi_s(\eta) - \eta \cdot \varphi_s(\xi) = \varphi_s\left([\xi, \eta]_{\mathfrak{p}}\right) + [\varphi_s(\xi), \varphi_s(\eta)]^{(s)}, \qquad (15)$$

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- Let $\mathcal{N}^R\left(\mathfrak{l}^{(s)}\right) = \{\xi \in \mathfrak{l} : a_s(\eta, \gamma, \xi) = 0 \text{ for all } \eta, \gamma \in \mathfrak{l}\}$.

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Theorem

If $\mathbb{L}$ is a G-loop, we have the following inclusions of Lie algebras

$$\ker \varphi_s = \mathfrak{h}_s \underset{ideal}{\subset} \operatorname{Ann}_{\mathfrak{p}} (\varphi_s) \subset \operatorname{Ann}_{\mathfrak{p}} (b_s) \cong \mathfrak{h}^{(s)} \oplus \mathcal{N}^R \left(\mathfrak{l}^{(s)} \right) \subset \mathfrak{p}. \tag{16}$$

Example

If $\mathbb{L}$ is the loop of unit octonions, then we know $\mathfrak{p} \cong \mathfrak{so}(7) \cong \Lambda^2 \left(\mathbb{R}^7\right)^*$ and $\mathfrak{l} \cong \mathbb{R}^7$, so φ_1 can be regarded as an element of $\mathbb{R}^7 \otimes \Lambda^2 \mathbb{R}^7$, and this is (up to a factor) a dualized version of the G_2-invariant 3-form φ, as used to project from $\Lambda^2 \left(\mathbb{R}^7\right)^*$ to $\mathbb{R}^7$. The kernel of this map is then the Lie algebra $\mathfrak{g}_2$. In this case, both b_s and φ_s are determined by the same object, but in general they have different roles.

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- Let M be a smooth manifold and suppose $s : M \longrightarrow \mathbb{L}$ is a smooth map. Using s we define a product on $\mathbb{L}$-valued maps from M and a corresponding bracket on $\mathfrak{l}$-valued maps.

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- Let $A, B : M \longrightarrow \mathbb{L}$ and $\xi, \eta : M \longrightarrow \mathfrak{l}$, then at each $x \in M$, define

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\begin{aligned}
A \circ_s B|_x &= A_x \circ_{s_x} B_x \in \mathbb{L} && \text{(17a)}\\
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- As for Lie groups, define the (right) *Darboux derivative* $\theta_s = s^* \theta \in \Omega^1(M, \mathfrak{l})$. At every $x \in M$,

$$\left. (\theta_s) \right|_x = \left(R_{s(x)}^{-1} \right)_* \left. ds \right|_x . \tag{18}$$

Theorem

Let M be a smooth manifold and let $x \in M$. Suppose $A, B, s \in C^{\infty}(M, \mathbb{L})$, then

$$d(A \circ_s B) = (dA) \circ_s B + A \circ_s (dB) + [A, B, \theta_s]^{(s)} \tag{19}$$

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The $\mathfrak{l} \otimes \mathfrak{p}^*$-valued map $\varphi_s : M \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ satisfies

$$d\varphi_s = \mathrm{id}_{\mathfrak{p}} \cdot \theta_s - [\varphi_s, \theta_s]^{(s)}. \tag{21}$$

Loop bundles

- Let $\mathbb{L}$ be a smooth loop, and let us define for brevity $\Psi^R(\mathbb{L}) = \Psi$, $\mathrm{Aut}(\mathbb{L}) = H$, and $\mathrm{PsAut}^R(\mathbb{L}) = G \supset H$, and $\mathcal{N}^R(\mathbb{L}) = \mathcal{N}$. As before, suppose $\Psi, H, G, \mathcal{N}$ are Lie groups.

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- Let M be a smooth manifold with a Ψ-principal bundle $\mathcal{P}$. Recall that if S is a set with an action of Ψ on it, then we can define an associated bundle $\mathcal{P} \times_{\Psi} S$, with sections being in an 1-1 correspondence with equivariant maps $\mathcal{P} \longrightarrow S$. Define the following bundles:

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Bundle	**Equivariant map**	**Equivariance**
$\mathcal{P}$	$k : \mathcal{P} \longrightarrow \Psi$	$k(ph) = h^{-1}k(p)$
$\mathrm{Ad}(\mathcal{P}) = \mathcal{P} \times_{\mathrm{Ad}_{\Psi}} \Psi$	$u : \mathcal{P} \longrightarrow \Psi$	$u(ph) = h^{-1}u(p)h$
$\mathring{Q} = \mathcal{P} \times_{\Psi} \mathring{\mathbb{L}}$	$r : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$	$r(ph) = h^{-1}(r(p))$
$Q = \mathcal{P} \times_{\Psi'} \mathbb{L}$	$q : \mathcal{P} \longrightarrow \mathbb{L}$	$q(ph) = \left(h^{-1}\right)'(q(p))$
$\mathcal{A} = \mathcal{P} \times_{\Psi'_{*}} \mathfrak{l}$	$\eta : \mathcal{P} \longrightarrow \mathfrak{l}$	$\eta(ph) = \left(h^{-1}\right)'_{*}\eta$

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- Given s, easy to show that corresponding maps b_s, a_s, and φ_s are also equivariant.

Connections and Torsion

- Suppose the principal Ψ-bundle $\mathcal{P}$ has a principal connection given by

$$T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP}$$

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- $d^{\mathcal{H}} f$ is an equivariant horizontal map and, given the section $\tilde{f}$ of $\mathcal{P} \times_{\Psi} S$ that corresponds to f, $d^{\mathcal{H}} f$ defines a unique map $d^{\mathcal{H}} \tilde{f} : TM \longrightarrow \mathcal{P} \times_{\Psi} TS$.

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- Denote by D the covariant derivative on $\mathbb{L}$-valued maps and by $\mathring{D}$ the derivative on $\mathring{\mathbb{L}}$-valued maps

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The *torsion* $T^{(s,\omega)}$ of s and ω is a horizontal $\mathfrak{l}$-valued 1-form on $\mathcal{P}$ given by

$$T^{(s,\omega)} = \theta_s \circ \mathrm{proj}_{\mathcal{H}} \tag{23}$$

where θ_s is the Darboux derivative of s. Equivalently, at $p \in \mathcal{P}$, we have

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Theorem

Let $\hat{\omega}^{(s)} = \varphi_s(\omega) \in \Omega^1(\mathcal{P}, \mathfrak{l})$. *Then,*

$$\theta_s = T^{(s,\omega)} - \hat{\omega}^{(s)}. \tag{25}$$

- We this see that the torsion is the horizontal part of the loop Darboux derivative θ_s, and hence it defines a 1-form with values in the bundle $\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$ over M. The vertical part of θ_s is $\hat{\omega}^{(s)} = \varphi_s(\omega)$.

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- This object is completely analogous to the torsion of a G_2-structure. If we take $\mathcal{P}$ to be the spin bundle over a 7-manifold M and ω the Levi-Civita connection of some metric on M, then it is easy to see that $T^{(s,\omega)}$ is precisely the torsion of the G_2-structure defined by the map s. Indeed, $S^7 \cong Spin(7)/G_2$, so an equivariant map $s : \mathcal{P} \longrightarrow Spin(7)/G_2$ defines a reduction of $\mathcal{P}$ to a G_2-subbundle (and hence a G_2-structure).

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- $T^{(s,\omega)} = 0$ if and only if $\theta_s = -\hat{\omega}^{(s)}$. For Lie groups, a Lie-algebra-valued 1-form is a Darboux derivative of some function if and only if it satisfies the Maurer-Cartan equation. For loops, such a characterization is in general more complicated and less clear.

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Theorem

Suppose $A, B : \mathcal{P} \longrightarrow \mathbb{L}$, *and* $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ *are equivariant, and let* $p \in \mathcal{P}$. *Then,*

$$D\left(A \circ_s B\right)\big|_p = \left(R_{B_p}^{(s_p)}\right)_* DA|_p + \left(L_{A_p}^{(s_p)}\right)_* DB|_p + \left[A_p, B_p, T^{(s,\omega)}\Big|_p\right]^{(s_p)}$$

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If $\xi, \eta : \mathcal{P} \longrightarrow \mathfrak{l}$ *are equivariant, then*

$$d^{\mathcal{H}}\left[\xi, \eta\right]^{(s)} = \left[d^{\mathcal{H}}\xi, \eta\right]^{(s)} + \left[\xi, d^{\mathcal{H}}\eta\right]^{(s)} + a_s\left(\xi, \eta, T^{(s,\omega)}\right). \tag{26}$$

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The $\mathfrak{l} \otimes \mathfrak{p}^*$*-valued map* $\varphi_s : \mathcal{P} \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ *satisfies*

$$d^{\mathcal{H}}\varphi_s = \mathrm{id}_{\mathfrak{p}} \cdot T^{(s,\omega)} - \left[\varphi_s, T^{(s,\omega)}\right]^{(s)}. \tag{27}$$

Curvature

- Curvature $F^{(\omega)} \in \Omega^2(\mathcal{P}, \mathfrak{p})$ of the connection ω on $\mathcal{P}$ is given by

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- Define $\hat{F}^{(s,\omega)} = \varphi_s\left(F^{(\omega)}\right) \in \Omega^2(\mathcal{P}, \mathfrak{l})$. This is a basic (i.e. horizontal and equivariant) 2-form on $\mathcal{P}$ with values in $\mathfrak{l}$, and thus also defines a 2-form on M with values in the bundle $\mathcal{A}$. It is easy to see that $\hat{F}^{(s,\omega)} = d^{\mathcal{H}}\hat{\omega}^{(s)}$.

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$\hat{F}^{(s,\omega)}$ and $T^{(s,\omega)}$ satisfy the following structure equation

$$\hat{F}^{(s,\omega)} = d^{\mathcal{H}}T^{(s,\omega)} - \frac{1}{2}\left[T^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)}, \tag{29}$$

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where wedge product between the 1-forms $T^{(s,\omega)}$ is implied. Equivalently,

$$d\hat{\omega}^{(s)} + \frac{1}{2}\left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}\right]^{(s)} = \hat{F}^{(s,\omega)} - d^{\mathcal{H}}\varphi_s \wedge \omega. \tag{30}$$

- Equation (29) is precisely the analog of the well-known "G_2 Bianchi identity" for the torsion of a G_2-structure:

$$\nabla_i T_j^{\ \ \alpha} - \nabla_i T_j^{\ \ \alpha} + 2T_i^{\ \beta} T_j^{\ \gamma} \varphi^{\alpha}{}_{\beta\gamma} = \frac{1}{4} \operatorname{Riem}_{ij}{}^{\beta\gamma} \varphi^{\alpha}{}_{\beta\gamma}. \tag{31}$$

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- In (29), the torsion differs by a sign, and we take $(\varphi_s)^a{}_{bc} = -\frac{1}{4}\varphi^a{}_{bc}$ and $(b_s)^a{}_{bc} = 2\varphi^a{}_{bc}$. Note that (27) and (26) then agree to give the standard expression for $\nabla\varphi$.

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- Equation (30) then shows that $-\hat{\omega}^{(s)}$ satisfies the Maurer-Cartan equation if and only if both $\hat{F}^{(s,\omega)}$ and $d^{\mathcal{H}}\varphi_s$ vanish. In the G_2 case, the latter implies the former and is actually equivalent to $T^{(s,\omega)} = 0$.

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Theorem

The quantity $\hat{F}^{(s,\omega)}$ satisfies the equation

$$d^{\mathcal{H}} \hat{F}^{(s,\omega)} = d^{\mathcal{H}} \varphi_s \wedge F = F \dot{\wedge} T^{(s,\omega)} - \left[\hat{F}^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)} \quad (32)$$

Deformations

Theorem

Suppose $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ *and* $u : \mathcal{P} \longrightarrow \Psi$ *are equivariant. Then,*

$$T^{(s,u^*\omega)} = T^{(s,\omega)} + \varphi_s\left(\left(u^*\theta_\Psi\right)^{\mathcal{H}}\right) = \left(u^{-1}\right)'_* T^{(u(s),\omega)} \tag{33a}$$

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Suppose $A : \mathcal{P} \longrightarrow \mathbb{L}$ *is equivariant. Then,*

$$T^{(As,\omega)} = \left(R_A^{(s)}\right)_*^{-1} DA + \left(\mathrm{Ad}_A^{(s)}\right)_* T^{(s,\omega)} \qquad (34a)$$

$$\hat{F}^{(As,\omega)} = \left(R_A^{(s)}\right)_*^{-1} \left(F' \cdot A\right) + \left(\mathrm{Ad}_A^{(s)}\right)_* \hat{F}^{(s,\omega)}, \qquad (34b)$$

where $F' \cdot A$ *denotes the infinitesimal action of* $\mathfrak{p}$ *on* $\mathbb{L}$.

Concluding remarks

- This is a work in progress, however we see a number of features of G_2-structures appearing in a general setting, suggesting that they are not that special.

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- The case of G_2-structures has very special properties: the properties of Moufang loops and also the fact the $\mathcal{P}$ corresponds to the spin structure.
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- The case of G_2-structures has very special properties: the properties of Moufang loops and also the fact the $\mathcal{P}$ corresponds to the spin structure.
- Even in the general setting we obtain a rich structure with a key role being played by the loop Maurer-Cartan equation.
- The lack of a suitable Bianchi identity for $\hat{F}^{(s,\omega)}$ precludes the possibility of defining characteristic classes in the usual sense, however there could be some weakened analogs.