Bryant-Salamon G_2 manifolds and coassociative fibrations

Spiro Karigiannis

Department of Pure Mathematics. University of Waterloo

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Joint work with Jason D. Lotay (Oxford), arXiv:2002.06444

Outline

Preliminaries

G_2 manifolds

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Definition

Let M be a 7-manifold. A G_2-structure on M is a 3-form φ on M that is *nondegenerate* in the sense that φ determines a metric and orientation via

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- G_2 manifolds are *Ricci-flat*, admit a *parallel spinor*, and have *holonomy* contained in G_2

Calibrated submanifolds

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- [associative submanifolds] $\leftrightarrow$ [J-holomorphic curves]
- [coassociative submanifolds] $\leftrightarrow$ [special Lagrangian submanifolds]
- SYZ conjecture: *mirror symmetry* via fibrations of CY by sLag.
- Motivates study of fibrations of G_2 manifolds by coassocatives

Cast of characters

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Definition

We say that (N, g_N) is asymptotically conical (AC) to C with rate $\lambda < 0$, if outside of a compact set $K \subseteq N$, we have $N \setminus K \cong (R, \infty) \times \Sigma$, and

$$g_N - g_C = \mathrm{O}(r^\lambda) \text{ as } r \to \infty$$

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Definition (Madsen–Swann)

The action G on (M, φ) admits a multi-moment map if there exist maps $\mu = (\mu_1, \mu_2, \mu_3)\colon M \to \mathbb{R}^3$ and $\nu\colon M \to \mathbb{R}$ such that

$$\varphi(X_2, X_3, \cdot) = d\mu_1, \quad \varphi(X_3, X_1, \cdot) = d\mu_2, \quad \varphi(X_1, X_2, \cdot) = d\mu_3,$$
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- We will exhibit partial results on existence in our situation.

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Lemma

Let (M, φ, g_φ) be a G_2 manifold and $N \subseteq M$ be coassociative. Suppose

$$\varphi = Fh_1 \wedge h_2 \wedge h_3 + h_1 \wedge \beta_1 + h_2 \wedge \beta_2 + h_3 \wedge \beta_3,$$

where $dh_k = 0$ on M and $h_k|_N = 0$. Then the triple $(\omega_1, \omega_2, \omega_3)$ defined by $\omega_k = \beta_k|_N$ is a hypersymplectic structure on N.

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Note: The "hypersymplectic metric" is not the induced metric $g_\varphi|_N$.

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In the "limit" as $\mathrm{Vol}(K) \to \infty$, each (M_c, φ_c) "converges" to $(\mathbb{R}^7, \varphi_{\mathbb{R}^7})$.

Case One: $\not{S}(\mathrm{S}^3)$

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Theorem (Coassociative fibration)

Let $M = (\$(\mathrm{S}^3), \varphi_c)$ and $M_0 = (\$(\mathrm{S}^3) \setminus \mathrm{S}^3, \varphi_0)$.

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(a) *The canonical projection $\pi\colon M \to \mathrm{S}^3$ is a coassociative fibration. All fibres are $\mathrm{SO}(4)$-invariant and diffeomorphic to $\mathbb{R}^4$.*

Group actions and coassociative fibration

Recall $M = \$(\mathrm{S}^3) \cong \mathbb{H} \times \mathrm{S}^3 \subseteq \mathbb{H} \times \mathbb{H}$ and $\mathrm{SU}(2) \cong \mathrm{S}^3$ is unit quaternions. We also have $\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm(1,1)\}$.

- $\mathrm{SU}(2)$ acts fibrewise on M and M_0 by $p \cdot (a, x) = (pa, x)$
- $\mathrm{SO}(4)$ acts fibrewise on M and M_0 by $(p, q) \cdot (a, x) = (pa\overline{q}, x)$

Theorem (Coassociative fibration)

Let $M = (\$(\mathrm{S}^3), \varphi_c)$ and $M_0 = (\$(\mathrm{S}^3) \setminus \mathrm{S}^3, \varphi_0)$.

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(b) *The canonical projection $\pi_0\colon M_0 \to \mathrm{S}^3$ is a coassociative fibration. All fibres are $\mathrm{SO}(4)$-invariant and diffeomorphic to $\mathbb{R}^4 \setminus \{0\}$.*

Multi-moment maps and hypersymplectic structures

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Proposition (Multi-moment map)

The $\mathrm{SU}(2)$ *action on* M *or* M_0 *admits a* *partial* *multi-moment map:*

$$\psi(X_1, X_2, X_3, \cdot) = d\nu \quad \text{where } \nu = 6(3c - r^2)(c + r^2)^{\frac{1}{3}} - 18c^{\frac{4}{3}}.$$

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However, the 1*-forms* $\alpha_k = \varphi(X_i, X_j, \cdot)$ *are not exact (nor even closed).*
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Up to a scale factor of $4\sqrt{3}$*, this hypersymplectic structure is the standard Euclidean hyperKähler structure on* $\mathbb{R}^4$ *or* $\mathbb{R}^4 \setminus \{0\}$*, respectively.*

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(a) *On M, the induced metric on the coassociative $\mathbb{R}^4$ fibres is conformally flat and asymptotically conical with rate -3 to the metric*

$$g_{\mathbb{R}^+\times\mathrm{S}^3} = dr^2 + \tfrac{4}{9}r^2 g_{\mathrm{S}^3}.$$

(b) *On M_0, the induced metric on the coassociative $\mathbb{R}^4 \setminus \{0\}$ fibres is exactly the above cone metric, which is conformally flat, but not flat, and so is not smooth at the origin.*

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As $\mathrm{Vol}(\mathrm{S}^3) \to \infty$, the coassociative fibration of M limits to the trivial coassociative $\mathbb{R}^4$ fibration of $(\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3, \varphi_0)$ over $\mathbb{R}^3$.

Case Two: $\Lambda^2_-(\mathrm{S}^4)$

Group action and coassociative fibration

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The space S^4 sits inside $\mathbb{R}^5$. Choose a splitting $\mathbb{R}^5 = \mathbb{R}^3 \oplus \mathbb{R}^2$. Then $\mathrm{SO}(3)$ acts on $\mathbb{R}^5$ by the standard action on $\mathbb{R}^3$ and trivially on $\mathbb{R}^2$.

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(a) *$\exists$ an $\mathrm{SO}(3)$-invariant projection $\pi_c \colon M \to \mathbb{R}^3$ with coassoc fibres. There is a critical circle $\mathrm{S}^1_c \subseteq \mathbb{R}^3$ such that the fibres $\pi_c^{-1}(x)$ are diffeomorphic to $T^*\mathrm{S}^2$ for $x \notin \mathrm{S}^1_c$ and to $(\mathbb{R}^+ \times \mathbb{RP}^3) \cup \{0\}$ for $x \in \mathrm{S}^1_c$.*

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where $\mathbf{p} \in \mathrm{S}^2 \subset \mathbb{R}^3$ and $\mathbf{q} \in \mathrm{S}^1 \subset \mathbb{R}^2$ are given by

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$$\mathbf{p} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi), \qquad \mathbf{q} = (\cos\beta, \sin\beta).$$

This determines an orthogonal frame Ω_1, Ω_2, Ω_3 for $\Lambda^2_-(\mathrm{S}^4)$ by

$$\begin{aligned}
\Omega_1 &= \sin\alpha d\alpha \wedge d\beta - \cos^2\alpha\sin\theta d\theta \wedge d\phi,\\
\Omega_2 &= \cos\alpha d\alpha \wedge d\theta - \sin\alpha\cos\alpha\sin\theta d\phi \wedge d\beta,\\
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\end{aligned}$$

thus determining fibre coordinates a_1, a_2, a_3. We let

$$a_1 = t, \qquad a_2 = s\cos\gamma, \qquad a_3 = s\sin\gamma.$$

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$$a_1 = t, \qquad a_2 = s\cos\gamma, \qquad a_3 = s\sin\gamma.$$

Thus we get local coordinates $(\alpha, \beta, \theta, \phi, s, t, \gamma)$ on our 7-manifold.

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Theorem (Nonlinear ordinary differential equations)

The nonlinear ODEs for $\mathrm{SO}(3)$-invariant coassociatives are $\dot{\beta} = 0$ and

$$0 = s \sin\alpha (t\dot{t} + s\dot{s}) + 2(c + s^2 + t^2) s \cos\alpha\,\dot{\alpha},$$

$$0 = t \sin^2\alpha (t\dot{t} + s\dot{s}) + (c + s^2 + t^2)(4t \sin\alpha \cos\alpha\,\dot{\alpha} - 2\cos^2\alpha\,\dot{t}).$$

From these we find the following constants on each coassociative fibres:

$$\beta, \qquad u = t\cos\alpha, \qquad v = 2(c + s^2 + t^2)^{\frac{1}{4}} \sin\alpha.$$

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The $\mathrm{SO}(3)$ action on M or M_0 admits a partial multi-moment map:

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$$\mathrm{diag}\Big(\frac{2(c+s^2+t^2)^{\frac{1}{2}}}{2c\cos^2\alpha+(s^2+t^2)(1+\cos^2\alpha)}, \frac{c+s^2+t^2}{2c\cos^2\alpha+(s^2+t^2)(1+\cos^2\alpha)}, 2(c+s^2+t^2)^{\frac{1}{2}}\sin^2\alpha\Big).$$

This hypersymplectic structure is not hyperKähler.

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(a) *All the coassociative fibres (smooth and singular) in M or M_0 are asymptotically conical, with rate at least -2, to the cone over $\mathbb{RP}^3$, equipped with the cone metric*

$$g_{\mathrm{AC}} = dr^2 + \tfrac{1}{4}r^2\sigma_1^2 + \tfrac{1}{2}r^2(\sigma_2^2 + \sigma_3^2).$$

(b) *All the singular coassociative fibres in M are conically singular to the cone over $\mathbb{RP}^3$, equipped with the cone metric*

$$g_{\mathrm{CS}} = dr^2 + \tfrac{1}{2}r^2(\sigma_1^2 + \sigma_2^2) + r^2\sigma_3^2.$$

(c) *The unique singular coassociative fibre in M_0 is exactly a Riemannian cone, equipped with the cone metric g_{AC} above.*

Flat limit

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Taking the direct sum with $\mathbb{R}$ gives a translation invariant and $\mathrm{SO}(3)$-invariant coassociative fibration of $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$.

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Base metric and harmonic 1-form on base

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- Joyce–K. [JDG'20] constructs new compact G_2 manifold from (M, φ)
- involves family of $T^*\mathrm{S}^2$ spaces fibred over an associative L^3 in M
- fibres equipped with Eguchi–Hanson metrics parametrized by a harmonic 1-form λ on L^3, obtained by integrating φ over S^2 "bolts"

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Let λ_c be the 1-form on $\mathbb{R}^3$ obtained by integrating φ_c over the S^2 bolts. Then λ_c is harmonic with respect to k_c.
Moreover, $|\lambda_c|_{k_c}$ gives the bolt size. Thus if $c > 0$, then λ_c vanishes only on the critical circle S^1_c, and if $c = 0$ then λ_0 vanishes only at the origin.

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The corresponding thimbles in M or M_0 are associative submanifolds.

Case Three: $\Lambda^2_-(\mathbb{CP}^2)$

Group action and coassociative "fibration"

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However, these "coassociative fibres" intersect in a codimension 4 set.

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(b) *The remaining fibres form a codimension 1 subfamily and are each diffeomorphic to $\mathbb{R}^+ \times \mathrm{S}^3$. Moreover, these $\mathbb{R}^+ \times \mathrm{S}^3$ fibres do not intersect any other fibres.*

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This determines a particular orthogonal frame Ω_1, Ω_2, Ω_3 for $\Lambda^2_-(\mathbb{CP}^2)$ (which we omit), thus determining fibre coordinates a_1, a_2, a_3. We let

$$a_1 = r\cos\gamma, \qquad a_2 = r\sin\gamma\cos\beta, \qquad a_3 = r\sin\gamma\sin\beta.$$

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Theorem (Nonlinear ordinary differential equations)

The nonlinear ODEs for $\mathrm{SU}(2)$-invariant coassociatives are $\dot\beta = 0$ and

$$0 = \cos\gamma\cos^2\alpha\dot r + (c+r^2)\sin^2\alpha(-\cos\gamma\dot r + r\sin\gamma\dot\gamma - 4r\cos\gamma\cot\alpha\dot\alpha),$$
$$0 = r^2\sin\gamma\cos\alpha(1+\cos^2\alpha)\dot r - 2(c+r^2)r\sin\gamma\sin\alpha(1+3\cos^2\alpha)\dot\alpha$$
$$-2(c+r^2)\sin^2\alpha\cos\alpha(r\cos\gamma\dot\gamma + \sin\gamma\dot r).$$

From these we find the following constants on each coassociative fibres:

$$\beta, \qquad v = 2(c+r^2)^{\frac{1}{4}}\cos\alpha\cot\gamma.$$

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However, for $c > 0$ we cannot compute induced hypersymplectic structure. [We can do it for $c = 0$ but it is extremely complicated.]

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With respect to a coframe σ_1, σ_2, σ_3 for $\mathrm{SU}(2) \cong \mathrm{S}^3$, define metric cones g_A and g_B on $\mathbb{R}^+ \times \mathrm{S}^3$ by

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$$g_B = dR^2 + \tfrac{1}{16} R^2 \sigma_1^2 + \tfrac{1}{4} R^2(\sigma_2^2 + \sigma_3^2).$$

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(b) *For M_c the generic smooth fibre $\mathcal{O}_{\mathbb{CP}^1}(-1)$ is asymptotically conical with unknown rate to the cone g_A.*

Induced Riemannian geometry

Theorem (Induced Riemannian geometry)

Each coassociative fibre N of M or M_0 inherits an induced Riemannian metric g_N by restriction.

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A non-generic set of smooth fibres $\mathcal{O}_{\mathbb{CP}^1}(-1)$ is asymptotically conical with rate -4 to the cone g_B.
The singular fibres $\mathbb{R}^+ \times \mathrm{S}^3$, are AC with unknown rate to g_A and unknown as $r \to 0$ (but likely CS).

Flat limit

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$$N_{\mathrm{HL}}(\varepsilon,\tau) = \{rq\varepsilon\overline{q} + (s\overline{q})e : q \in \mathrm{S}^3 \subset \mathbb{H},\ r \in \mathbb{R},\ s \geq 0,\ r(4r^2 - 5s^2)^2 = \tau\}$$

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Proposition (Flat limit)

As $\mathrm{Vol}(\mathbb{CP}^2) \to \infty$, the coassociative fibration of M limits to the Harvey–Lawson coassociative fibration of $(\mathbb{R}^7, \varphi_0)$.

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[M has a non-commuting $\mathrm{U}(1)$ action; makes limiting fibration nontrivial.]

Thank you for your attention.