Deformation theory of deformed Hermitian Yang–Mills connections and deformed Donaldson–Thomas connections

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Background

Deformed Hermitian Yang-Mills (dHYM) connections and deformed Donaldson-Thomas (dDT) connections appear in the context of mirror symmetry.

$$\{\text{special Lagrangian submanifolds}\} \leftrightarrow \{\text{dHYM connections}\},$$
$$\{\text{(co)associative submanifolds}\} \leftrightarrow \{\text{dDT connections}\}.$$

These correspondences are described explicitly if ambient manifolds are trivial torus bundles and submanifolds are graphical by the real Fourier–Mukai transform.

- dHYM connection $\cdots$ a Hermitian connection of a Hermitian line bundle L over a Kähler manifold introduced by Leung, Yau and Zaslow.
- dDT connection $\cdots$ a Hermitian connection of a Hermitian line bundle L over a G_2-manifold introduced by Lee and Leung.

Today, we focus on dDT connections on a G_2-manifold.

Definition

- X^7: a manifold with a G_2-structure $\varphi \in \Omega^3$,
- $(L, h) \to X$: a smooth complex Hermitian line bundle.

A Hermitian connection ∇ of (L, h) is called a deformed Donaldson–Thomas (dDT) connection if

$$\frac{1}{6}F_\nabla^3 + F_\nabla \wedge *\varphi = 0,$$

where $F_\nabla \in \sqrt{-1}\Omega^2$ is a curvature of ∇.

- dDT connection: "nonlinear analogue" of G_2-instanton: $F_\nabla \wedge *\varphi = 0$.
- We expect that dDT connections will have similar properties to associative submanfolds and G_2-instantons.

Common properties of associative submanifolds and G_2-instantons:

- The moduli space is a smooth 0-dim. manifold for generic G_2-structures.
- The moduli space is orientable. ([Joyce, Joyce-Upmeier])
- There is a challenging problem to define an enumerative invariant of 7-manifolds by counting associative submanifolds or G_2-instantons.

Almost nothing is known about the moduli space of dDT connections. Even whether or not the expected dim. of the moduli space is finite dimensional is unknown. In this talk, we show that

- The deformation of dDT connections is controlled by (a subcomplex of) the elliptic complex. (This makes the expected dim. of the moduli space is finite dimensional and the moduli space is a smooth manifold for generic G_2-structures.)
- The moduli space is orientable.

Optimistically, can we define enumerative invariants by counting dDT connections?

Deformation theory

- X^7: a compact connected 7-manifold with a coclosed G_2-structure φ,
  - φ need not to be torsion-free. $\exists$ many explicit examples (e.x. S^7).
- $(L, h) \to X$: a smooth complex Hermitian line bundle.

$$
\begin{aligned}
\mathcal{A}_0 :&= \{\, \nabla \mid \text{a Hermitian connection of } (L, h) \,\} \\
&= \nabla + \sqrt{-1}\Omega^1 \cdot \mathrm{id}_L \qquad (\nabla \in \mathcal{A}_0 \text{ is fixed}),
\end{aligned}
$$

$\mathcal{G}_U$:the group of unitary gauge transformations of (L, h) acting on $\mathcal{A}_0$ by $(\lambda, \nabla) \mapsto \lambda^{-1} \circ \nabla \circ \lambda$.

Define a map $\mathcal{F}_{G_2} : \mathcal{A}_0 \to \sqrt{-1}\Omega^6$ by

$$\mathcal{F}_{G_2}(\nabla) = \frac{1}{6}F_\nabla^3 + F_\nabla \wedge *\varphi.$$

Each element of $\mathcal{F}_{G_2}^{-1}(0)$ is a dDT connection.

$\mathcal{M}_{G_2} := \mathcal{F}_{G_2}^{-1}(0)/\mathcal{G}_U$:moduli space of dDT connections of (L, h).

Fix any $\nabla \in \mathcal{F}_{G_2}^{-1}(0)$. We want to know whether $\mathcal{M}_{G_2}$ is a smooth manifold near $[\nabla]$.

The strategy is standard. Roughly,

- Compute the linearization $\delta_\nabla \mathcal{F}_{G_2}$ of $\mathcal{F}_{G_2}$ at ∇.
- Take the slice to the $\mathcal{G}_U$-action. i.e. find a vector space V such that $V \overset{locally}{\cong} \mathcal{A}_0/\mathcal{G}_U$ near $[\nabla]$.
- If $\delta_\nabla \mathcal{F}_{G_2}$ is surjective, the implicit function theorem implies that $\mathcal{M}_{G_2}$ is a smooth manifold near $[\nabla]$.

The most nontrivial part is to describe $\delta_\nabla \mathcal{F}_{G_2}$ "nicely". (I will explain the idea later.)

$\Rightarrow$

the infinitesimal deformation and the obstruction space (i.e. vanishing of this $\Rightarrow$ $\delta_\nabla \mathcal{F}$ is surjective.) are interpreted as the first and the second cohomology of (the subcomplex of) an elliptic complex, respectively (as in common with other deformation theories).

Slice

$$\mathcal{A}_0 = \{ \nabla \mid \text{a Hermitian conn. of } (L,h) \} = \nabla + \sqrt{-1}\Omega^1 \cdot \mathrm{id}_L,$$
$$\mathcal{G}_U = \{ \text{the group of unitary gauge transformations of } (L,h) \}$$
$$= \{ f \cdot \mathrm{id}_L \mid f \in C^\infty(X, \mathbb{C}),\ |f| = 1 \}.$$

The action $\mathcal{G}_U \times \mathcal{A}_0 \to \mathcal{A}_0$ is defined by

$$(\lambda, \nabla) \mapsto \lambda^{-1} \circ \nabla \circ \lambda = \nabla + f^{-1} df \cdot \mathrm{id}_L,$$

where $\lambda = f \cdot \mathrm{id}_L$. Then, $T_\nabla(\mathcal{G}_U \cdot \nabla) = \sqrt{-1} d\Omega^0$.

Lemma (slice)

Setting $\Omega^1_{d^*} = \{ d^*\text{-closed 1-forms on } X \}$

$$\sqrt{-1}\Omega^1_{d^*} \to \mathcal{A}_0 / \mathcal{G}_U, \quad A \mapsto [\nabla + A \cdot \mathrm{id}_L]$$

is a local homeomorphism. Hence, $\mathcal{M}_{G_2}$ *is locally homeomorphic to* $(\mathcal{F}_{G_2}|_{\nabla + \Omega^1_{d^*}})^{-1}(0)$.

Recall

$$\mathcal{F}_{G_2} : \mathcal{A}_0 = \nabla + \sqrt{-1}\Omega^1 \cdot \mathrm{id}_L \to \sqrt{-1}\Omega^6, \quad \mathcal{F}_{G_2}(\nabla) = \frac{1}{6} F_\nabla^3 + F_\nabla \wedge *\varphi.$$

Lemma

If $\mathcal{M}_{G_2} \neq \emptyset$, the image of $\mathcal{F}_{G_2}$ is contained in $\sqrt{-1}d\Omega^5$.

Given $\forall \nabla, \nabla' \in \mathcal{A}_0$, $F_{\nabla'} - F_\nabla \in \sqrt{-1}d\Omega^1$. Then,

$$\mathcal{F}_{G_2}(\nabla') - \mathcal{F}_{G_2}(\nabla) \in \sqrt{-1}d\Omega^5.$$

If $\nabla \in \mathcal{F}_{G_2}^{-1}(0)$, the statement follows.

Then, if the linearization $\delta_\nabla \mathcal{F}_{G_2} : \sqrt{-1}\Omega^1 \to \sqrt{-1}d\Omega^5$ is surjective, the implicit function theorem tells us that $\mathcal{M}_{G_2}$ is a smooth manifold near $[\nabla]$.

linearization

Recall $\mathcal{F}_{G_2} : \mathcal{A}_0 = \nabla + \sqrt{-1}\Omega^1 \cdot \mathrm{id}_L \to \sqrt{-1}d\Omega^5$ is given by

$$\mathcal{F}_{G_2}(\nabla) = \frac{1}{6}F_\nabla^3 + F_\nabla \wedge *\varphi.$$

Naively, $\delta_\nabla \mathcal{F}_{G_2} : \sqrt{-1}\Omega^1 \to \sqrt{-1}d\Omega^5$ is

$$\delta_\nabla \mathcal{F}_{G_2}(\sqrt{-1}b) = \left(\frac{1}{2}F_\nabla^2 + *\varphi\right) \wedge (\sqrt{-1}db).$$

We have to manage the strange term $\frac{1}{2}F_\nabla^2 + *\varphi$.

If $F_\nabla = 0$,

$$\delta_\nabla \mathcal{F}_{G_2}(\sqrt{-1}b) = *\varphi \wedge (\sqrt{-1}db).$$

Set

$$0 \to \sqrt{-1}\Omega^0 \xrightarrow{d} \sqrt{-1}\Omega^1 \xrightarrow{\delta_\nabla \mathcal{F}_{G_2}} \sqrt{-1}d\Omega^5 \to 0. \qquad (\#_\nabla)$$

This is a subcomplex of

$$0 \to \sqrt{-1}\Omega^0 \xrightarrow{d} \sqrt{-1}\Omega^1 \xrightarrow{d(*\varphi\wedge\bullet)} \sqrt{-1}\Omega^6 \xrightarrow{d} \sqrt{-1}\Omega^7 \to 0,$$

which is the canonical complex introduced by Reyes Carrión. This complex is elliptic. ($\Rightarrow \dim H^i(\#_\nabla) < \infty$.)
Since $T_\nabla(\mathcal{G}_U\text{-orbit}) = \sqrt{-1}d\Omega^0$,

- $H^1(\#_\nabla) \cong T_{[\nabla]}\mathcal{M}_{G_2}$,
- $H^2(\#_\nabla)$ is the obstruction space:
  ($H^2(\#_\nabla) = \{0\} \Rightarrow \mathcal{M}_{G_2}$ is a smooth manifold near $[\nabla]$.)

If $F_\nabla \neq 0$, we can do a similar thing by introducing a new G_2-structure defined by φ and F_∇.

Lemma (Main Lemma)

For any $\nabla \in \mathcal{F}_{G_2}^{-1}(0)$*, we have* $1 + \langle F_\nabla^2, *\varphi \rangle/2 \neq 0$ *and we define a new coclosed* G_2*-structure* $\tilde{\varphi}_\nabla$ *by*

$$\tilde{\varphi}_\nabla = f_\nabla \cdot (\mathrm{id}_{TX} + (-\sqrt{-1}F_\nabla)^\sharp)^* \varphi,$$

where

$$f_\nabla = \left| 1 + \frac{1}{2} \langle F_\nabla^2, *\varphi \rangle \right|^{-3/4},$$

$$(-\sqrt{-1}F_\nabla)^\sharp \in \Gamma_{\mathrm{skew}}(\mathrm{End}\, TX) \quad (\Rightarrow \det(\mathrm{id}_{TX} + (-\sqrt{-1}F_\nabla)^\sharp) > 0).$$

Then,

$$\frac{1}{2} F_\nabla^2 + *\varphi = \pm \tilde{*}_\nabla \tilde{\varphi}_\nabla.$$

The sign agree with that of $1 + \langle F_\nabla^2, *\varphi \rangle/2$.

The proof is mainly given by pointwise computations, but this is surprising to us.

Then,

$$(\delta_\nabla \mathcal{F}_{G_2})(\sqrt{-1}b) = \pm\sqrt{-1}db \wedge \tilde{*}_\nabla \tilde{\varphi}_\nabla.$$

For each $\nabla \in \mathcal{F}_{G_2}^{-1}(0)$, set

$$0 \to \sqrt{-1}\Omega^0 \xrightarrow{d} \sqrt{-1}\Omega^1 \xrightarrow{d(\tilde{*}_\nabla \tilde{\varphi}_\nabla \wedge \bullet)} \sqrt{-1}d\Omega^5 \to 0. \qquad (\#_\nabla)$$

This is a subcomplex of the canonical complex

$$0 \to \sqrt{-1}\Omega^0 \xrightarrow{d} \sqrt{-1}\Omega^1 \xrightarrow{d(\tilde{*}_\nabla \tilde{\varphi}_\nabla \wedge \bullet)} \sqrt{-1}\Omega^6 \xrightarrow{d} \sqrt{-1}\Omega^7 \to 0,$$

which is the canonical complex introduced by Reyes Carrión. This complex is elliptic. ($\Rightarrow \dim H^i(\#_\nabla) < \infty$.)
Since $T_\nabla(\mathcal{G}_U\text{-orbit}) = \sqrt{-1}d\Omega^0$,

- $H^1(\#_\nabla) \cong T_{[\nabla]}\mathcal{M}_{G_2}$,
- $H^2(\#_\nabla)$ is the obstruction space:
  ($H^2(\#_\nabla) = \{0\} \Rightarrow \mathcal{M}_{G_2}$ is a smooth manifold near $[\nabla]$.)

We can also show:

Lemma

The expected dimension of $\mathcal{M}_{G_2}$ is given by

$$\dim H^1(\#_\nabla) - \dim H^2(\#_\nabla) = b^1.$$

Then, we can apply the standard method to obtain:

Theorem (K.-Yamamoto)

- *If* $H^2(\#_\nabla) = \{0\}$ *for* $[\nabla] \in \mathcal{M}_{G_2}$, $\mathcal{M}_{G_2}$ *is a* b^1*-dim. smooth manifold near* $[\nabla]$.
- *Suppose that* $[\nabla] \in \mathcal{M}_{G_2}$ *and*
    1. *the* G_2*-structure* φ *is torsion-free, or*
    2. ∇ *satisfies* $F^3_\nabla \neq 0$.

  Then, the moduli space close to $[\nabla]$ *is a smooth* b^1*-dimensional manifold (or empty) if we perturb* φ *generically in the space of coclosed* G_2*-structures.*

Orientation of $\mathcal{M}_{G_2}$

We show the following.

Theorem (K.-Yamamoto)

Suppose that $H^2(\#_\nabla) = \{0\}$ for any $[\nabla] \in \mathcal{M}_{G_2}$. Then, $\mathcal{M}_{G_2}$ is an orientable manifold.

Denote by

$$D(\#_\nabla) = (d(\tilde{*}_\nabla \tilde{\varphi}_\nabla \wedge \bullet), d^*) : \sqrt{-1}\Omega^1 \to \sqrt{-1}d\Omega^5 \oplus \sqrt{-1}\Omega^0$$

the two term complex associated to $(\#_\nabla)$.
Suppose that $H^2(\#_\nabla) = \{0\}$ ($\Leftrightarrow \mathrm{Coker} D(\#_\nabla) = \mathbb{R}$) for any $[\nabla] \in \mathcal{M}_{G_2}$. Then,

$$\Lambda^{\mathrm{top}} T\mathcal{M}_{G_2} = \bigsqcup_{[\nabla] \in \mathcal{M}_{G_2}} \Lambda^{\mathrm{top}} \ker D(\#_\nabla) = \bigsqcup_{[\nabla] \in \mathcal{M}_{G_2}} \det D(\#_\nabla).$$

$$\mathcal{M}_{G_2} : \text{ orientable} \iff \Lambda^{\mathrm{top}} T\mathcal{M}_{G_2} : \text{ trivial}.$$

The idea is:

- For any (not necessarily coclosed) G_2-structure $\phi \in \Omega^3$ on X,

$$D(\phi) = (d(*_\phi \phi \wedge \bullet), d^*) : \sqrt{-1}\Omega^1 \to \sqrt{-1}d\Omega^5 \oplus \sqrt{-1}\Omega^0,$$

  is Fredholm.
- $D(\#_\nabla) = D(\tilde{\varphi}_\nabla)$.
- Recall $\tilde{\varphi}_\nabla = f_\nabla \cdot (\mathrm{id}_{TX} + (-\sqrt{-1}F_\nabla)^\sharp)^* \varphi$.
  $\Rightarrow \tilde{\varphi}_\nabla$ is connected to φ in the space of G_2-structures.
- $\exists$ a homotopy between $\{D(\#_\nabla) = D(\tilde{\varphi}_\nabla)\}_{[\nabla] \in \mathcal{M}_{G_2}}$ and $\{D(\varphi)\}_{[\nabla] \in \mathcal{M}_{G_2}}$ through Fredholm operators.

$$\Rightarrow \Lambda^{\mathrm{top}} T\mathcal{M}_{G_2} = \bigsqcup_{[\nabla] \in \mathcal{M}_{G_2}} \det D(\#_\nabla) \cong \mathcal{M}_{G_2} \times \det D(\varphi).$$

dHYM connections

deformed Hermitian Yang-Mills (dHYM) connection $\cdots$ a Hermitian connection of a Hermitian line bundle L over a Kähler manifold, which corresponds to special Lagrangian submanifolds via mirror symmetry.

- Introducing a new balanced (i.e. the Kähler form is coclosed) Hermitian metric, we can show that the deformation is controlled by a subcomplex of the canonical complex introduced by Reyes Carrión.
- The moduli space is always a smooth b^1-dimensional affine manifold.
  (This is because dHYM conn. is mirror to special Lagrangian?)
- The moduli space is orientable.

Summary

- dDT connections correspond to (co)associative submanifolds via mirror symmetry.
- dDT connections are "nonlinear analogue" of G_2-instantons.

dDT connections have the following common properties with associative submanifolds and G_2-instantons:

- The moduli space is a smooth b^1-dim. manifold for generic G_2-structures.
- The moduli space is orientable.

Optimistically, can we define enumerative invariants by counting dDT connections?

Future work

- Deformation theory of dDT connections for $\mathrm{Spin}(7)$-manifolds (work in progress)
- Construction of nontrivial examples of dDT connections
- "Singular" dDT connections?
  No idea.
- Compactness theorem for dDT connections
  - associative Smith map [Cheng-Karigiannis-Madnick]
  - G_2-instanton [Tian]

  might be helpful?