Extremally Ricci-pinched G_2-structures

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G_2-structures

M^7 differentiable manifold.

G_2-structure: $\varphi \in \Omega^3 M$ definite (or positive), i.e., at any $p \in M$,

$$\varphi_p = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

w.r.t. some basis $\{e_1, \dots, e_7\}$ of T_pM, or equivalently, when φ determines a Riemannian metric g on M and an orientation by

$$g(X,Y)\,\mathrm{vol} = \frac{1}{6}\iota_X(\varphi) \wedge \iota_Y(\varphi) \wedge \varphi, \qquad \forall X, Y \in \mathfrak{X}(M).$$

Assume φ closed: $d\varphi = 0$, in which case,

$$\tau := -*d*\varphi \in \Omega^2_{14}M, \quad d*\varphi = \tau \wedge \varphi, \quad d\tau = \Delta\varphi.$$

$\tau = 0$ (i.e., parallel or torsion-free) $\Rightarrow \mathrm{Hol}(M,g) \subset G_2$.

ERP structures

Theorem ([Bryant 92])

If M^7 is compact and φ is closed, then

$$\int_M \mathrm{Scal}_g^2 *1 \leq 3 \int_M |\operatorname{Ric}_g|_g^2 * 1,$$

where equality holds if and only if $d\tau = \frac{1}{6}|\tau|^2\varphi + \frac{1}{6} * (\tau \wedge \tau)$ (called Extremally Ricci Pinched (ERP)).

[Bryant 92]: For any ERP φ, M compact (or locally homogeneous),

- $|\tau|_g$ constant in M.
- $\tau \wedge \tau \wedge \tau = 0$.
- $d(\tau \wedge \tau) = 0$.
- $d * (\tau \wedge \tau) = 0$.
- $\mathrm{Spec}(\operatorname{Ric}_g) = \{-\frac{1}{6}|\tau|^2, -\frac{1}{6}|\tau|^2, -\frac{1}{6}|\tau|^2, 0, 0, 0, 0\}$.

ERP G_2-structures: $d\varphi = 0$ and $d\tau = \frac{1}{6}|\tau|^2\varphi + \frac{1}{6} * (\tau \wedge \tau)$.

- [Bryant 92] Example on $M = G/K = \mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2/\mathrm{SU}(2)$ ([Cleyton-Ivanov 08] can also be presented on a (non-unimodular) solvable Lie group). Compact quotients.
- [L 16] Example on a (unimodular) solvable Lie group (different). [Kath-L 20] Compact quotients.
- [Fino-Raffero 18] The space of ERP G_2-structures is invariant under the Laplacian flow $\frac{\partial}{\partial t}\varphi(t) = \Delta\varphi(t)$ and the solutions are always eternal (i.e., $t \in (-\infty, \infty)$). New examples. Only one unimodular Lie algebra allowed.
- [L-Nicolini 19] Classification on Lie groups (only five structures).
- [Ball 19, 20] Complete non-homogeneous examples. Partial classifications up to local equivalence (Cartan's method of exterior differential systems and the moving frame). Classification in the homogeneous case completed.

Theorem (Structure [L-Nicolini 19])

Every left-invariant ERP G_2-structure on a Lie group is equivariantly equivalent to a (G,φ) with torsion $\tau = e^{12} - e^{56}$, where

$$\begin{aligned}\varphi =& e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} \\ =& \omega_3 \wedge e^3 + \omega_4 \wedge e^4 + \omega_7 \wedge e^7 + e^{347},\end{aligned}$$

$\omega_7 := e^{12} + e^{56}$, $\omega_3 := e^{26} - e^{15}$ *and* $\omega_4 := e^{16} + e^{25}$, *and if* $\mathfrak{g} = Lie(G)$, *then*

(i) $\mathfrak{h} := \mathrm{sp}\{e_1, \dots, e_6\}$ *is a unimodular ideal of* $\mathfrak{g}$.

(ii) $\mathfrak{g}_0 := \mathrm{sp}\{e_7, e_3, e_4\}$ *is a Lie subalgebra of* $\mathfrak{g}$ *and* $[e_3, e_4] = 0$.

(iii) $\mathfrak{g}_1 := \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ *is an abelian ideal of* $\mathfrak{g}$ *(i.e.,* $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$*).*

(iv) $\theta(\mathrm{ad}\, e_7|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_7$, $\theta(\mathrm{ad}\, e_3|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_3$ *and* $\theta(\mathrm{ad}\, e_4|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_4$.

(v) $\theta(\mathrm{ad}\, e_7|_{\mathfrak{g}_1})\omega_7 + \theta(\mathrm{ad}\, e_3|_{\mathfrak{g}_1})\omega_3 + \theta(\mathrm{ad}\, e_4|_{\mathfrak{g}_1})\omega_4 = \tau + (\mathrm{tr}\,\mathrm{ad}\, e_7|_{\mathfrak{g}_0})\omega_7$.

Conversely, if $\mathfrak{g}$ *satisfies (i)-(v), then* (G,φ) *is an ERP* G_2*-structure.*

Theorem (Structure [L-Nicolini 19])

Every left-invariant ERP G_2-structure on a Lie group is equivariantly equivalent to a (G,φ) with torsion $\tau = e^{12} - e^{56}$, and if $\mathfrak{g} = Lie(G)$, then

(i) $\mathfrak{h} := \mathrm{sp}\{e_1, \dots, e_6\}$ *is a unimodular ideal of* $\mathfrak{g}$.

(ii) $\mathfrak{g}_0 := \mathrm{sp}\{e_7, e_3, e_4\}$ *is a Lie subalgebra of* $\mathfrak{g}$ *and* $[e_3, e_4] = 0$.

(iii) $\mathfrak{g}_1 := \mathrm{sp}\{e_1, e_2, e_5, e_6\}$ *is an abelian ideal of* $\mathfrak{g}$ *(i.e.,* $\mathfrak{g} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$*).*

(iv) $\theta(\mathrm{ad}\, e_7|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_7$, $\theta(\mathrm{ad}\, e_3|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_3$ *and* $\theta(\mathrm{ad}\, e_4|_{\mathfrak{g}_1})\tau = \frac{1}{3}\omega_4$.

(v) $\theta(\mathrm{ad}\, e_7|_{\mathfrak{g}_1})\omega_7 + \theta(\mathrm{ad}\, e_3|_{\mathfrak{g}_1})\omega_3 + \theta(\mathrm{ad}\, e_4|_{\mathfrak{g}_1})\omega_4 = \tau + (\mathrm{tr}\,\mathrm{ad}\, e_7|_{\mathfrak{g}_0})\omega_7$.

Conversely, if $\mathfrak{g}$ satisfies (i)-(v), then (G,φ) is an ERP G_2-structure.

Summarizing: $\mathfrak{g} = \mathrm{sp}\{e_7, e_3, e_4\} \ltimes \mathrm{sp}\{e_1, e_2, e_5, e_6\}$, solvable,

$$A_1 := \mathrm{ad}\, e_7|_{\mathrm{sp}\{e_3,e_4\}}, \quad A := \mathrm{ad}\, e_7|_{\mathfrak{g}_1}, \quad B := \mathrm{ad}\, e_3|_{\mathfrak{g}_1}, \quad C := \mathrm{ad}\, e_4|_{\mathfrak{g}_1}.$$

JACOBI !! $\qquad$ $\mathfrak{n}$: nilradical of $\mathfrak{g}$, $\dim \mathfrak{n} = 4, 5, 6$.

Corollary

Every left-invariant ERP G_2-structure on a Lie group is a steady Laplacian soliton, i.e.,

$$\Delta\varphi = \mathcal{L}_{X_D}\varphi, \quad D \in Der(\mathfrak{g}),$$

and an expanding Ricci soliton, i.e.,

$$\mathrm{Ric}_g = cg + \mathcal{L}_{X_D}g, \quad D \in Der(\mathfrak{g}), \quad c < 0.$$

Corollary

Every left-invariant ERP G_2-structure on a non-unimodular Lie group is exact; indeed,

$$\varphi = d\left(3\tau - (\operatorname{tr} A_1)^{-1} e^{34}\right).$$

Theorem (Classification [L-Nicolini 19])

Any left-invariant ERP G_2-structure on a Lie group is equivalent to (G_μ, φ), where μ is exactly one of the following Lie algebras:

$$\mu_B, \quad \mu_{M1}, \quad \mu_{M2}, \quad \mu_{M3}, \quad \mu_J.$$

Moreover, any left-invariant ERP G_2-structure on a Lie group is equivariant equivalent to exactly on of the following:

$$\mu_B, \quad \mu_{M1}, \quad \mu_{M2}, \quad \mu_{M3}, \quad \mu_J, \quad \mu_{rt}, \quad (r,t) \neq (0,0).$$

The structures $(G_{\mu_{rt}}, \varphi)$ ([Fino-Raffero 18]) are all equivalent to (G_{μ_B}, φ) and the family of Lie algebras μ_{rt}, $r, t \in \mathbb{R}$ is pairwise non-isomorphic.

Key ingredients of the proof: Structure theorem; $\theta : \mathfrak{sl}_4(\mathbb{R}) \xrightarrow{\simeq} \mathfrak{so}(3,3)$;
$U_{\mathfrak{h},\tau} := \{h \in G_2 : h(\mathfrak{h}) \subset \mathfrak{h},\ h \cdot \tau = \tau\} \simeq \mathrm{S}^1 \times \mathrm{S}^1$,
$U_{\mathfrak{g}_1,\tau} := \{h \in G_2 : h(\mathfrak{g}_1) \subset \mathfrak{g}_1,\ h \cdot \tau = \tau\} \simeq \mathrm{U}(2)$.

Example (μ_B, $\dim \mathfrak{n} = 6$, $\mathfrak{n}$ 2-step [Bryant 92], [Cleyton-Ivanov 08])

$$(A_1)_B = \frac{1}{3}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad A_B = \frac{1}{6}\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

$$B_B = \frac{1}{3}\begin{bmatrix} & & & 0 \\ & & 0 & \\ & 1 & & \\ 1 & & & \end{bmatrix}, \quad C_B = \frac{1}{3}\begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 1 & 0 & & \\ 0 & -1 & & \end{bmatrix}.$$

Example (μ_{M1}, $\dim \mathfrak{n} = 6$, $\mathfrak{n}$ 4-step [L-Nicolini 19])

$$(A_1)_{M1} = \frac{1}{30}\begin{bmatrix} \sqrt{30} & 0 \\ 0 & 2\sqrt{30} \end{bmatrix}, \quad A_{M1} = \frac{1}{60}\begin{bmatrix} -10-\sqrt{30} & 0 & -2\sqrt{5} & 0 \\ 0 & -10+\sqrt{30} & 0 & -2\sqrt{5} \\ -2\sqrt{5} & 0 & 10-\sqrt{30} & 0 \\ 0 & -2\sqrt{5} & 0 & 10+\sqrt{30} \end{bmatrix},$$

$$B_{M1} = \frac{1}{30}\begin{bmatrix} 0 & -\sqrt{5} & 0 & 5-\sqrt{30} \\ 5\sqrt{5} & 0 & 5 & 0 \\ 0 & 5+\sqrt{30} & 0 & \sqrt{5} \\ 5 & 0 & -5\sqrt{5} & 0 \end{bmatrix}, \quad C_{M1} = \frac{1}{30}\begin{bmatrix} -\sqrt{5} & 0 & 5-\sqrt{30} & 0 \\ 0 & \sqrt{5} & 0 & -5+\sqrt{30} \\ 5+\sqrt{30} & 0 & \sqrt{5} & 0 \\ 0 & -5-\sqrt{30} & 0 & -\sqrt{5} \end{bmatrix}.$$

Example (μ_{M2}, $\dim \mathfrak{n} = 5$, $\mathfrak{n}$ 3-step [L-Nicolini 19])

$$(A_1)_{M2} = \frac{1}{3}\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}, \quad A_{M2} = \frac{1}{3}\begin{bmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix},$$

$$B_{M2} = \frac{1}{6}\begin{bmatrix} -1 & & & \\ & 1 & 2 & \\ & 2 & 1 & \\ & & & 1 \end{bmatrix}, \quad C_{M2} = \frac{1}{3}\begin{bmatrix} 0 & & & \\ -1 & 0 & & \\ 1 & 0 & 0 & \\ 0 & -1 & 1 & 0 \end{bmatrix}.$$

Example (μ_{M3}, $\dim \mathfrak{n} = 5$, $\mathfrak{n}$ 2-step [L-Nicolini 19])

$$(A_1)_{M3} = \frac{1}{6}\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{6} \end{bmatrix}, \quad A_{M3} = \frac{1}{12}\begin{bmatrix} -2 & 0 & -\sqrt{2} & 0 \\ 0 & -2 & 0 & -\sqrt{2} \\ -\sqrt{2} & 0 & 2 & 0 \\ 0 & -\sqrt{2} & 0 & 2 \end{bmatrix},$$

$$B_{M3} = \frac{1}{6}\begin{bmatrix} 0 & \sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & -\sqrt{2} \\ 1 & 0 & -\sqrt{2} & 0 \end{bmatrix}, \quad C_{M3} = \frac{1}{12}\begin{bmatrix} -\sqrt{2} & 0 & 2-\sqrt{6} & 0 \\ 0 & \sqrt{2} & 0 & -2+\sqrt{6} \\ 2+\sqrt{6} & 0 & \sqrt{2} & 0 \\ 0 & -2-\sqrt{6} & 0 & -\sqrt{2} \end{bmatrix}.$$

Example (μ_J, $\dim \mathfrak{n} = 4$, $\mathfrak{n}$ abelian [L 17])

$$A_J = \frac{1}{6}\begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 3 & \\ & & & -1 \end{bmatrix}, B_J = \frac{1}{6}\begin{bmatrix} 0 & -\sqrt{2} & 0 & 2 \\ -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}, C_J = \frac{1}{6}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}.$$

Example (μ_{rt}, $\dim \mathfrak{n} = 6$, $\mathfrak{n}$ 2-step [Fino-Raffero 18])

$$(A_1)_{rt} = \frac{1}{3}\begin{bmatrix} 1 & -r \\ r & 1 \end{bmatrix}, \quad A_{rt} = \frac{1}{6}\begin{bmatrix} -1 & -2t & & \\ 2t & -1 & & \\ & & 1 & 2(r+t) \\ & & -2(r+t) & 1 \end{bmatrix},$$

$$B_{rt} = \frac{1}{3}\begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 1 & & \\ 1 & 0 & & \end{bmatrix}, \quad C_{rt} = \frac{1}{3}\begin{bmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 1 & 0 & & \\ 0 & -1 & & \end{bmatrix}.$$

$(\mu_B = \mu_{00})$

Automorphism group of homogeneous ERP G_2-structures

M^7, ERP G_2-structure: $d\varphi = 0$ and $d\tau = \frac{1}{6}|\tau|^2\varphi + \frac{1}{6} * (\tau \wedge \tau)$.

$\text{Aut}(M,\varphi) := \{f \in \text{Diff}(M) : f^*\varphi = \varphi\} \subset \text{Isom}(M,g)$, Lie group.

$(M,\varphi) = (G,\varphi)$ simply connected Lie group endowed with a left-invariant G_2-structure,

$$(\text{Aut}(\mathfrak{g}) \cap G_2) \ltimes G \subset \text{Aut}(G,\varphi), \quad (\text{Aut}(\mathfrak{g}) \cap \mathrm{O}(7)) \ltimes G \subset \text{Isom}(G,g),$$

and they coincide if G unimodular and completely solvable. [Alekseevskii 71], [Gordon-Wilson 88].

- [L-Nicolini 19] Compute $\text{Aut}(\mathfrak{g}) \cap G_2$ and $\text{Aut}(\mathfrak{g}) \cap \mathrm{O}(7)$ for each of the five structures in the classification.
- [Ball 20] Completes the classification in the homogeneous case: same list of five. Computes the connected component $\text{Aut}(G,\varphi)_0$ for each of them.

[L-Nicolini 19] first and second column; [Ball 20] third column.

	$\mathrm{Aut}(\mu) \cap G_2$	$\mathrm{Aut}(\mu) \cap \mathrm{O}(7)$	$\mathrm{Aut}(G_\mu, \varphi)_0$	dim
μ_B	$S^1 \times S^1$	$\mathbb{Z}_2 \ltimes (S^1 \times S^1)$	$(\mathrm{S}^1 \times \mathrm{SL}_2(\mathbb{C})) \ltimes \mathbb{C}^2$	11
μ_{M1}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	G_{M1}	7
μ_{M2}	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbb{R} \times \mathrm{SO}(2,1)) \ltimes \mathbb{R}^4$	8
μ_{M3}	$\mathbb{Z}_4$	$D_4 \times \mathbb{Z}_2$	$(\mathbb{R} \times \mathrm{SL}_2(\mathbb{R})) \ltimes \mathbb{R}^4$	8
μ_J	$\mathrm{SL}_2(\mathbb{Z}_3)$	$S_4 \ltimes \mathbb{Z}_2^4$	G_J	7

Table: Symmetries.

Compact ERP G_2-structures

M^7 compact, ERP G_2-structure: $d\varphi = 0$ and $d\tau = \frac{1}{6}|\tau|^2\varphi + \frac{1}{6} * (\tau \wedge \tau)$.

Only two examples known:

- [Bryant 92] $M = \left(\mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2/\mathrm{SU}(2)\right)/\Gamma$, $\Gamma \subset \mathrm{SL}_2(\mathbb{C}) \ltimes \mathbb{C}^2$ lattice.
- [Kath-L 20] G_J/Γ, where Γ is a lattice of the unimodular solvable G_J. This is a counterexample to a conjecture made in [Cleyton-Ivanov 08] (they proved that Bryant's example is the only ERP G_2-structure whose intrinsic torsion is parallel with respect to the canonical G_2 connection).

[Kath-L 20] $\mathfrak{g}_J = \mathfrak{a} \ltimes \mathbb{R}^4$, where $\mathfrak{a} \subset \mathfrak{sl}_4(\mathbb{R})$ is the subspace of all diagonal matrices, i.e., $G_J = \exp(\mathfrak{a}) \ltimes \mathbb{R}^4$. Consider the matrices in $\mathrm{SL}_4(\mathbb{Z})$,

$$A_1 := \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -4 & -5 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 1 & 5 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 3 & -1 & -1 & -1 \\ -4 & -1 & -5 & -5 \\ 0 & 0 & 3 & -1 \\ 1 & 1 & 1 & 4 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 4 & 1 & 2 & 3 \\ 3 & 8 & 9 & 14 \\ -1 & -1 & 0 & -3 \\ -1 & -2 & -3 & -3 \end{bmatrix}.$$

There exists $\phi \in \mathrm{SL}_4(\mathbb{R})$ such that $\phi A_j \phi^{-1}$, $j = 1, 2, 3$ are all diagonal, positive and generate a lattice Λ of $\exp(\mathfrak{a})$.

Since they leave invariant the subset $\phi(\mathbb{Z}^4) \subset \mathbb{R}^4$, the set

$$\Gamma := \Lambda \ltimes \phi(\mathbb{Z}^4)$$

is a subgroup of G_J. Moreover, Γ is a lattice in G_J since $\Lambda \subset \exp(\mathfrak{a})$ and $\phi(\mathbb{Z}^4) \subset \mathbb{R}^4$ are both discrete and cocompact.

Remark. The three matrices correspond to the action by multiplication of three multiplicatively independent units in a totally real quartic number field K on the ring $\mathcal{O}_K$ of integers.

Many thanks for your attention !!