Instantons on Joyce-Karigiannis manifolds

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Abstract: Joyce constructed compact G_2-manifolds through the generalised Kummer construction on T^7. Walpuski constructed G_2-instantons on these manifolds by gluing flat connections on the orbifold to pullbacks of ASD-instantons on the gluing region. In 2017, Joyce and Karigiannis explained a further generalisation of the Kummer construction that can resolve orbifold singularities on non-flat orbifolds. In the talk, I will review the construction of Joyce and Karigiannis and present one simplification of the proof of existence of a torsion-free G_2-structure. I will then explain how to construct G_2-instantons on these manifolds by gluing together G_2-instantons on the orbifold and Fueter sections and will present one example of this construction.

The exceptional holonomy group G_2

- $(x_1, \dots, x_7)$ standard coordinates on $\mathbb{R}^7$, $G_2 := \mathrm{Stab}_{\mathrm{GL}(7)}(\varphi)$, where

$$\varphi = \mathrm{d}x_{123} - \mathrm{d}x_{145} - \mathrm{d}x_{167} - \mathrm{d}x_{246} + \mathrm{d}x_{257} - \mathrm{d}x_{347} - \mathrm{d}x_{356}$$

- Given M^7, $\varphi \in \Omega^3(M)$ is called *G_2-structure*, if for all $x \in M$ there exists an isomorphism $T_x M \simeq \mathbb{R}^7$ that sends φ_x to φ
- $G_2 \subset \mathrm{SO}(7)$, so φ induces a metric g_φ on M

Theorem (Fernández-Gray '82)

$\mathrm{Hol}(M, g_\varphi) \subset G_2$ *if and only if* $\mathrm{d}\varphi = 0$ *and* $\mathrm{d}^*\varphi = 0$.

- In this case, φ is called torsion-free
- Example: (T^7, φ)

Product G_2 structures

- $\varphi = dx_{123} - dx_{145} - dx_{167} - dx_{246} + dx_{257} - dx_{347} - dx_{356}$
- Identify $\mathbb{R}^7 \simeq \mathbb{R}^3 \oplus \mathbb{H}$ with coordinates $((x_1, x_2, x_3), (y_1, y_2, y_3, y_4))$, then

$$\varphi = dx_{123} - \sum_{i=1}^{3} dx_i \wedge \omega^i, \text{ where}$$

$$\omega^1 = dy_{12} + dy_{34}, \qquad \omega^2 = dy_{13} + dy_{42}, \qquad \omega^3 = dy_{14} + dy_{23},$$

$$\psi = *\varphi = \frac{1}{2}\omega^1 \wedge \omega^1 - \sum_{\substack{(1,2,3),(2,3,1),\\(3,1,2)}} dx_{ij} \wedge \omega^k$$

- Linear model extends to product manifolds: $T^3 \times X^4$, X Calabi-Yau manifold

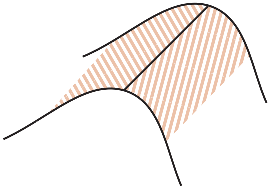
The Joyce-Karigiannis construction: the smooth manifold

G_2-manifold (M,φ), $\iota : M \to M$ such that $\iota^2 = \mathrm{Id}$, $\iota^*\varphi = \varphi$

$L = \mathrm{fix}(\iota)$ associative, assume there is $\lambda \in \Omega^1(L)$ closed+co-closed, nowhere 0

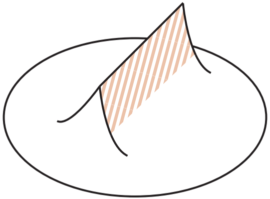
$$N_t = (M/\langle\iota\rangle \setminus L) \cup (\exp_t \circ \rho)^{-1}(U)/\sim$$

for $t \in (0,1)$



$P = \mathrm{Fr}(\nu) \times_{\mathrm{U}(2)} X_{\mathrm{EH}}$, with fibre Eguchi-Hanson space

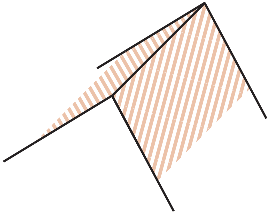
$\downarrow \rho$



$M/\langle\iota\rangle$

U neighbourhood of L

$\xleftarrow{\exp_t = \exp \circ (\cdot t)}$



$\nu/\{\pm 1\}$ normal bundle over L

complex structure $I(\cdot) = \frac{\lambda}{|\lambda|} \times (\cdot)$

$\nu/\{\pm 1\} = \mathrm{Fr}(\nu) \times_{\mathrm{U}(2)} (\mathbb{C}^2/\{\pm 1\})$

The Joyce-Karigiannis construction: the G_2-structure

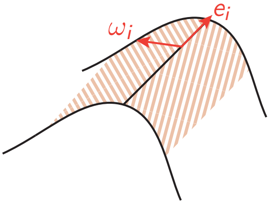
$\widehat{\omega}^i \in \Omega^2(\nu/\{\pm 1\})$, $\omega_i \in \Omega^2(P)$ s.t.

for $x \in L$: $\omega_i|_{P_x}$ blowup of $\widehat{\omega}^i|_{(\nu/\{\pm 1\})_x}$

$r = d_\varphi(L,\cdot) : N_t \to \mathbb{R}$

$$\varphi_t^N = \begin{cases} \widetilde{\varphi}_t^P & \text{, where } r \leq t^{7/8} \\ \varphi & \text{, where } r \geq 2t^{7/8} \end{cases}$$

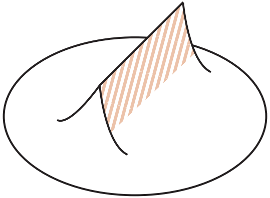
and interpolate. ψ_t^P, $\widetilde{\psi}_t^P$, ψ_t^N analog



$$\varphi_t^P = e_{123} - t^2 \sum_{i=1}^3 e_i \wedge \omega_i$$

$$\widetilde{\varphi}_t^P = \varphi_t^P + \xi \text{ closed}$$

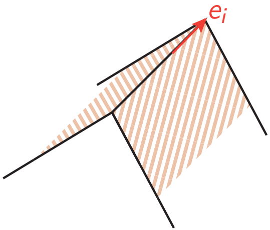
$$\downarrow \rho$$



$\varphi/\langle\iota\rangle$

$(e_1 = \frac{\lambda}{|\lambda|}, e_2, e_3)$ orthonormal frame on L

$\xleftarrow{\exp_t = \exp\circ(\cdot t)}$



$x \in L : \varphi|_x = e_{123} - \sum_{i=1}^3 e_i \wedge \widehat{\omega}_i$

get $\widehat{\omega}^i \in \Omega^2(\nu/\{\pm 1\})$ parallel on fibres

The Joyce-Karigiannis construction: the torsion-free G_2-structure

Theorem (Joyce '96)

$(M^7, \varphi, g_\varphi)$ with $\mathrm{d}\varphi = 0$. $\mathrm{inj} \geq ct, |\,\mathrm{Riem}\,| \leq ct^{-2}$. $\psi \in \Omega^4(M)$ such that

$$||{*\varphi} - \psi||_{L^2} \leq ct^{7/2+\epsilon} \text{ for some } c, \epsilon > 0$$

(plus C^0 and L^{14}_1 estimate). Then: for small t, ex. torsion-free G_2-structure $\widetilde{\varphi}$ on M.

But: $\left|\left|{*\varphi^N_t} - \psi^N_t\right|\right|_{L^2} \leq ct^{18/8}$. Solution: norms adapted to manifold (r dist. from L)

$$||\cdot||_{C^{k,\alpha}_{\beta;t}} \text{ with weight function } (t+r)^{-\beta} \Rightarrow \left|\left|\mathrm{d}({*\varphi^N_t} - \psi^N_t)\right|\right|_{C^{0,\alpha}_{-4;t}} \leq ct^{25/8}$$

Theorem

$(N_t, \varphi = \varphi^N_t, g_\varphi)$. $\psi \in \Omega^4(M)$ such that

$$||\mathrm{d}({*\varphi} - \psi)||_{C^{0,\alpha}_{\beta-2;t}} \leq ct^{1-\beta+\alpha+\epsilon} \text{ for some } c, \epsilon > 0, \alpha \in (0,1), \beta \in (-4,-2)$$

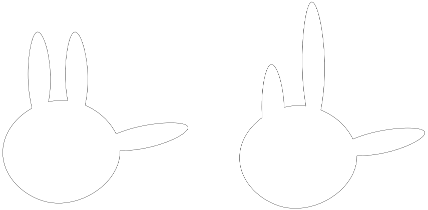
(plus $C^{0,\alpha}$ estimate). Then: for small t, ex. torsion-free G_2-structure $\widetilde{\varphi}$ on M.

Proof of the correction theorem

1. Ex. unique $\mu \in \Omega^2(X_{\mathrm{EH}})$ harmonic which decays faster than r^{-2}
2. Let $\alpha_1, \dots, \alpha_k$ be basis of $\mathcal{H}^2(M/\langle\iota\rangle)$
3. $\dim \operatorname{Ker} \Delta_{N_t} = b^2(N_t) = b^2(M/\langle\iota\rangle) + b^0(L)b^2(X)$ by Künneth formula. This motivates definition of approximate kernel of same dimension:
   $\mathcal{K}_{\mathrm{ap}} = \langle \text{cut-offs of } \alpha_i \rangle_{i=1,\dots,k} \oplus \langle \text{cut-offs of } \mu \text{ on connected cpts of } P \rangle$
4. Claim: for $\beta \in (-4,-2)$, $a \perp \mathcal{K}_{\mathrm{ap}} : ||a||_{C^{2,\alpha}_{\beta;t}} \leq c\,||\Delta a||_{C^{0,\alpha}_{\beta-2;t}}$.
   Proof: Assume not, then there exists a_i such that $||a_i||_{C^{2,\alpha}_{\beta;t}} = 1$, $||\Delta a_i||_{C^{0,\alpha}_{\beta-2;t}} \to 0$.
   - Zoom into P: get a limit $a^* \in \Omega^2(X_{\mathrm{EH}})$ which is harmonic and perpendicular to μ. $\Rightarrow a^* = 0$, because μ is the only harmonic form with decay faster than r^{-2} (use $\beta < -2$ here)
   - Zoom into $M/\langle\iota\rangle$: get a limit $a^* \in \Omega^2(M/\langle\iota\rangle \setminus L)$ harmonic, which extends to $M/\langle\iota\rangle$ as a distribution because $\beta > -4$. $\Rightarrow a^* = 0$, because $a^* \perp \alpha_i$.
5. $\operatorname{Im}\left(\Delta_{N_t}|_{\mathcal{K}_{\mathrm{ap}}^\perp}\right) = \operatorname{Im} \Delta_{N_t}$
6. $\vartheta = \psi - *\varphi$, solve $\Delta\eta = \mathrm{d}^*\vartheta + \mathrm{d}^*(f\vartheta) + \text{h.o.t.}$, where $f\frac{7}{3} = \langle \varphi, \mathrm{d}\eta\rangle$, using the estimate for the linearisation from 4.
7. Solution $\varphi + \mathrm{d}\eta$ is torsion-free (cf. [5])

Two consequences

1. Because $\text{inj} \geq ct$ and $|\text{Riem}| \leq ct^{-2}$ don't appear in the new theorem, can resolve connected components of L at different length scales



$\rightsquigarrow$ $b^0(L)$-dimensional family of torsion-free G_2-structures

2. Denote by φ_t^N, ψ_t^N the corrected forms from Joyce-Karigiannis, then:

$$||\widetilde{\varphi} - \varphi||_{C^0} \leq ct^{1-\epsilon}, \text{ for any } \epsilon > 0 \text{ (previously: } t^{1/8}).$$

On T^7, with extra correction:

$$||\widetilde{\varphi} - \varphi||_{C^0} \leq ct^{4/3-\epsilon}, \text{ for any } \epsilon > 0 \text{ (previously: } t^{1/2}).$$

Example of the construction

- $C \subset \mathbb{CP}^2$ curve of degree 6, $\pi : X \to \mathbb{CP}^2$ double cover branched over C
- $\Rightarrow X$ is K3-surface $\Rightarrow T^3 \times X$ has torsion-free G_2-structure φ
- $\alpha : X \to X$ swapping sheets; $\beta : X \to X$ lift of $[z_0 : z_1 : z_2] \mapsto [\overline{z}_0 : \overline{z}_1 : \overline{z}_2]$
- Extend to $\alpha, \beta : T^3 \times X \to T^3 \times X$ such that $\alpha^* \varphi = \varphi$, $\beta^* \varphi = \varphi$
- $\mathrm{fix}(\alpha) = 2 \cdot (S^1 \times C)$, $\mathrm{fix}(\beta) = 2 \cdot (S^1 \times S^2)$ all disjoint
- $L = \mathrm{fix}(\alpha) \cup \mathrm{fix}(\beta)$, $\lambda \in \Omega^1(L)$ coordinate in the S^1 direction

Theorem (Joyce-Karigiannis '17)

The resolution N of $T^3 \times X / \langle \alpha, \beta \rangle$ carries a torsion-free G_2-structure. $\pi_1(N) = 0$, so this G_2-structure has holonomy equal to G_2.

G_2-instantons

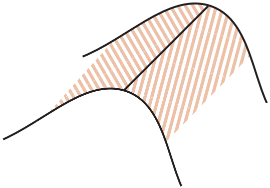
- Connection A on bundle over $(M, \varphi, \psi = *\varphi)$ is G_2-instanton if $F_A \wedge \psi = 0$
- Example on $\pi : \mathbb{R}^3 \times \mathbb{H} \to \mathbb{H}$: A anti-selfdual instanton over $\mathbb{H}$, i.e. $*F_A = -F_A$
  $\omega^1, \omega^2, \omega^3$ self-dual $\Rightarrow F_A \wedge \omega^i = 0$

$$F_{\pi^* A} \wedge \psi = F_{\pi^* A} \wedge \left(\frac{1}{2}\omega^1 \wedge \omega^1 - \sum_{\substack{(1,2,3),(2,3,1),\\(3,1,2)}} \mathrm{d}x_{ij} \wedge \omega^k \right) = 0$$

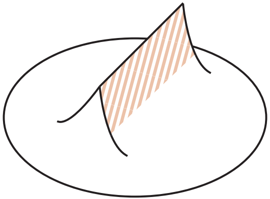
- Example on $\pi : \mathbb{R} \times \mathbb{C}^3 \to \mathbb{C}^3$: A Hermitian Yang-Mills connection over $\mathbb{C}^3$, then $F_{\pi^* A} \wedge \psi = 0$
- Extends to $T^3 \times \mathrm{K3}$ and $S^1 \times \mathrm{CY}^6$

The instanton gluing construction

(assume $\nu \to L$ trivial for easier notation)
M_{ASD} moduli of ASD-instantons on X_{EH}
if holonomy of θ at L and holonomy of $s(A)$ at $\infty \in P$ agree, then can glue to (E_t, A_t) bundle and connection over N_t

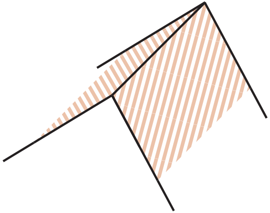


$s : L \to M_{\mathrm{ASD}}$
$\rightsquigarrow (s(E), s(A)) \to P$ bundle and connection: $s(A)|_{P(x)} = s(x)$
$\downarrow \rho$



$(E_0, \theta) \to (M/\langle\iota\rangle) \setminus L$ bundle and connection

$\xleftarrow{\exp_t = \exp \circ (\cdot t)}$



Solutions to the instanton equation

X_{EH} Hyperkähler $\Rightarrow$ M_{ASD} Hyperkähler. Fueter equation for $s : L \to M_{\mathrm{ASD}}$

$$I_1 \, \mathrm{d}s(e_1) + I_2 \, \mathrm{d}s(e_2) + I_3 \, \mathrm{d}s(e_3) = 0$$

Assume θ is a G_2-instanton and s satisfies the Fueter equation

Theorem

If θ is infinitesimally rigid and s is a constant section, taking a rigid ASD-instanton as its value, then ex. a_t such that $A_t + a_t$ is G_2-instanton w.r.t. $\widetilde{\psi}_t^N$.

Theorem

If θ and s (as a Fueter section) are infinitesimally rigid and $\left|\left|F_{A_t} \wedge \widetilde{\psi}_t^N\right|\right|_{C^{0,\alpha}_{-2,t}} \leq ct^2$, then ex. a_t such that $A_t + a_t$ is G_2-instanton with respect to $\widetilde{\psi}_t^N$.

Remark: If θ flat, $M = T^7$, have $\left|\left|F_{A_t} \wedge \psi_t^N\right|\right|_{C^{0,\alpha}_{-2,t}} \leq ct^2$

Need good estimate for $\widetilde{\psi}_t^N - \psi_t^N$ to satisfy pregluing estimate

Example I

ingredient 1:
$\pi : X \to \mathbb{CP}^2$ K3, $\beta : X \to X$ antiholomorphic involution, fix $E \to X$ complex v.b.

Theorem (Donaldson '85, S. Wang '90)

$$\left\{ \begin{matrix} \textit{HYM } A \textit{ on } E \\ \textit{invariant under } \beta \end{matrix} \right\} / \mathcal{G} \leftrightarrow \left\{ \begin{matrix} \textit{stable holo. structures on } E \\ \textit{such that } \beta^* E \simeq \overline{E}^* \textit{ as holo. bundles ("real")} \end{matrix} \right\} / \textit{iso}$$

e.g. $\mathcal{E} = \pi^*(T^*\mathbb{CP}^2)$, only stable rank 2 bundle $\Rightarrow \mathcal{E} \simeq \beta^* \overline{\mathcal{E}}^*$
$\mathrm{End}_0(\mathcal{E}) = F_{\mathbb{C}}$ for SO(3)-bundle $F \to X$, also β-invariant
$\Rightarrow$ pullback to $T^3 \times X$ is rigid α, β-invariant G_2-instanton
ingredient 2:

Theorem (Gocho-Nakajima '92, Walpuski '13)

$G = U(n)$ or SO(3). For any monodromy at ∞ ex. infinitesimally rigid ASD-instanton on X_{EH} with this monodromy.

s constant Fueter section with value this compatible ASD-instanton $\rightsquigarrow$ glue together

(Conjectural) Example II

ingredient 1:
Flat T^7, $\Gamma = \langle \alpha, \beta, \gamma \rangle$ acting on T^7, $\pi_1(T^7/\Gamma \setminus L) = \langle \alpha, \beta, \gamma, \tau_1, \dots, \tau_7 \rangle$
Flat SU(2)-bundle with monodromy representation $\rho : \alpha, \beta, \gamma \mapsto -1$
Infinitesimally rigid

ingredient 2:
T^7/Γ, $\mathrm{fix}(\Gamma) = 16 \cdot T^3$, let (x_1, x_2, x_3) coordinates on T^3

Theorem (Kronheimer-Nakajima '90)

There exists a connected component C of M_{ASD} isomorphic to X_{EH}.

Weierstrass function $\wp : T^2 = (x_2, x_3) \to \mathbb{CP}^1 \subset X_{\mathrm{EH}}$ holomorphic
Then $s : T^3 \to X_{\mathrm{EH}}$, $s(x_1, x_2, x_3) = \wp(x_2, -x_3)$ has $ds(\partial x_2) - I\, ds(\partial x_3) = 0$
Applying J shows: $s : T^3 \to M_{\mathrm{ASD}}$ is a Fueter section, not rigid
Perturb G_2-structure for rigid Fueter section
(cf. conjectural Floer theory for Fueter sections)

Up next

1. Can find more real, stable, holomorphic bundles on $\mathbb{CP}^2$ via monad construction:

$$A \xrightarrow{a} B \xrightarrow{b} C, \qquad \mathcal{E} = \text{Ker } b / \text{ Im } a$$

$$\mathcal{E} \text{ real bundle} \Leftrightarrow \begin{array}{lccccc} \mathcal{E}^* : & C^* & \xrightarrow{b_z^*} & B^* & \xrightarrow{a_z^*} & A^* \\ & \downarrow & & \downarrow & & \downarrow \\ \beta^*\overline{\mathcal{E}} : & \overline{A} & \xrightarrow{a_{\beta z}} & \overline{B} & \xrightarrow{b_{\beta z}} & \overline{C} \end{array}$$

$\mathcal{E}$ stable $\Leftarrow$ monad of form $A \to B \to A$ plus algebraic condition
To do: describe moduli of real, stable bundles

2. To do: Kuranishi model for gluing non-rigid ingredients
3. The manifold example over $T^3 \times X / \langle \alpha, \beta \rangle$ is suspected to also come from the twisted connected sum construction
To do: do instanton constructions on these manifolds construct the same instantons?

Thank you for the attention!

- S. K. Donaldson. "Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles". In: *Proc. London Math. Soc. (3)* 50.1 (1985), pp. 1–26. ISSN: 0024-6115. DOI: 10.1112/plms/s3-50.1.1. URL: https://doi.org/10.1112/plms/s3-50.1.1.
- M. Fernández and A. Gray. "Riemannian manifolds with structure group G_2". In: *Ann. Mat. Pura Appl. (4)* 132 (1982), 19–45 (1983). ISSN: 0003-4622. DOI: 10.1007/BF01760975. URL: https://doi.org/10.1007/BF01760975.
- Toru Gocho and Hiraku Nakajima. "Einstein-Hermitian connections on hyper-Kähler quotients". In: *J. Math. Soc. Japan* 44.1 (1992), pp. 43–51. ISSN: 0025-5645. DOI: 10.2969/jmsj/04410043. URL: https://doi.org/10.2969/jmsj/04410043.
- D. Joyce and S. Karigiannis. "A new construction of compact G_2-manifolds by gluing families of Eguchi-Hanson spaces". In: *ArXiv e-prints* (July 2017). arXiv: 1707.09325 [math.DG].

- Dominic D. Joyce. "Compact Riemannian 7-manifolds with holonomy G_2. I, II". In: *J. Differential Geom.* 43.2 (1996), pp. 291–328, 329–375. ISSN: 0022-040X. URL: http://projecteuclid.org/euclid.jdg/1214458109.
- Peter B. Kronheimer and Hiraku Nakajima. "Yang-Mills instantons on ALE gravitational instantons". In: *Math. Ann.* 288.2 (1990), pp. 263–307. ISSN: 0025-5831. DOI: 10.1007/BF01444534. URL: https://doi.org/10.1007/BF01444534.
- Thomas Walpuski. "G_2-instantons on generalised Kummer constructions". In: *Geom. Topol.* 17.4 (2013), pp. 2345–2388. ISSN: 1465-3060. DOI: 10.2140/gt.2013.17.2345. URL: https://doi.org/10.2140/gt.2013.17.2345.
- Shuguang Wang. "Moduli spaces over manifolds with involutions". In: *Math. Ann.* 296.1 (1993), pp. 119–138. ISSN: 0025-5831. DOI: 10.1007/BF01445098. URL: https://doi.org/10.1007/BF01445098.