Some questions apropos of Kirchhoff's theorem

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Kirchhoff's theorem

Theorem (Kirchhoff 1947)

If the sphere S^n admits an almost complex structure, then S^{n+1} is parallelizable.

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If the sphere S^n *admits an almost complex structure, then* S^{n+1} *is parallelizable.*

Obs: it is not an existence theorem, it is a constructive one.

A little of history.

1) 1947 S^n is almost complex $\implies$ S^{n+1} is parallelizable. (Kirchhoff)

2) 1951 S^n is almost complex $\implies$ $n = 0, 2, 6$ (Borel-Serre)

3) 1958 S^{n+1} is parallelizable $\implies$ $n = 0, 2, 6$ (Kervaire, Bott-Milnor)

4) 1960 S^{n+1} is an H-space $\iff$ $n = 0, 2, 6$ (Adams)

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Almost complex:	S^0, S^2, S^6	Parallelizable:	S^1, S^3, S^7
Complex manifolds:	S^0, S^2, S^6 ?	Lie groups:	S^1, S^3

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Lie groups spheres are the integrable parallelizable spheres.

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In the hypothetical case that there is no integrable almost complex structure on S^6 the following statement is true a fortiori:

S^n is a complex manifold if and only if S^{n+1} is a Lie group

What is the bridge? Kirchhoff's theorem

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When a parallelism comes from a Lie group structure?

parallelism = $\{e\}$-structure

$\{X_1, \cdots, X_n\}$ smooth global frame on M, there is associated a flat connection ∇^c (the canonical connection)

$\nabla^c_Y \left(\sum f^i X_i\right) = \sum Y(f^i) X_i$ for $Y \in \mathbf{X}(M)$

The structure equations of ∇^c are:

$$d\theta^i = \tfrac{1}{2} T^i{}_{jk}\, \theta^j \wedge \theta^k \quad \text{and} \quad \Omega^i_j = 0,$$

$\{\theta^1, \cdots, \theta^n\}$ is the coframe dual of $\{X_1, \cdots, X_n\}$.

Torsion tensor of ∇^c:

$$T^c(X_j, X_k) = \sum_{i=1}^{n} T^i{}_{jk}\, X_i = -[X_j, X_k].$$

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A parallelism on M (compact and simply connected) is integrable if, and only if, its structure functions $T^i{}_{jk}$ are constant, in which case M is a Lie group.

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Classical parallelism of the seven sphere

Fix the canonical basis of the octonions $\mathbb{O}$: 1 and e_i, $i = 1, \cdots, 7$ with multiplication rule: $e_i e_j = -\delta_{ij} + a_{ijk} e_k$, the structure constants a_{ijk} are totally antisymmetric in the three indices.

We construct seven linearly independent vector fields X_i on the sphere $S^7 \subset \mathbb{O}$ of octonions of norm one:

$X_i(x) = e_i x$ for $x \in S^7$, $i = 1, \cdots, 7$.

Computing the structure functions of this global frame:

$$\begin{aligned}[X_i, X_j](x) &= e_i(e_j x) - e_j(e_i x) \\ &= 2a_{ijk} e_k x - 2[e_i, e_j, x] \\ &= 2\left(a_{ijk} - \langle [e_i, e_j, x], e_k x \rangle\right) X_k(x),\end{aligned}$$

$[a, b, c] := (ab)c - a(bc)$ is the associator, $\langle a, b \rangle$ is the scalar product in $\mathbb{O}$.

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- The non-associativity of the octonions $([a, b, c] \neq 0)$ causes the non-constancy of the structure functions of the parallelism on S^7. The non-commutativity of the algebra causes the non-vanishing of the torsion.
- Note the structure functions coincide with the structure constants of the algebra at the north and south pole, i.e., at 1 and -1.
- We used the alternativity of the octonionic product to prove the second equality.

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How the multiplication in the octonions induces an almost complex structure on S^6

$Im\,\mathbb{O} \subset \mathbb{O}$ hyperplane of imaginary octonions orthogonal to $1 \in \mathbb{O}$.

$S^6 \subset Im\,\mathbb{O}$ sphere of imaginary octonions of norm one.

Right multiplication by $y \in S^6$ induces an orthogonal linear transformation:

$$R_y : \mathbb{O} \to \mathbb{O} \quad \text{such that} \quad (R_y)^2 = -\mathrm{Id}\,.$$

R_y preserves the plane spanned by 1 and y, $\quad (1 \to y,\, y \to -1)$.

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R_y preserves its orthogonal six dimensional plane $\langle 1, y\rangle^{\perp}$, which can be identified with $T_y S^6 \subset \mathbb{O}$.

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$Im\,\mathbb{O} \subset \mathbb{O}$ hyperplane of imaginary octonions orthogonal to $1 \in \mathbb{O}$.

$S^6 \subset Im\,\mathbb{O}$ sphere of imaginary octonions of norm one.

Right multiplication by $y \in S^6$ induces an orthogonal linear transformation:

$$R_y : \mathbb{O} \to \mathbb{O} \quad \text{such that} \quad (R_y)^2 = -\,\mathrm{Id}\,.$$

R_y preserves the plane spanned by 1 and y, $(1 \to y,\, y \to -1)$.

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R_y preserves its orthogonal six dimensional plane $\langle 1, y\rangle^\perp$, which can be identified with $T_yS^6 \subset \mathbb{O}$.

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Nijenhuis tensor corresponding to this almost complex structure on S^6

$$N(X,Y) = [JX, JY] - [X,Y] - J[X,JY] - J[JX,Y]$$

We are in euclidean space, then we can compute the Lie brackets of two vector fields $X : S^6 \to \mathbb{R}^7$, $Y : S^6 \to \mathbb{R}^7$ by $[X,Y] = dY(X) - dX(Y)$, where dX and dY denote the differential of X and Y respectively as maps

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For $b, c \in T_a S^6$ we get:

$$\begin{aligned} N_a(b,c) &= c \cdot (b \cdot a) - b \cdot (c \cdot a) - (c \cdot b) \cdot a + (b \cdot c) \cdot a \\ &= 2[a,b,c]. \end{aligned}$$

- The non-associativity of the octonions is responsible for the non-integrability of this almost complex structure.
- To establish the last equality we used that the algebra of octonions is alternative.

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Kirchhoff's construction is modeled on the previous example, in fact its proof reverses this process, he reconstructs the 'multiplication' of $\mathbb{R}^8$ from the almost complex structure on S^6.

The Kirchhoff's construction can be divided in two parts:

1) To extend the almost complex structure J_y on $y \in S^6$ to an almost complex structure $\hat{J}_y$ on $\mathbb{R}^8$.
2) To construct a global frame σ on S^7 from $\hat{J}$.

Notation:

$\mathbb{R}^8 = \langle e_8 \rangle \oplus \mathbb{R}^7$, $S^7 \subset \mathbb{R}^8$,

$S^6 \subset \mathbb{R}^7$ the equator of S^7 with respect to the north pole $e_8 \in S^7$.

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Given $y \in S^6$ denote by V_y the 6-dimensional vector subspace of $\mathbb{R}^8$ parallel to the tangent space $T_y(S^6)$ in $\mathbb{R}^8$.

Define a linear transformation $\hat{J}_y : \mathbb{R}^8 \to \mathbb{R}^8$ by:

$$\hat{J}_y(e_8) := y, \quad \hat{J}_y(y) := -e_8 \quad \hat{J}_y(z) := J_y(z) \text{ for } z \in V_y.$$

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To construct a global frame σ on S^7 from $\hat{J}$

Let $x \in \mathbb{R}^8$, then it can be written uniquely as follows:

$$x = \alpha e_8 + \beta y, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \geq 0, \quad \text{and} \quad y \in S^6.$$

Define the linear transformation:

$$\sigma_x : \mathbb{R}^8 \to \mathbb{R}^8, \quad \sigma_x := \alpha\, Id + \beta \hat{J}_y$$

$\hat{J}_y^2 = -Id \implies \sigma_x$ is an isomorphism. Note also that $\sigma_x(e_8) = x$.

$$\sigma_x|_{\mathbb{R}^7} : \ \mathbb{R}^7 \to T_x(S^7) \quad x \in S^7,$$

we get the desire linear frame.

In fact, $\mathbb{R}^7 = \langle y, V_y \rangle$, $\sigma_x(y) \perp x$ and $\sigma_x(z) \perp x$, $z \in V_y$, $x \in \mathbb{R}^8$, then can be considered as elements of $T_x(S^7)$.

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Some remarks about Kirchhoff's theorem

- Note the linear frame σ is smooth at all points of S^7 except at e_8 and $-e_8$, where it is only continuous.
- Kirchhoff's theorem does not assume any additional condition on the almost complex structure J.
- If we start with an almost hermitian structure (g, J) on S^6, we obtain a Kirchhoff's global frame $\sigma \in SO(8, \hat{g})$, where $\hat{g}$ is a metric extension of g to $\mathbb{R}^8$ being compatible with the extended almost complex structure $\hat{J}$.
- The vector fields $\{X_i(x) := \sigma_x(e_i)\}_{i=1,\cdots,7}$ defining the parallelism in Kirchhoff's theorem can be written explicitly as:

$$X_i(x) = x_8 e_i - x_i e_8 + \beta(x) J_y \left(e_i - \langle y, e_i \rangle y\right),$$

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Main Question

To what extent does the integrability of an almost complex structure J on S^6 imply the integrability of the associated parallelism on S^7?

A framework to approach the Main Question

Recently Loubeau and Sá-Earp in *'Harmonic flow of geometric structures'* arXiv:1907.06072 propose a twistorial interpretation of geometric structures on Riemannian manifolds.

They interpreted a geometric G-structure on (M, g) as a section of the homogeneous fibre bundle $\pi : N := P/G \to M$, which admits a natural notion of torsion.

They formulated a general theory of harmonicity for geometric structures on a Riemannian manifold (using a Dirichlet energy of sections of π)

Various torsion regimes for a geometric section fit in a logical chain:

$$d^{\mathcal{V}}\sigma = 0 \implies \text{super-flat} \implies \text{totally geodesic} \implies$$
$$\implies \text{harmonic map} \implies \text{harmonic section.}$$

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In particular, for parallelisms on a sphere $\sigma : (S^n, g) \to SO(n+1, g)$, it is not hard to check that eg. the Hopf frame on round S^3 is harmonic as a section and integrable, but $d^{\mathcal{V}}\sigma \neq 0$, since it is non-Abelian.

Question: For a parallelism on a sphere (S^n, g), what are the explicit conditions of (1) for σ ?

Question: How to express the integrability condition $\nabla^c T = 0$ in terms of the vertical torsion $d^{\mathcal{V}}\sigma$? Eg. for almost-complex structures, the vanishing of the Nijenhuis tensor is equivalent to $J^{\mathcal{V}} \circ d^{\mathcal{V}}\sigma = d^{\mathcal{V}}\sigma \circ J$.

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Homotopy approach

Definition: an H-space is a topological space M which admits a continuous multiplication $m : M \times M \longrightarrow M$ with a two-sided identity element.

By a celebrated theorem of Adams(1960) the only spheres that admit an H-space structure are S^1, S^3 and S^7.

We can rephrase Kirchhoff's theorem as follows:

If S^n admits an almost complex structure J then S^{n+1} is a an H-space

This follows from Kirchhoff's theorem and the well known fact that a parallelizable sphere is an H-space.

The point is that the induced multiplication on S^{n+1} is written explicitly in terms of J.

$$m : S^7 \times S^7 \longrightarrow S^7, \quad m(x,y) := \sigma_x(y) / \|\sigma_x(y)\|$$

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Theorem (James 1957) There exists no homotopy-associative multiplication on S^n unless $n = 1$ or 3.

As we have already seen the non-associativity of the octonions causes the non-integrability of the almost complex structure induced on S^6 by the octonions.

We would like to relate the probable non-existence of complex structure on S^6 with the lack homotopy associative multiplications on S^7.

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