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CONTRIBUTIONS TO STOCHASTIC DIFFERENTIAL GEOMETRY AND ROUGH PATHS THEORY

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This habilitation thesis provides an overview of my research activity, starting with my PhD thesis works, in 2006. This activity has developped in two main directions, and a side one. Starting with my PhD subject, I have kept a constant interest in addressing probabilistic questions in a geometric setting. This is obviously clear from my work on relativistic diffusions **[P1,3,7,9,10]** and sub-elliptic diffusions **[P19, 21, 22]**, but geometry is also one of the main sources of inspiration of my research activity in rough paths theory and related topics. Before turning recently to the latter business, I have spent four wonderful years of post-doc in Cambridge, with James (Norris), working on different aspects of Smoluchowski's coagulation equation **[P4,5,6,8]**, which I have chosen not to present in this memoir, so as to keep its structure coherent. My apologies to my co-authors on some of these works. Somewhat ironically, Peter (Friz) had a position in Cambridge at that time, but I only started working on rough paths theory in early 2012, after I arrived in Rennes. I have thus chosen to divide this habilitation thesis in two parts, the first one dealing with different problems in stochastic differential geometry, the second one being centered on rough paths theory and its outgrowth, with a blend of geometry. (Three other works **[P2,11,22]** will not be described in the following manuscript. The reader is invited to consult the original works if she/he is interested in the topics they investigate.) Given its size, this memoir is to be understood as an introduction to the different works I have done, and is aimed at putting the problems in their setting and describing the results I have obtained with almost no proofs. The reader is referred to the original works for the details.

Given the relatively wide scope of subjects addressed here, a number of 'intermezzo' frames have been inserted inside the text, to provide some general background that may not be part of the common knowledge of the working probabilist. Each chapter has been given its own bibliography, so they can be read independently from one another. A number of perspectives have been included inside the text, in sections entitled "What next?" This is by no means all I have in mind for future research!

[P1] **Poisson boundary of a relativistic diffusion**
I. Bailleul
Probability Theory & Related Fields, **141**(1) (2008), 283–329.

[P2] **Une preuve simple d'un résultat de Dufresne**
I. Bailleul
Séminaire de Probabilités de Paris, **Vol. XLI** (2008), 203–213.

[P3] **Poisson boundary of a random walk on Poincaré group**
I. Bailleul and A. Raugi
ESAIM, P & S. **14** (2010) 16–52.

[P4] **Coupling algorithm for calculating sensitivities in Smoluchowski equation**
I. Bailleul, M. Kraft, P. Man and J. Norris
SIAM Journal on Scientific Computing **32**(2) (2010), 635–655.

[P5] **A stochastic algorithm for sensitivity in Smoluchowski equation**
I. Bailleul, M. Kraft, P. Man
SIAM Journal on Numerical Analysis **48**(3) (2010), 1064–1086.

[P6] **Sensitivity for Smoluchowski equation**
I. Bailleul
J. of Physics A. **44**(24) (2011).

[P7] **A stochastic approach to relativistic diffusions**
I. Bailleul
Annales de l'Institut H. Poincaré **46**(3) (2010), 760–795.

[P8] **Spatial coagulation with bounded coagulation rate**
I. Bailleul
J. of Evolution Equations **11**(3) (2011), 675–686.

[P9] **Non-explosion criteria for relativistic diffusions**
I. Bailleul and J. Franchi
Annals of Probability **40**(5) (2012), 2168–2196.

[P10] **A probabilistic view on singularities**
I. Bailleul
J. Math. Phys. **52** (2012), 023520.

[P11] **General relativistic Boltzmann equation**
I. Bailleul and F. Debbasch
Classical and Quantum Gravity **29**(6) (2012), 065020.1-10.

[P12] **Flows driven by rough paths**
I. Bailleul
To appear in *Revista Iberoam. Math.*, 29 pages (2015).

[P13] **Flows driven by Banach space valued rough paths**
I. Bailleul
Séminaire de Probabilités de Paris, **Vol. XLVI** (2014), 195–205.

[P14] **Rough integrators on Banach manifolds**
I. Bailleul
Submitted to *Bull. Sc. Math.*, 13 pages (2015)

[P15] **Regularity of the Itô-Lyons map**
I. Bailleul
To appear in *Confluentes Mat.*, 11 pages (2015)

[P16] **Rough flows**
I. Bailleul and S. Riedel
Submitted to *Annals of Probability*, 60 pages (2015)

[P17] **The inverse problem for rough controlled differential equations**
I. Bailleul and J. Diehl
To appear in *SIAM J. Opt. Control*, 19 pages (2014)

[P18] **Rough drivers**
I. Bailleul and M. Gubinelli
To appear in *Ann. Fac. Sc. Toulouse*, 29 pages (2015)

[P19] **Kinetic Brownian motion**
J. Angst, I. Bailleul and C. Tardif
Submitted to *Elec. J. Probability*, 42 pages (2015)

[P20] **Heat semigroups and singular PDEs**
I. Bailleul and F. Bernicot
Submitted to *J. Func. Anal.*, 72 pages (2015)

[P21] Small time fluctuations for Riemannian and sub-Riemannian diffusion processes
I. Bailleul, J. Norris and L. Mesnager
Submitted to *J. Diff. Geom.*, 53 pages (2015)

[P22] Large deviation principle for bridges of sub-Riemannian diffusions
I. Bailleul
Submitted to *Séminaire de Probabilités*, 8 pages (2015)

Although they are not to be considered on an equal footing with the above works, I would like to mention here the following two original sets of lecture notes that I wrote for a Part III course in Cambridge, and an M2 course in Rennes, respectively.

- **Advanced probability** (95 pages)
- **A flow-based approach to rough paths** (63 pages)

Acknowledgments

Un grand merci à Xue-Mei, Fabrice et Samy pour avoir accepté de rapporter ce mémoire, ainsi qu'aux membres du jury qui me font l'honneur de leur présence en cette occasion. Il va sans dire que je me sens tout petit face à un tel jury : charge à moi de vous montrer dans le futur que la reconnaissance que vous m'accordez aujourd'hui pour mes travaux passés sera toute justifiée par mes travaux à venir. L'envie est là, ainsi qu'un monde de questions passionnantes.

Un autre grand merci à mes co-auteurs pour avoir partagé avec moi un temps de vie autours de diverses questions mathématiques. Je suis impatient de continuer l'aventure !

Rien que pour le plaisir de l'écrire, merci à Jürgen, Florent, Hélène, Jean-Christophe, Sébastien, Ludovic et tous les autres membres des équipes de Théorie ergodique et Probabilités de l'IRMAR pour leur amitié, et l'ambiance chaleureuse qui fait des journées que je passe au bureau des moments précieux.

Je dédie ce travail à ma muse Delphine, et à nos deux petites pépettes Hanaé et Prune.

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Part 1

Stochastic differential geometry

THIS part of the memoir gives an introduction to those of my works that deal with stochastic differential geometry. All of them share the feature of dealing with some *non-elliptic* diffusion processes.

Chapter 1 is about **relativistic diffusions [P1,3,7,9,10]**. This is a class of diffusion processes that somehow play in the Lorentzian setting the role played by Brownian motion in a Riemannian setting. Unlike Brownian motion which is truly 'rough', the sample paths of a relativistic diffusion are $\mathcal{C}^1$, while being intrinsically associated with the geometry of the manifold though. Their description in the model setting of Minkowski spacetime is elementary, as they are fully described by saying that the velocity process of the path is a Brownian motion on the (upper half-) unit sphere of Minkowski space, and that the path itself is obtained as the integral of its velocity. The hypoelliptic nature of these diffusions, their finite speed of propagation, and the fact they propagate in one 'direction', make the study of some of their properties fairly non-trivial.

Section 1.1 is dedicated to investigating some of the properties of the relativistic process that was first introduced in Minkowski spacetime by Dudley [**1**] in 1966, and extended to the setting of Lorentzian manifolds by Franchi and Le Jan some fourty years later, in 2007 [**3**]. Section 1.1.1 gives an overview of the results of my PhD thesis, in which I gave a full description of the **Poisson boundary** of Dudley's diffusion in Minkowski spacetime, relating it to the large scale structure of this space – its conformal boundary. The Lorentzian analogue of Dudley's diffusion is defined in section 1.1.2 as a random perturbation of the timelike geodesic flow. We investigate in section 1.1.3 the question of **stochastic incompleteness** of Dudley's diffusion in the light of the incompleteness theorems of Lorentzian geometry first proved by Penrose and Hawking. A general class of relativistic diffusions is introduced in section 1.2, and two of their fundamental properties are investigated; the existence and properties of a **one-particle distribution function [P7]**, and the **stochastic completeness** problem **[P9]**.

I gathered in Chapter 2 two works that deal with other classes of non-elliptic diffusions on manifolds. Section 2.1 is about the Riemannian analogue of Dudley's diffusion. It is given as a one parameter family of diffusion processes, collectively called **kinetic Brownian motion** in a joint work with J. Angst and C. Tardif **[P18]**. It provides a family of diffusion processes that interpolates between geodesic and Brownian flows, in the line of Bismut's hypoelliptic Laplacian; this point will be described in Chapter 3, in Part 2 on rough paths theory. Despite the a priori difficulty linked to its hypoelliptic character, we were able to give a full description of the Poisson boundary of kinetic Brownian motion in the model setting of rotationally invariant manifolds.

Section 2.2 presents a joint work with J. Norris and L. Mesnager on the **small time asymptotics of bridges of subelliptic diffusion processes** on manifolds. A **large deviation principle** in path space is obtained under essentially optimal conditions on the endpoints of the bridges, and gives the projection of a bicharacteristic as deterministic first order limit path. When its endpoints lie further outside the intrinsic cut-locus of the generator of the diffusion, we establish convergence, to a **Gaussian** limit, of the **fluctuations** of the process about the limit path. The Gaussian limit is characterized in terms of the second variation of the energy functional on paths at a minimum, the formulation of which is new in the sub-elliptic case.

Chapter 1

Relativistic diffusions

Relativistic diffusions were first introduced by Dudley [**1**] in 1966 as the set of those random processes with values in Minkowski spacetime which enjoy the same features as Brownian motion in a Euclidean space, that is, continuity, Markovianity and invariance in law of the action of any affine isometry. However, Minkowski spacetime has an additional causal structure related to the non-positive definite character of its metric. Requiring from the sample paths of the processes that they describe the idealized motion in spacetime of a massive object happens to specify completely the law of the process, as discovered by Dudley. A further work of his [**2**] was dedicated to showing that all of these processes escape to infinity in a random direction. The subject was left at that point in 1973, and it was not investigated any further before Le Jan and Franchi's work [**3**] on a Lorentzian analogue of Dudley's diffusion in 2006(!), after some related process was introduced and studied independently by Debbasch and co-authors [**4**] in 1997, from a mathematical physics point of view, for applications to relativistic statistical physics.

This chapter offers an introduction to the different problems on relativistic diffusions I have worked on, starting with my PhD work on the Poisson boundary of Dudley's diffusion. Read the titles of the subsections for an overview of the different topics addressed here.

Intermezzo : Minkowski spacetime

Let $(\epsilon_0, \epsilon_1, \dots, \epsilon_d)$ stand for the canonical basis of $\mathbb{R}^{1+d}$, and $\dot{m}^i$ stand for the coordinates of a generic point $\dot{m} \in \mathbb{R}^{1+d}$. Minkowski's quadratic form is defined by the formula

$$g(\dot{m}, \dot{m}) = \dot{m}_0^2 - \sum_{i=1}^{d} \dot{m}_i^2.$$

Although g has signature $(1, -d)$, its restriction to the upper half unit sphere

$$\mathbb{H} = \left\{\dot{m} \in \mathbb{R}^{1+d}\,;\, \dot{m}_0 > 0,\ g(\dot{m}, \dot{m}) = 1\right\}$$

is definite negative, so $\mathbb{H}$ inherits from the ambiant space a Riemannian structure that actually turns it into a model of the d-dimensional hyperbolic space. **Minkowski spacetime** $\mathbb{R}^{1,d}$ is the affine space $\mathbb{R}^{1+d}$ equipped with the Lorentzian constant metric g on its tangent bundle. The set of all affine isometries of $\mathbb{R}^{1,d}$ is called **Poincaré group**. Given the non-definite character of the metric g, one distinguishes three kinds of $\mathbb{R}^{1,d}$-valued paths $\gamma_\bullet$. If $g\big(\gamma_s - \gamma_r, \gamma_s - \gamma_r\big) > 0$ at all times $r < s$, the **path** $\gamma_\bullet$ is said to be **timelike**; it represents the motion in spacetime of an object moving with a speed less than the speed of light. If $g\big(\gamma_s - \gamma_r, \gamma_s - \gamma_r\big) < 0$ at all times $r < s$, the **path** $\gamma_\bullet$ is said to be **spacelike**; it represents the motion in spacetime of an object moving with a speed greater than the speed of light. Last, if $g\big(\gamma_s - \gamma_r, \gamma_s - \gamma_r\big) = 0$ at all times $r < s$, the **path** $\gamma_\bullet$ is said to be **lightlike**; it represents the motion in spacetime of an object moving with the speed of light. Due to the cone constraint satisfied by a timelike path, such a path is always Lipschitz continuous, so it is in particular differentiable at almost-every time. A timelike path $\gamma_\bullet$ can always be reparametrized by its well-defined arclength, so as to have almost-everywhere unit speed; we say in that case that it is parametrized by its **proper time**. We shall say that it is **future-oriented** if $\dot\gamma \in \mathbb{H}$, and **past-oriented** if $(-\dot\gamma) \in \mathbb{H}$.

Despite its elementary geometry, Minkowski spacetime has a non-trivial geometry at infinity, related to its conformal structure. We define the past of a point $m \in \mathbb{R}^{1,d}$

$$\mathrm{Past}(m) := \big\{m' \in \mathbb{R}^{1,d}\,;\, q(m - m') > 0\big\}$$

as the set of points m' from which one can emit a signal propagating in spacetime from m' to m at a speed less than the speed of light. The set $\mathrm{Past}(\gamma_s)$ increases along a future-oriented timelike path $\gamma_\bullet$. Given that a point in $\mathbb{R}^{1,d}$ is uniquely determined by its past, it seems natural to say that two future-oriented causal curves $\gamma_\bullet$ and $\gamma'_\bullet$, leaving every compact, converge towards the same point at infinity if they have the same past

$$\bigcup_{s\geqslant 0} \mathrm{Past}(\gamma_s) = \bigcup_{s\geqslant 0} \mathrm{Past}(\gamma'_s).$$

This equivalence relation defines the **causal boundary** of Minkowski spacetime. This conformal notion of boundary is best understood by embedding Minkowski spacetime into a historically important example of Lorentzian manifold called Einstein universe. Minkowski spacetime is densely embedded in the latter, with a boundary $\mathbf{C}$ diffeomorphic to a pinched cylinder $\mathbb{S}^{d-1} \times \big(\mathbb{R} \cup \{\pm\infty\}\big)$, with $\mathbb{S}^{d-1} \times \{\pm\infty\}$ identified to a single point $\mathbf{p}_0$. Moreover, if one identifies $\mathbb{S}^{d-1}$ to the corresponding subset of the spacelike space $\mathbb{R}^d \subset \mathbb{R}^{1,d}$, a future-oriented timelike path of Minkowski spacetime converges to a boundary point with parameters $(\sigma, \ell) \in \mathbb{S}^{d-1} \times \mathbb{R}$ if, and only if, $q\big(\gamma_s, \epsilon_0 + \sigma\big)$ tends to ℓ, as s tends to infinity.

1.1 Dudley's diffusion

1.1.1. Dudley's diffusion in Minkowski spacetime

Dudley's starting point in his work [**1**] is the following fact. In a Euclidean space $\mathbb{R}^d$, there is a unique continuous Markov process whose (family of) law(s $(\mathbb{P}_x)_{x\in\mathbb{R}^d}$) is invariant by the action any affine Euclidean isometry. What about Minkowski spacetime? Is there a unique $\mathbb{R}^{1,d}$-valued continuous Markov process whose law is invariant by the action of Poincaré group? The answer is *no* in such a generality, but a little twist can save the situation. If one is interested in random timelike paths $\gamma_\bullet$, parametrized by their proper time say, we have

$$\gamma_s = \gamma_0 + \int_0^s \dot\gamma_r\, dr,$$

so the randomness needs to be on the velocity process $\dot\gamma_\bullet$. Dudley showed in [**1**] that there exists essentially a unique $\mathbb{R}^{1,d} \times \mathbb{H}$-valued continuous Markov process $(\gamma_s, \dot\gamma_s)_{s\geqslant 0}$ whose law is invariant by the action of Poincaré group. It corresponds to taking for velocity process $(\dot{\gamma}_s)_{s\geqslant 0}$ a Brownian motion on $\mathbb{H}$, run at speed σ^2, for some positive constant σ.

My PhD work addressed the question of determining the Poisson boundary of Dudley's diffusion, from a probabilistic/analytic and geometric point of view **[P1,2]**. To describe the result, denote by $(r_s;\theta_s)\in(0,\infty)\times\mathbb{S}^{d-1}$ the polar coordinates of the velocity process $\dot{\gamma}_s$ of Dudley's diffusion, seen for the point $\epsilon_0\in\mathbb{H}$. As is well-known for Brownian motion in hyperbolic space, θ_s converges almost surely to some random variable $\theta_\infty\in\mathbb{S}^{d-1}$.

Theorem 1 (Poisson boundary of Dudley's diffusion **[P1,3]**). (1) The $\mathbb{S}^{d-1}\times\mathbb{R}$-valued random variable $\big(\theta_s, g(\xi_s,\epsilon_0+\theta_\infty)\big)$ converges almost surely to some random variable $\big(\theta_\infty,\ell_\infty\big)$ whose associated σ-algebra coincides with the tail σ-algebra of Dudley's diffusion, up to null sets.

(2) The $\mathbb{R}^{1,d}$-valued process γ_s converges almost surely to a random point γ_∞ of the causal boundary $\mathbf{C}$ of Minkowski spacetime, whose associated σ-algebra coincides with the tail σ-algebra of Dudley's diffusion, up to null sets.

The problem is far from obvious from an analytical point of view, as it involves finding the set of all bounded harmonic functions for the hypoelliptic generator L of Dudley's diffusion. Two proofs of point (1) were given, based respectively on stochastic calculus and coupling technics, and random walk methods.

The first approach takes as a starting point the fact that θ_s converges to some limit point $\theta_\infty\in\mathbb{S}^{d-1}$ to investigate the law of the conditioned process $\big(\gamma_s,\,\dot{\gamma}_s\big|\theta_\infty\big)$. Given the invariance of the diffusion laws by the action of Poincaré group, there was no loss of generality in working with the process conditioned on the event $\{\theta_\infty=\epsilon_1\}$, whose generator is some explicit h^{ϵ_1}-transform of L. The convergence of $g(\gamma_s,\epsilon_0+\epsilon_1)$ to some random variable ℓ_∞, as s tends to ∞, for the conditioned process was clear on its dynamics – and led to an enlighting proof of a result of Dufresne on some Brownian functional **[P2]**. Denote by $h^{\epsilon_1}_\ell$ the (well-defined) density of the distribution of ℓ_∞ with respect to Lebesgue measure $d\ell$ on $\mathbb{R}$. Then, using some coupling argument together with some uniform continuity property coming from the fact that Dudley's (conditioned) diffusion is the projection on $\mathbb{R}^{1,d}\times\mathbb{H}$ of a left invariant diffusion in Poincaré group, I was able to prove that bounded $L^{h^{\epsilon_1}}$-harmonic functions depend only on a 2-dimensional variable. (We denote by L^g Doob's g-transform of L, for any non-negative function g.) An elementary invariance property implied the same conclusion for bounded $L^{h^{\epsilon_1}h^{\epsilon_1}_\ell}$-harmonic functions.

Arrived at that step, I had a decomposition

$$\mathbf{1}=\int_{\mathbb{S}^{d-1}\times\mathbb{R}} h^\sigma h^\sigma_\ell\, d\sigma d\ell$$

of the constant function $\mathbf{1}$, in terms of the L-harmonic functions $h^\sigma h^\sigma_\ell$. Building on some Harnack inequality and Choquet's representation theorem, point (1) of theorem 1 was equivalent to proving that the L-harmonic functions $h^\sigma h^\sigma_\ell$ are extremal, that is, to prove that bounded $L^{h^\sigma h^\sigma_\ell}$-harmonic functions are constant. From a probabilistic point of view, I had to prove that one could couple two 2-dimensional diffusions started from different points. Fortunately, the dynamics of the objects at hand provided an automatic shift coupling of two independent processes, which was sufficient for our needs.

The algebraic approach to point (1) of theorem 1 builds on the fact that Dudley's diffusion is the projection of a left invariant diffusion $(\mathbf{e}_s)_{s\geqslant 0}$ on Poincaré group, whose tail and invariant σ-algebras coincide. This puts us in a position to use Raugi's very general result on the Poisson boundaries of random walks on locally compact groups, here $(\mathbf{e}_n)_{n\geqslant 0}$. Rather than invoking this general machinery, I implemented in **[P3]** its core ideas in the setting of Poincaré group $\mathcal{G}$. Its basic scheme reads as follows.

- One finds a decomposition $\mathcal{D}^- \times \mathcal{D}^+ \times \mathcal{K}$ of $\mathcal{G}$, given by a diffeomorphism $(\mathbf{d}^-, \mathbf{d}^+, \mathbf{k}) \mapsto \mathbf{d}^-\mathbf{d}^+\mathbf{k}$, with $\mathcal{K}$ compact, and such that the $\mathcal{D}^-$-component $\mathbf{d}_n^-$ of $\mathbf{e}_n$ converges almost surely to some random variable $\mathbf{d}_\infty^- \in \mathcal{D}^-$, with $\mathbf{d}_n^+$ diverging and $\mathbf{k}_n$ ergodic.
- One proves that $\mathcal{D}^-$-left invariant bounded harmonic functions are constant.

This second step somehow plays the role of the coupling step for $L^{h^\sigma h_\ell^\sigma}$-diffusions in the above stochastic analysis proof. Starting from Iwasawa's decomposition $\mathcal{NAK}$ of the semisimple group $SO_0(1,d)$, with $\mathcal{A}$ of dimension 1, and representing $\mathcal{G}$ as the semidirect product $\mathbb{R}^{1+d} \times SO_0(1,d)$, one sets

$$\mathcal{D}^- := \mathbb{R}\,(\epsilon_0 + \epsilon_1) \times \mathcal{N}, \qquad \mathcal{D}^+ := \Big(\mathbb{R}\,(\epsilon_0 - \epsilon_1) \otimes \big\langle \epsilon_2, \dots, \epsilon_d \big\rangle\Big) \times \mathcal{A}$$

and, with a slight abuse of notations,

$$\mathcal{K} := \{0\} \times \mathcal{K}.$$

The convergence of the $\mathcal{D}^-$-component $\mathbf{d}_n^-$ of $\mathbf{e}_n$ was obtained by looking at the action of $\mathcal{D} := \mathcal{D}^-\mathcal{D}^+$ on the set of polynomials on the Lie algebra $\mathfrak{d}^-$ of $\mathcal{D}^-$, of degree at most 2, for an appropriate notion of degree for monomials. It ultimately rested on the fact that, writing $\mathbf{e}_n = (\gamma_n, g_n) \in \mathbb{R}^{1+d} \times SO_0(1,d)$, and $N(x_n)A(t_n)K_n$, with $x_n \in \mathbb{R}^{d-1}, t_n \in \mathbb{R}$, for Iwasawa decomposition of g_n, the component x_n converges in $\mathbb{R}^{d-1}$ and $\frac{t_n}{n}$ converges almost surely to $-\frac{d-1}{2}$. The proof that bounded $\mathcal{D}^-$-left invariant harmonic functions are constant is based on a simple, though very efficient, criterion for a point $\mathbf{e}$ in $\mathcal{G}$ to be a left period, meaning $h(\mathbf{e}\bullet) = h(\bullet)$. It suffices that $\mathbf{e}$ satisfies the following condition. For any $\mathbf{e}' \in \mathcal{G}$, one has $\mathbf{e}\mathbf{e}'\mathbf{e}_n = \mathbf{e}'\mathbf{e}_n\mathbf{z}_n$, for some random sequence $\mathbf{z}_n$ in $\mathcal{G}$, having almost surely a converging subsequence. The precise choice of group $\mathcal{D}^-, \mathcal{D}^+, \mathcal{K}$ as above, with specific behaviour of the components $\mathbf{d}_n^+, \mathbf{k}_n$, is important for proving that $\mathcal{D}^-$-left invariant bounded harmonic functions are constant.

At the time that Dudley introduced his process, in 1966, the tools of stochastic calclus were not advanced enough for making sense and working with the analogue in Lorentzian manifolds of Dudley's diffusion [**3**]. The subject was somehow forgotten, and we had to wait fourty years before Franchi and Le Jan define Dudley's diffusion in a Lorentzian manifold and study its behaviour in Schwarzchild spacetime.

Intermezzo : Lorentzian manifolds

A $(1+d)$-dimensional **Lorentzian manifold** is a manifold $\mathcal{M}$ equipped with a smooth $(0,2)$-tensor field g of signature $(1,-d)$. In other words, each tangent space is equipped with a scalar product that turns it into a Minkowski spacetime. They provide mathematical models for spacetime in the setting of Einstein's general relativity. All the usual local objects from Riemannian geometry, such as Riemann or Ricci tensors, or Levi-Civitta connection, are defined in the very same way in Lorentzian geometry as in Riemannian geometry. Classical, historically important, spacetime models are

- de Sitter spacetime is the sphere $\{m \in \mathbb{R}^{1+(d+1)}\,;\, g(m,m) = -1\}$ of negative radius in a $(2+d)$-dimensional Minkowski spacetime. It admits coordinates $(t, r, \sigma) \in \mathbb{R} \times (-1, 1) \times \mathbb{S}^{d-1}$ in which the metric tensor is
$$ds^2 = (1 - r^2)\, dt^2 - (1 - r^2)^{-1}\, dr^2 - r^2 d\sigma^2,$$
with $d\sigma^2$ the usual metric on the sphere $\mathbb{S}^{d-1}$.
- Robertson-Walker metric was one of the first model spacetime for the entire universe. It is given by the manifold $(a, b) \times N$, with a Lorentzian metric of the form
$$ds^2 = dt^2 - f(t)^2 dx^2,$$
with dx^2 a Riemannian metric on N.
- Schwarzchild spacetime is a model of gravity field around an isolated star. It is given by the manifold $\mathbb{R} \times (r_S, \infty) \times \mathbb{S}^{d-1}$, with a Lorentzian metric of the form
$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right) dr^2 - r^2 d\sigma^2,$$
with $d\sigma^2$ the usual metric on the sphere $\mathbb{S}^{d-1}$.

What distinguishes fundamentally the Lorentzian from the Riemannian world is the additional **causal structure** of Lorentzian manifolds. A $\mathcal{C}^1$ path $\gamma_\bullet$ with values in a Lorentzian manifold $(\mathcal{M}, g)$, is said to be timelike, resp. causal, if $g(\dot\gamma_s, \dot\gamma_s) > 0$, resp. $\geqslant 0$, at all times. The possibility to have closed timelike paths on a generic Lorentzian manifold causes some modelization problems. A whole range of causality conditions has been elaborated along the years to deal with a number of pathologies that may happen in different cases. The strongest of all these conditions is **global hyperbolicity**, that roughly states that $(\mathcal{M}, g)$ is diffeomorphic to a Lorentzian manifold of the form $(a, b) \times N$, with $-\infty \leqslant a < b \leqslant +\infty$ and N a d-dimensional manifold, with a Lorentzian metric that turns any path $s \mapsto (s, x)$, with $x \in N$ fixed, into a timelike path. So the first coordinate $(t, x) \mapsto t$, provides an absolute, extrinsic, notion of time. A much weaker assumption is **strong causality**. Every point of a strongly causal spacetime has arbitrary small (connected) neighbourhoods which no causal path intersect more than once.

We shall assume in the sequel that $\mathcal{M}$ is time-oriented, which means that the unit tangent bundle $T^1\mathcal{M} := \{(m, \dot m) \in T\mathcal{M}\,;\, g_m(\dot m, \dot m) = 1\}$ has two components. We choose arbitrarily one of them and will refer to it as $T^1\mathcal{M}$ in the sequel.

1.1.2. Dudley's diffusion on a Lorentzian manifold

Let $(\mathcal{M}, g)$ be a time-oriented Lorentzian manifold. Dudley's diffusion on $\mathcal{M}$ is constructed by using the method of Cartan's moving frame, rolling without slipping Dudley's diffusion in Minkowski spacetime on $\mathcal{M}$, using Levi-Civitta connection [**3**]. To give a more formal description of Cartan's machinery, let us introduce here the following notations, that will be used throughout that part of the manuscript. With a slight abuse of notations, we shall talk about orthonormal frames for pseudo-orthonormal frames.

- We shall use Einstein's well-known summation convention and shall denote the pseudo-orthonormal frame bundle of $(\mathcal{M}, g)$ by $O\mathcal{M}$. We shall denote by $z = (x, e)$

a generic point of $O\mathcal{M}$, with $x \in \mathcal{M}$ and $e : \mathbb{R}^{1,d} \to T_x\mathcal{M}$, an orthonormal frame of $T_x\mathcal{M}$; we write $\pi : O\mathcal{M} \to \mathcal{M}$ for the canonical projection map.

- Denote by $(V_i)_{1\leqslant i\leqslant d}$, the canonical vertical vector fields on $O\mathcal{M}$, associated with the Lie elements $v_i := \epsilon_i \otimes \epsilon_0^* - \epsilon_0 \otimes \epsilon_i^*$ of the orthonormal group of $\mathbb{R}^{1,d}$. The Levi-Civita connection on $T\mathcal{M}$ defines a unique horizontal vector field H_0 on $O\mathcal{M}$ such that $(\pi_* H_0)(z) = e(\epsilon_0)$, for all $z = (x, e) \in O\mathcal{M}$. The flow of this vector field is the natural lift to $O\mathcal{M}$ of the (locally defined) *timelike geodesic flow*. Taking local coordinates $(x^i)_{i=0..d}$ on $\mathcal{M}$ induces canonical coordinates on $O\mathcal{M}$ by writing
$$e_i := e(\epsilon_i) = \sum_{j=0}^{d} e_i^j \partial_{x^j}.$$
Denoting by Γ_{ij}^k the Christoffel symbol of the Levi-Civita connection associated with the above coordinates, the vector fields $(V_i)_{1\leqslant i\leqslant d}$ and H_0 have the following expressions in these local coordinates
$$\begin{aligned} V_i(z) \quad &= e_i^k \frac{\partial}{\partial e_0^k} - e_0^k \frac{\partial}{\partial e_i^k}, \\ H_0(z) \quad &= e_0^i \frac{\partial}{\partial x_i} - \Gamma_{ij}^k(x) e_0^i e_l^j \frac{\partial}{\partial e_l^k}. \end{aligned}$$

Let w be a d-dimensional Brownian motion. The following Stratonovich differential equation is easily seen to define an $SO(d)$-invariant diffusion process $(z_s)_{0\leqslant s<\zeta}$ in $O\mathcal{M}$ whose projection in $T^1\mathcal{M}$ is the development in $(\mathcal{M}, g)$ of Dudley's diffusion in Minkowski spacetime

$$\circ dz_s = H_0\big(z_s\big)\, ds + \sigma \sum_{j=1}^{d} V_j\big(z_s\big) \circ dw_s^j. \tag{1.1.1}$$

It describes a random perturbation of the geodesic flow whose intuitive meaning is the following. To get z_{s+ds} out of $z_s = (x_s, e_s)$, transport first e_s parallelly along the geodesic starting from x_s in the timelike direction $e_s(\epsilon_0)$, during an amount of time ds; this gives you an orthonormal frame f_s of $T_{x_{s+ds}}\mathcal{M}$. You get e_{s+ds} by making in each spacelike 2-plane of f_s some independent hyperbolic rotations of angle a centered normal random variable with variance $\sigma^2\, ds$. We shall also use the expression Dudley diffusion for this $O\mathcal{M}$-valued process.

Equation (1.1.1) has a unique strong solution defined up to its explosion time ζ; Dudley's diffusions on $O\mathcal{M}$ and $T^1\mathcal{M}$ can be seen to have the same lifetime. The diffusion $(z_s)_{0\leqslant s<\zeta}$ in $O\mathcal{M}$ has a generator given by the formula
$$L := H_0 + \frac{\sigma^2}{2} \sum_{j=1}^{3} V_j^2.$$
Given the dynamical definition of Dudley's diffusion in $\mathcal{M}$ via moving frames, the process is expected to behave locally as its Minkowskian model. This is obviously not the case at larger scales, where geometry has a dramatic influence on its behaviour. The simplest illustration of this sensitivity to the geometric background is given by the study of the stochastic completeness problem of Dudley's diffusion in $\mathcal{M}$. This question gets here a particular flavour in so far as this diffusion is a random perturbation of the geodesic

dynamics. The latter has been paid much attention in a Lorentzian setting as geodesic incompleteness is presently understood as the main sign of singularity of a spacetime model, and that this phenomenon was shown to be somewhat unavoidable.

Intermezzo : Singularity theorems

The introduction of the concepts of singularities and boundary of a spacetime both have their source in the observation that Einstein's geometrical picture of a spacetime does not prevent the existence of some undesirable features like the existence of regions where some geometrical scalar explodes along some path, or the existence of physically relevant incomplete paths (geodesics, paths with bounded acceleration, etc). Following the pioneering works of Penrose [**5**] and Hawking [**6**], [**7**], [**8**] and recent works of Christodoulou, it was proved that most (if not all) reasonnable solutions to Einstein's equations define Lorentzian manifolds that are inevitably singular, in so far as they have incomplete causal geodesics. The seeds of these incompleteness theorems are a combination of energy, causality and boundary/initial conditions, that usually combine as follows to give birth to incompleteness.

- Using causality conditions, one constructs a useful maximal geodesic. At the same time, the energy and initial/boundary conditions induce conjugate or focal points along any geodesic, provided time is allowed to run long enough; as this cannot happen along a maximal geodesic, its time parameter has to be bounded.
- Supposing the spacetime is geodesically complete, one constructs a compact proper achronal boundary whose existence prevents the existence of an open Cauchy hypersurface.

The above mentioned energy conditions are curvature conditions that get their physical meaning from Einstein's equations relating the energy-momentum tensor T of matter to curvature of spacetime.

- Weak energy condition. We have $T(u,u) \geqslant 0$, for all timelike vector fields u.
- Dominant energy condition. In addition to the weak energy condition holding true, the mass-energy can never be observed to flow faster than light, that is the vector field corresponding to the one-form $-T(u)$ is a future-oriented causal vector field, for any future-oriented causal vector field u.
- Strong energy condition. We have $Ric(u,u) \geqslant 0$, for all future-oriented timelike vector field u.

The last condition quantifies the fact that gravity is attractive, maybe not in all directions, but in mean. As an example, for a perfect fluid with

$$T = (\rho + p)\, u \otimes u + p\, g,$$

with density ρ and pressure p, the weak energy condition reads $\{\rho \geqslant 0,\ \rho + p \geqslant 0\}$, the dominant energy condition reads $\{\rho \geqslant |p|\}$, while the strong energy condition becomes $\{\rho + p \geqslant 0,\ \rho + 3p \geqslant 0\}$.

1.1.3. A probabilistic view on singularities We addressed in the work **[P10]** the following question,

Can timelike geodesic completeness be destroyed by a random perturbation of its dynamics?

by putting forward different kinds of energy and 'energy flow' conditions that are sufficient to ensure the stochastic incompleteness of Dudley's diffusion. This provides an unusual type of conclusion for stochastic perturbations of the geodesic flow, under a non-common set of conditions where no causality assumption is required, nor any boundary or initial condition. Interestingly, they also give some valuable purely geometric results such as proposition 4 below.

Probabilistic incompleteness theorems differ somewhat from their deterministic analogue. To wit, if there exists an initial condition $z \in O\mathcal{M}$ such that $\mathbb{P}_z(\zeta < \infty) > 0$, then explosion occurs in an arbitrarily small time with positive probability $\mathbb{P}_z(\zeta < \epsilon) > 0$, for all $\epsilon > 0$. Also, since the function $z \mapsto \mathbb{P}_z(\zeta < \infty)$ is L-harmonic, if $\mathbb{P}_{z_0}(\zeta < \infty) = 1$ at some point $z_0 = (x, e) \in O\mathcal{M}$, we shall actually have $\mathbb{P}_{z'}(\zeta < \infty) = 1$ for any $z' = (x', e')$ with x' in the chronological future of x.

a) A first probabilistic incompleteness theorem. Let $R(\cdot,\cdot)\cdot$ be Riemann curvature tensor and denote by Ric the restriction to $T^1\mathcal{M}$ of Ricci curvature tensor; we identify trivially Ric to a function on $O\mathcal{M}$. Write R for the scalar curvature and $\mathbf{T} = \mathrm{Ric} - \frac{1}{2}\,R\,g$ for the restriction to $T^1\mathcal{M}$ of energy-momentum tensor.

Theorem 2 ([P10]). Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Suppose

(1) (Pointwise energy condition.) $\forall\,(m, \dot m) \in T^1\mathcal{M},\ \ \mathbf{T}_m(\dot m, \dot m) \geqslant 0$,
(2) the function Ric is non-identically constant, positive at some point $(m_0, \dot m_0)$, and bounded above,
(3) (Energy flow condition.) there exists a constant $c \in \mathbb{R}$ such that $H_0\,\mathrm{Ric} \geqslant c\,\mathrm{Ric}$.

Write $c = C - 2\,\sigma^2$ for some positive constants C and σ. Then Dudley's diffusion with diffusivity σ^2 explodes with $\mathbb{P}_{(m_0,\dot m_0)}$-positive probability.

The simple proof of this statement makes a crucial use of Khasminski's well-known explosion criterion and the following elementary lemma, which we write down explicitly here as we use it in the next paragraph.

Lemma 3 ([P10]). We have $L\mathrm{Ric} = H_0\,\mathrm{Ric} + 2\,\sigma^2\,\mathrm{Ric} + 2\,\sigma^2\,\mathbf{T}$.

b) A purely geometric result. Let A be any subset of $\mathcal{M}$. The past domain of dependence of A is the set $D^-(A)$ of points m of $\mathcal{M}$ from which any future directed timelike path starting from m eventually hits A. The future domain of dependence $D^+(A)$ is defined similarly using past directed timelike paths. The domain of dependence of A is $D(A) = D^-(A) \cup D^+(A)$; it is globally hyperbolic. Note that if A is a relatively compact spacelike hypersurface, and m a point of $D^-(A)$, then there exists a constant $T(m)$, depending on m, such that all timelike path starting from m hits A before proper time $T(m)$.

Given $A \subset \mathcal{M}$, it is well-known that if the dominant energy condition holds on its domain of dependence $D(A)$ and the energy momentum tensor vanishes on A then it vanishes on the whole of $D(A)$, **[9]**. Lemma 3 provides for free a similar result under different hypotheses.

Proposition 4 ([P10]). Let A be a relatively compact spacelike hypersurface of a spacetime $(\mathcal{M}, g)$. Suppose the following conditions hold on $D^-(A)$:

- strong and weak energy conditions,
- there exists a constant $c \in \mathbb{R}$ such that $H_0 \,\mathrm{Ric} \geqslant c\, \mathrm{Ric}$.

If $\mathrm{Ric} = 0$ on A then $\mathrm{Ric} = 0$ on $D^-(A)$.

Note that the strong and weak energy conditions together do not imply the dominant energy condition, which is the reason why the above statement is interesting.

Proof – Combining the hypotheses of the proposition and lemma 3 we get the inequality

$$L\mathrm{Ric} \geqslant (\sigma^2 + c)\, \mathrm{Ric} \geqslant 0,$$

for a big enough σ. The function Ric is thus a non-negative L^σ-sub-harmonic function on $O\mathcal{M}$. Given $z = (m, z)$ with $m \in D^-(A)$, the stopping time $\tau = \inf\{s \geqslant 0\,;\, \pi\big(z_s\big) \in A\}$ is almost surely bounded by a constant depending only on m, and

$$\mathbb{E}_z\big[\mathrm{Ric}(z_\tau)\big] = \mathrm{Ric}(z) + \mathbb{E}_z\Big[\int_0^\tau \big(L\mathrm{Ric}\big)(z_r)\, dr\Big] \geqslant \mathrm{Ric}(z) \geqslant 0,$$

by optional stopping. As $\mathrm{Ric}(a_\tau) \equiv 0$ this proves that $\mathrm{Ric}(z) = 0$, and implies the result as z is any point of $\pi^{-1}\big(D^-(A)\big) \subset O\mathcal{M}$. ▷

A similar result holds on the future domain of dependence $D^+(A)$ of A, if we have $H_0\mathrm{Ric} \leqslant c\,\mathrm{Ric}$, for some constant $c \in \mathbb{R}$.

c) A second probabilistic singularity theorem. We use in this section a refined version of Khasminski's explosion criterion for which no boundedness assumption is needed.

Lemma 5 ([P10]). Suppose there exists two non-null, non-negative smooth functions f, h on $O\mathcal{M}$, with $f \leqslant h$, and two constants $0 \leqslant c' < c$ such that

$$Lf \geqslant c\, f \quad \text{and} \quad Lh \leqslant c'\, h.$$

Let $z_0 \in O\mathcal{M}$ be such that $f(z_0) > 0$. Then the basic relativistic diffusion started from z_0 explodes with positive probability.

The next probabilistic singularity theorem is obtained by applying the above explosion criterion, lemma 5, to the functions $f = \mathrm{Ric}$ and $h = \mathrm{Ric} + k$, where

$$k(z) := 2 \int_{T^1_m\mathcal{M}} G(e_0, y)\mathrm{Ric}_m(y, y)\, dy, \qquad \text{with } z = (m, e), \text{ and } e_0 := e(\epsilon_0),$$

is defined in terms of the Green kernel G of the Laplace-Beltrami operator on $T^1_m\mathcal{M}$, so the function k is a solution of the equation

$$\frac{1}{2}\sum_{j=1}^{3} V_j^2\, k = -2\, \mathrm{Ric}.$$

Theorem 6 ([P10]). Let $(\mathcal{M}, g)$ be a Lorentzian manifold satisfying the following conditions.

(1') **Static energy conditions.** Ric is non-negative and non-identically null, and $R \leqslant 0$.

(2') **Regularity condition**. The function Ric is exp-bounded in each $T^1_m\mathcal{M}$ and there exist some constants $0 < \alpha < 1,\ 0 \leqslant c' < c$ and $\sqrt{\frac{c}{2}} < \sigma < \sqrt{\frac{c'}{2\,\alpha}}$, such that

$$\frac{1-\alpha}{\alpha}\,\mathrm{Ric} \leqslant k.$$

(3') **Dynamical energy conditions.** *(i)* $H_0\,\mathrm{Ric} \geqslant (c - 2\,\sigma^2)\,\mathrm{Ric}$,
(ii) $H_0\,h \leqslant (c' - 2\,\alpha\,\sigma^2)\,h$.

Let $z_0 \in O\mathcal{M}$ be such that $\mathrm{Ric}(z_0) > 0$. Then the basic relativistic diffusion with diffusivity σ^2, started from z_0, explodes with positive probability.

Note that our choice of constants gives $c' - 2\,\alpha\,\sigma^2 > 0$ and $c - 2\,\sigma^2 < 0$, which gives a non-trivial character to conditions (3'). The interest of this singularity theorem is that it does not require any causality assumption, in contrast to most classical singularity theorems.

1.2

A general class of relativistic diffusions

The setting of Minkowski spacetime, with its trivial local geometry and unique relativistic diffusion was somewhat misleading and it took a little bit of time before I realized that one can actually define a whole family of diffusion processes on $T^1\mathcal{M}$ on a generic Lorentzian manifold, in terms only of non-trivial geometric tensors such as Ricci or scalar curvature tensors. Le Jan and Franchi also realized that point at the very same time [**10**]. I introduced in **[P7]** a class of relativistic diffusions encompassing all the example studied previously, including [**10**]. This family of processes can be understood as the output of the meeting of two different stories, the first one consisting on the body of knowledge on Dudley's diffusion.

The other story was born immediately after Einstein's theory of relativity and gravitation was accepted and spread in the scientific community. It deals with the extension of Boltzmann theory of gases to the relativistic framework. Although Boltzmann model is primarily a particle model of gases, most of the works have been on understanding the macroscopic behaviour of relativistic gases through the study of the relativistic Boltzmann equation. One had to wait the nineties and the article [**4**] of F. Debbasch, K. Mallick and J.P. Rivet to see the introduction of a probabilistic mesoscopic model of diffusion of a particle in a fluid, under the form of a special relativistic counterpart of Ornstein-Uhlenbeck process. Generalisations of this model to the framework of general relativity have been given in later articles by these authors and their collaborators.

These two stories met with the proposition, made in the article [**11**] of C. Chevalier and F. Debbasch, to define a class of random processes including Dudley's process and the relativistic Ornstein-Uhlenbeck process, and characterized by the following property. There exists at each proper time of the moving particle a local rest frame where the acceleration of the particle is Brownian in any spacelike direction of the frame, when computed using the time of the rest frame. This point of view was adopted in **[P7]**, where I proved that the one-particle function of each diffusion of this class satisfies some remarkable equation linked to the study of Martin boundary of the generator of the diffusion, and where an

H-theorem on the increasing character of some related entropy was proved. We describe shortly the results of this work in this section.

1.2.1. Definition of relativistic diffusions As said above, relativistic diffusions should be considered as a class of toy models of diffusion in relativistic media. We define them as random perturbations of the geodesic flow

$$dz_s = H_0(z_s)ds.$$

We model the action of the surrounding fluid on the moving object $(z_s)_{s\geqslant 0} = \big((x_s, e_s)\big)_{s\geqslant 0}$ by the datum of an $O\mathcal{M}$-valued *previsible* process $(\mathfrak{z}_s)_{s\geqslant 0}$ such that $\mathfrak{z}_s(z_\bullet) = \big(x_s, f_s\big)$, for some orthonormal basis $f_s = \big(f_s^0, f_s^1, \dots, f_s^d\big)$ of $T_{x_s}\mathcal{M}$. The random perturbation induced by the medium on the dynamics results in adding to the deterministic acceleration a random part which is determined by the following requirement. *When computed in the rest frame $\mathfrak{z}_s$, i.e. using its associated time*, the acceleration of x_s has a deterministic part and a random part which is Brownian in any spacelike direction belonging to $\mathrm{span}(f_s^1, \dots, f_s^d)$. To complete this description, we shall ask the vectors $e_s(\epsilon_1), \dots, e_s(\epsilon_d)$ to be transported parallelly along the "Brownian" increment of $e_s(\epsilon_0)$.

An example in Minkowski spacetime will clarify this verbal description. Let then work in $\mathbb{R}^{1,d}$, and look at the heuristic description of what happens when $V = 0$ and the action process $\mathfrak{z}_.$ is constant, equal to the canonical basis, *i.e.* $f_s = \{\epsilon_0, \dots, \epsilon_d\}$, at all times s.

Denote by $z_s = (x_s, e_s)$ the associated $O\mathbb{R}^{1,d}$-valued process and write t_s for the ϵ_0-component of x_s. As we have in that case $dx_s = e_s(\epsilon_0)\, ds$, the function $s \mapsto t_s$ is a $\mathcal{C}^1$ increasing function that can be used as a parameter of the process. Given $t \in \mathbb{R}$, set $\tau_t = \inf\big\{s \geqslant 0\,;\, t_s = t\big\}$ and look at the re-parametrized process $\widehat{z}_t = \big(\widehat{x}_t, \widehat{e}_t\big) := z_{\tau_t}$, for $t \geqslant g(\epsilon_0, x_0)$. The above description of the action of the surrounding medium on the dynamics means that the projection on $\mathrm{span}(\epsilon_1, \dots, \epsilon_d)$-part of $d\widehat{e}_t(\epsilon_0)$ parallely to ϵ_0 is a Brownian increment $d\widehat{w}_t$ in $\mathrm{span}(\epsilon_1, \dots, \epsilon_d)$. Let us adopt here the convenient notation $\widehat{e}_t^i$ for $\widehat{e}_t(\epsilon_i)$, for $i = 0..d$. If we write $\circ d\widehat{e}_t^0 = \sum_{i=1..d} \widehat{e}_t^j \circ d\widehat{\beta}_t^j$, then

$$\circ d\widehat{w}_t^i = -\sum_{i=1..d} g(\epsilon_i, \widehat{e}_t^j) \circ d\widehat{\beta}_t^j.$$

For $e \in SO_0(1,d)$ and $e^j := e(\epsilon_j)$, denote by $A(e)$ the $d \times d$ matrix with coefficients $(i,j) \in [\![1,d]\!]^2$ equal to $g(\epsilon_i, e^j)$. This matrix being invertible, we have

$$\circ d\widehat{\beta}_t = -A(\widehat{e}_t)^{-1} \circ d\widehat{w}_t. \tag{1.2.1}$$

Back to the proper time s of the process, we shall write $\circ de_s^0 = \sum_{i=1..d} e_s^i \circ d\varpi_s^j$. Writing A_s for $A(e_s)$, identity (1.2.1) implies that we have

$$\circ d\varpi_s = g(\epsilon_0, e_s^0)^{\frac{1}{2}}\, A_s^{-1} \circ dw_s$$

for some d-dimensional Brownian motion w. The $\mathbb{R}^d$-valued process ζ is the process that really drives the dynamics. Last, we shall ask the vectors $e_s^1, \dots, e_s^d$ to be parallelly transported along the paths $(e_s^0)_{s\geqslant 0}$ in $\mathbb{H}$.

In the setting of a Lorentzian manifold $(\mathcal{M}, g)$, with generic datum $\mathfrak{z}$, and given an $O\mathcal{M}$-valued process $z_s = (x_s, e_s)$, define A_s as the $d \times d$ matrix-valued random process with coefficient $(i,j) \in [\![1,d]\!]^2$, equal to $g(f_s^i, e_s^j)$ at time s, and set

$$\circ d\varpi_s = q(f_s^0, \mathbf{g}_s^0)^{\frac{1}{2}}\, A_s^{-1} \circ dw_s.$$

Definition. An **$\mathfrak{z}$-diffusion** is an $O\mathcal{M}$-valued process $z_s = (x_s, e_s)$ satysfying the stochastic differential equation

$$\circ dz_s = H_0(z_s)ds + V_i(z_s) \circ d\varpi_s^i,$$

up to its explosion time ζ.

In case $\mathfrak{z}$ is Markovian, with $f_s = f(e_s)$ for some function f, the $(V, \mathfrak{z})$-diffusion has generator

$$L = H_0 + V + \frac{1}{2} V_i\Big(\gamma\, \Theta^{ij} V_j\Big),$$

where $\gamma(e) = g\big(f(e)^0, e^0\big)$, and $A(e)^{ij} = g\big(f(e)^i, e^j\big)$, and $\Theta = (A^{-1})^t A^{-1}$. Working with orthogonal bases f rather than orthonormal bases, the above framework includes as a particular case the setting of Le Jan and Franchi, where $\Theta(e) = \theta(e^0)\,\mathrm{Id}$, for a function $\theta(e^0)$ typically equal to a constant multiple of the scalar curvature, or the Ricci curvature in the direction of e^0. This corresponds in both cases to taking for frame f a varying multiple of the frame e. We shall work in the next section with Markovian relativistic diffusions.

1.2.2. One-particle distribution function

Fix here the starting point z_0 of a relativistic diffusion. Given a point $z = (x, e) \in O\mathcal{M}$, different from z_0, let $\mathcal{H}_z$ stand for the set of spacelike hypersurfaces of $\mathcal{M}$ contained in a small enough neighbourhood of x, and such that $x \in \mathbb{V}$ and $T_x\mathbb{V} = \big(e^0\big)^{\perp}$. Associate to any such hypersurface $\mathbb{V} \in \mathcal{H}_z$ the hitting time

$$\tau := \inf\{s \geqslant 0\,;\, x_s \in \mathbb{V}\}.$$

Given any point x' in $\mathcal{M}$ we shall denote by $\textsc{Vol}_{x'}(dg)$ the Haar measure on $O_{m'}\mathcal{M}$, normalized in such a way that its projection on $T^1_{x'}\mathcal{M}$ is the Riemannian volume element induced by g. Recall the definition of $O\mathbb{V} = \big\{\widehat{z} = (\widehat{x}, \widehat{e}) \in O\mathcal{M}\,;\, \widehat{x} \in \mathbb{V}, \widehat{e} \in O_m\mathcal{M}\big\}$; we denote by$\textsc{Vol}_{O\mathbb{V}}$ its natural volume measure.

Proposition 7 ([P7]). Let $\mathbb{V} \in \mathcal{H}_z$.

(1) The random variable $z_\tau\, \mathbf{1}_{\tau<\infty}$ has a smooth density $f_{\mathbb{V}}(z_0\,;\, \widehat{z})$ with respect to the measure $\textsc{Vol}_{O\mathbb{V}}(d\widehat{z})$ on $O\mathbb{V}$.

(2) We have $f_{\mathbb{V}'}(z_0\,;\, z) = f_{\mathbb{V}}(z_0\,;\, z)$, for any other $\mathbb{V}'$ in $\mathcal{H}_z$.

So this quantity $f_{\mathbb{V}}(z_0\,;\, z)$ is independent of $\mathbb{V} \in \mathcal{H}_z$; call it the value at point z of the **one-particle distribution function of the relativistic diffusion started from** z_0. We shall denote it by $f_{z_0}(z)$; it is defined for $z \neq z_0$.

The next proposition shows that the one-particle distribution function is all we need to recover the hitting distribution on any spacelike hypersurface $\mathbb{V}$ with $z_0 \notin O\mathbb{V}$. Given a point $\widehat{z} = \big(\widehat{x}, \widehat{e}\big) \in O\mathbb{V}$, denote by $\varpi_{\mathbb{V}}(\widehat{z})$ the future-oriented unit timelike vector orthogonal to $T_{\widehat{x}}\mathbb{V}$.

Proposition 8 ([P7]). For any bounded real-valued function F on $O\mathbb{V}$, we have

$$\mathbb{E}_{z_0}\big[\mathrm{F}(z_\mathcal{T})\mathbf{1}_{\mathcal{T}<\infty}\big] = \int_{O\mathbb{V}} \mathrm{F}(\widehat{z})\, g\big(\widehat{e}^0, \varpi_{\mathbb{V}}(\widehat{z})\big)\, f_{z_0}(\widehat{z})\, \mathrm{Vol}_{O\mathbb{V}}(d\widehat{z}).$$

The next theorem has a crucial importance from a probabilistic point of view.

Theorem 9 ([P7]). We have

$$L^* f_{z_0} = 0,$$

in $\mathbb{O}\mathrm{M}\backslash\{z_0\}$.

It gives us a wealth of invariant measures for the L-diffusion, and elementary L^*-harmonic functions. Note the elementary character of the dynamics of L^*-diffusions, as we have

$$L^* = -H_0 + \frac{1}{2}\sum_{i=1}^{d} V_i^2.$$

It is clear that the paths of an L^*-diffusion started from $z_0 = (x_0, e_0)$ take values in the chronological past $\pi^{-1}\big(I^-(x_0)\big)$ of z_0. Any open set of $\pi^{-1}\big(I^-(x_0)\big)$ is visited by the process with positive probability, and any of its spacelike hypersurfaces $\mathbb{V}$ is hit with positive probability. As noted in proposition 8, these hitting distributions are determined by the one-particle distribution function of the L^*-diffusion process $\overleftarrow{z}_s$; denote it by $\overleftarrow{f}_{z_0}$.

Towards Martin boundary. The present situation is reminiscent of results by Constantinescu & Cornea and J. Taylor who identified (in a potential theoretic language) the Martin boundary of Brownian (or more general diffusion processes) to the exterior harmonic boundary of their adjoint generator. Making the parallel between the setting of Taylor's theorem and the present situation, the definition of the one-particle distribution function, proposition 8 and theorem 9 make the following conjecture very likely to hold true although I did not manage to prove it so far.

Conjecture 10. The following statements are equivalent.

(1) The sequence $(z_n)_{n\geqslant 0}$ is fundamental for L in $(\mathcal{M}, g)$.

(2) For any spacelike smooth hypersurface $\mathbb{V}$ of $\mathcal{M}$ the sequence of conditional distributions

$$\mathbb{P}_{z_n}\Big(\overleftarrow{z}_{\tau_\mathbb{V}} \in \cdot\, \big|\, \tau_\mathbb{V} < \infty\Big)$$

converges.

(3) For any $z' = (x', e') \in O\mathcal{M}$, the sequence of L-harmonic functions $\left(\frac{\overleftarrow{f}_{z_n}(\cdot)}{\overleftarrow{f}_{z_n}(z')}\right)_{n\geqslant 0}$ converges uniformly on compact subsets of $\pi^{-1}\big(I^-(x')\big)$.

This fact would explain why the causal boundary of $(\mathcal{M}, g)$ is likely to appear in the study of the Martin boundary of L. In order for the above conditional distributions to converge, the support of each of these probabilities has to converge, for any spacelike hypersurface $\mathbb{V}$. This cannot happen unless the chronological past of z_n converges, *i.e.* unless the sequence $(x_n)_{n\geqslant 0}$ has a limit in the causal boundary of $(\mathcal{M}, g)$. This result would give a nice characterization of the Martin boundary without providing any geometrical characterization for it though.

The geometrical identification of the Poisson boundary of Dudley's diffusion in large classes of spacetimes is already quite a challenge. The first step made in **[P1]**, in the setting of Minkowski spacetime, suggested that the Poisson boundary may be identified with the causal boundary of spacetime. Franchi' study of Dudley's diffusion in Gödel spacetime, which has trivial causal boundary and non-trivial Poisson boundary, showed that causality conditions should be assumed if ever the above picture was to hold. The study by Tardif of what happens in de Sitter and anti-de Sitter spacetimes fitted the framework, however Angst and Tardif [**13**] proved that the Poisson boundary may be richer than the causal boundary in some Robertson-Walker spacetimes, which are as reasonnable as can be from a causal point of view. Characterizing the Poisson boundary in geometrical terms remains thus an exiting open problem.

1.2.3. Non-explosion criteria for relativistic diffusions

The results of the previous section were somewhat of a local nature and did not require the relativistic diffusions to be defined for all times, as we essentially paid attention to hitting distributions. Together with J. Franchi, I addressed in the work **[P9]** the question of finding conditions ensuring the non-explosion of relativistic diffusions. In addition to being a natural question, the completeness issue is strongly related to the above mentionned incompleteness theorems in mathematical general relativity, in so far as relativistic diffusions are intrinsic perturbations of the geodesic flow; their completeness/incompleteness appears as a distinguishing feature of a spacetime. Given an $SO(d)$-invariant non-negative real-valued function Θ on $O\mathcal{M}$, we concentrated on the stochastic completeness issue for Θ-diffusions, with generator

$$L = H_0 + \frac{1}{2} V_i\big(\Theta V_i\big).$$

Building on the same kind of Lyapounov method as in section 1.1.3, some simple non-explosion criteria of Khasminski can be used to ensure the following kind of result. Assume we are working with a globally hyperbolic spacetime $\mathcal{M}$ diffeomorphic to $I \times S$, for some interval I of $\mathbb{R}$, and whose metric tensor has the form

$$g_x(\dot{x}, \dot{x}) = a_x^2 \left|\dot{x}^0\right|^2 - h_x(\dot{x}^S, \dot{x}^S)\,, \tag{1.2.2}$$

where $\dot{x}^0$ is the image of $\dot{x} \in T_x^1\mathcal{M}$ by the differential of the first projection $I \times S \to I$, and $\dot{x}^S$ the image of $\dot{x}$ by the differential of the second projection $I \times S \to S$. Write $x = (t, y) \in I \times S$. The function a is a positive $\mathcal{C}^1$ function on $\mathcal{M}$, assumed to be bounded on any subset $I' \times S$ where I' is bounded from above, and h_x is a positive-definite scalar product on T_yS, depending on x in a $\mathcal{C}^1$ way. This class of spacetimes contains all Robertson-Walker spacetimes. The following result applies in particular when the spacetime has a bounded non-negative scalar curvature Θ.

Proposition 11 ([P9]). Let $\mathcal{M} = I \times S$ denote a generalized warped product spacetime and let Θ be a bounded non-negative function on $\mathcal{M}$. Then the Θ-diffusion does not explode if ∇a is everywhere non-spacelike and future-directed.

As we are going to see now, this kind of result can be significantly improved by adapting to the Lorentzian setting Takeda' strategy [**14, 15**] for proving stochastic completeness of Brownian motion on a Riemannian manifold, as improved by Hsu and Qin [**16**]. This is however far from being straightforward, since we are working in a non-symmetric, non-elliptic setting, where the main ingredients of Takeda's method have no obvious Lorentzian counterpart.

Intermezzo : Takeda's method for stochastic completeness

Using an idea of Lyons and Zheng, **[18]** , Takeda devised in **[14]**, **[15]**, a remarkably simple and sharp non-explosion criterion for Brownian motion on a Riemannian manifold $\mathcal{M}$. Loosely speaking, his reasonning works as follows.

Suppose we have a diffusion $x_\bullet$ on $\mathcal{M}$ which is symmetric, with respect to the Riemannian volume measure Vol say, and conservative. Denote by L its generator, and by $\mathbb{P}_{\mathrm{Vol}}$ the measure $\int \mathbb{P}_x \,\mathrm{Vol}(dx)$ on the path space, where $\mathbb{P}_x$ stands for the law of the L-diffusion started from x. Fix a time $T>0$ and let f be a smooth real-valued function on M. As the time-reversed process $(x_{T-s})_{0\leqslant s\leqslant T}$ is also an L-diffusion under $\mathbb{P}_{\mathrm{Vol}}$, applying Itô's formula to both $f(x_s)$ and $f(x_{T-s})$ provides two martingales M and $\widetilde{M}$, with respect to two different filtrations, such that

$$f(x_s) = f(x_0) + M_s + \int_0^s Lf(x_r)\,dr\,,$$

$$f(x_s) = f(x_{T-(T-s)}) = f(x_T) + \widetilde{M}_{T-s} + \int_0^{T-s} Lf(x_{T-r})\,dr.$$

It follows that

$$f(x_s) = \frac{f(x_0)+f(x_T)}{2} + \frac{M_s+\widetilde{M}_{T-s}}{2} + \int_0^T Lf(x_s)\,dr,$$

and consequently

$$f(x_T) - f(x_0) = \frac{1}{2}\big(M_T - \widetilde{M}_T\big).$$

If $\frac{d\langle M\rangle_s}{ds}$ and $\frac{d\langle \widetilde{M}\rangle_s}{ds}$ are bounded above, by 1 say, the previous identity provides a control of the increment $f(x_T)-f(x_0)$ by the supremum of the absolute value of a Brownian motion over the time interval $[0,T]$.

Back to the non-explosion problem for Brownian motion on $\mathcal{M}$, fix a point $m\in\mathcal{M}$ and a radius $r>1$, and consider Brownian motion $(x_s)_{s\geqslant 0}$ reflected on the boundary of the Riemannian ball $B(m;r)$, started under its invariant measure $\mathbf{1}_{B(m;r)}\mathrm{Vol}$. It is a symmetric conservative diffusion, with law denoted by $\overline{\mathbb{P}}_{B(m;R)}$. Using the Dirichlet forms approach to symmetric diffusions one can apply the above reasonning to the non-smooth, but 1-Lipschitz, Riemannian distance function $f(\bullet) := d(m,\bullet)$, and obtain the upper bound

$$\overline{\mathbb{P}}_{B(m;r)}\left(x_0\in B(m;1),\ \sup_{s\leqslant T} d(m,x_s) = r\right) \lesssim \mathrm{Vol}\big(B(m;r)\big)\,\mathbb{P}\left(\sup_{s\leqslant T}|B_s|>r\right).$$

But as the Brownian motion on $\mathcal{M}$ behaves in the ball $B(m;r)$ as the Brownian motion reflected on the boundary of $B(m;r)$, the above inequality also gives an upper bound for the probability that Brownian motion on $\mathcal{M}$, started uniformly from $B(m;1)$, exits the ball $B(m;r)$ before time T. Combining this estimate with Borel-Cantelli lemma, Takeda proved that Brownian motion on $\mathcal{M}$ is conservative provided

$$\liminf_{r\to\infty} r^{-2}\log \mathrm{Vol}\big(B(m;r)\big) < \infty\,,$$

reproving in a simple way a criterion due to Karp and Li.

Takeda's method has been refined by several authors, culminating with Hsu and Qin's recent work **[16]**, who give an elegant and simple proof of a sharp non-explosion criterion, due to Grigor'yan **[17]**, for Brownian motion on a Riemannian manifold in terms of volume growth of geodesic balls.

Endow $O\mathcal{M}$ with the sub-Riemannian structure induced by the free family of vector fields $H_0, V_1, \dots, V_d$. Given that the Lie algebra generated by these vectors span the whole tangent space to $O\mathcal{M}$ everywhere, the sub-Riemannian distance function d_{SR} is finite. Given any fixed reference point $z_{\rm ref}$, the following assumption does not depend on the reference point.

(H) Completeness hypothesis. *The closed sub-Riemannian boxes* $B_r := \big\{d_{SR}(z_{ref},\cdot) \leqslant r\big\}$ *in* $O\mathcal{M}$ *are compact for any radius* $r > 0$.

Theorem 12 ([P9]). Let $(\mathcal{M}, g)$ be a strongly causal Lorentzian manifold satisfying the completeness hypothesis **(H)**. Denote by $\textsc{Vol}$ the natural volume form on $O\mathcal{M}$, and set

$$\Theta_r := \sup_{z\in B_r} \Theta(z)$$

for any $r > 0$. If

$$\int^{\infty} \frac{r\,dr}{\Theta_r\,\log\big(\Theta_r\,\textsc{Vol}(B_r)\big)} = \infty\,, \tag{1.2.3}$$

then the Θ-diffusion has almost surely an infinite lifetime, from any starting point.

Condition (1.2.3) has the form of the classical non-explosion condition for Brownian motion: $\displaystyle\int^{\infty} \frac{r\,dr}{\log \textsc{Vol}(B_r)} = \infty$, first proved by Grigor'yan in [**17**]; it has precisely that form for Θ bounded. Note that no topological assumption on $\mathcal{M}$ is needed, contrary to the results stated in proposition 11. One can give a quantitative version of the above theorem by providing an upper rate function.

Corollary 13 ([P9]). Let $\mathcal{M}$ be a strongly causal Lorentzian manifold satisfying the completeness hypothesis **(H)**. Set $h(\rho) \equiv \rho$ if $\Theta \equiv 0$; otherwise, pick a constant R_0 such that $\Theta_{R_0} > 0$ and set for $\rho > 0$

$$h(\rho) := \inf\left\{R > R_0 \,\Big|\, \int_{R_0}^{R} \frac{r\,dr}{\Theta_r\,\log\big(\Theta_r\,\textsc{Vol}(B_r)\big)} > \rho\right\}.$$

Then, given any $z_0 \in O\mathcal{M}$, there exist $R_0 > 0$ and a positive constant c such that we have $\mathbb{P}_{z_0}$-almost surely

$$d_{SR}(z_0, z_s) \leqslant C\,h(c\,s).$$

The main difficulty in implementing Takeda's approach is in finding what can play the role of the pair "*Riemannian distance function – reflected Brownian motion*" in our Lorentzian, hypoelliptic framework. The sub-Riemannian distance function d_{SR} introduced above presents itself for playing the role of the distance function. Unfortunately, unlike its Riemannian analogue, d_{SR} is not smooth in any neighbourhood of $z_{\rm ref}$; however, it is a viscosity solution of the equation

$$|H_0\mathcal{D}|^2 + |V_1\mathcal{D}|^2 + \cdots + |V_d\mathcal{D}|^2 = 1$$

on $\mathbb{O}\mathrm{M}\backslash\{z_{\rm ref}\}$. We used that quantitative information under the classical form given in the following proposition, which is useful to bound above the equivalent in our setting of the bracket terms $\frac{d\langle M\rangle_s}{ds}$ and $\frac{d\langle \widetilde{M}\rangle_s}{ds}$ in Takeda's method.

Proposition 14 ([P9]). Fix $\lambda > 0$. One can associate to any positive constant η a smooth function $f : O\mathcal{M} \to \mathbb{R}_+$ such that

$$\max_{z\in B_\lambda} \big|f(z) - d_{SR}(z_{\mathrm{ref}}, z)\big| \leqslant \eta$$

and we have on B_λ

$$|H_0 f|^2 + |V_1 f|^2 + \cdots + |V_d f|^2 \leqslant 2.$$

Proof – Let us introduce the Riemannian metric g_ϵ on $O\mathcal{M}$ for which $H_0, H_1, \dots, H_d$ and the $(V_{ij})_{0\leqslant i<j\leqslant d}$ are *orthogonal*, with H_0 and the $V_{0j}(= V_j)$ of norm 1 and the other vectors of norm ϵ^{-1}. With a slight abuse of notations, denote by $d_\epsilon(\bullet) = d_\epsilon(z_{\mathrm{ref}}, \cdot)$ the distance function associated with g_ϵ. It is a 1-Lipschitz-continuous function, with respect to the distance function d_ϵ, which is differentiable almost-everywhere, by Rademacher's theorem, and has a gradient of norm 1 almost-everywhere

$$\big|H_0 d_\epsilon\big|^2 + \big|V_1 d_\epsilon\big|^2 + \cdots + \big|V_d d_\epsilon\big|^2 + \epsilon^{-2}\Bigg(\sum_{i=1}^d \big|H_i d_\epsilon\big|^2 + \sum_{1\leqslant i<j\leqslant d} \big|V_{ij} d_\epsilon\big|^2\Bigg) = 1. \tag{1.2.4}$$

(Indeed, the set of conjugate points to z_0 in B_λ is closed and has null measure. In the complementary, relatively open, set the distance is attained along a unique geodesic whose unit tangent vector at the final point is the gradient of the distance function to z_0.) The function d_ϵ is easily seen to converge uniformly to the function $d_{SR}(z_{\mathrm{ref}}, \cdot)$ on the compact box B_λ, and this is where we need these boxes to be compact. As we have almost-everywhere

$$\big|H_0 d_\epsilon\big|^2 + \big|V_1 d_\epsilon\big|^2 + \cdots + \big|V_d d_\epsilon\big|^2 \leqslant 1,$$

by (1.2.4), a standard regularization procedure yields the conclusion. ▷

With this function in hands, it remained to find a candidate for playing the role of reflected Brownian motion. This was done by introducing a non-standard reflection mechanism, which we describe here for a Brownian in a Riemannian manifold, so as to avoid technical details.

Brownian motion reflected on the boundary of a ball $B(m;r)$ is the simplest diffusion process which coincides with Brownian motion on the ball $B(m;r)$ and has a state space with finite volume. One cannot take a smaller state space if the former property is to be satisfied. Yet, one can make different choices if one is ready to loose the minimality property. To explain that fact, let us suppose that $(\mathcal{M}, g)$ is a Cartan-Hadamard manifold. Given a point $m \in \mathcal{M}$, use the exponential map $\exp_m$ at m as a global chart on $\mathcal{M}$; this identifies the geodesic ball $B(m;r)$ on $\mathcal{M}$ to the Euclidean-shaped ball $B'(0;r)$ in $T_m\mathcal{M}$. Given $\epsilon > 0$, let us modify the metric on $B'(0;r+\epsilon)\backslash B'(0;r)$ so as to interpolate smoothly between $\exp_m^* g$ on $B'(0;r)$ and the constant metric g_m outside $B'(0;r+\epsilon)$ – primed balls refer to the pull-back metric $\exp_m^* g$. Denote by $\widetilde{g}$ the restriction to $B'(0;r+2\epsilon)$ of this modified metric, and define the compact space $\mathbf{K}$ as the quotient of the closed ball $\overline{B}'(0;R+2\epsilon)$ by the identification of $m' \in \partial\overline{B}'(0;r+2\epsilon)$ and $-m'$. Then the $\widetilde{g}$-Brownian motion on $\mathbf{K}$ coincides with the $\exp_m^* g$-Brownian motion on $B'(0;r)$ and has a state space with finite $\widetilde{g}$-volume $\mathrm{Vol}_{\widetilde{g}}(\mathbf{K}) = (1+o(\epsilon))\,\mathrm{Vol}_g\big(B(m;r)\big)$.

1.2.4. What next? Besides the open question about the Poisson and Martin boudaries of relativistic diffusions, the question that seems to me most important in this field is to construct some random dynamics whose law is invariant by any conformal change of the metric. The distribution of Dudley's diffusion is indeed not conformally invariant in that sense; neither is the notion of geodesic itself, except that of null-geodesic, up to reparametrization. There is however a notion of conformal circles/geodesics that is a conformal invariant, and whose dynamics generates $\mathcal{C}^3$ paths on the base manifold $\mathcal{M}$ that solve third order ordinary differential equations [**19, 20**]. From a higher level point of view, these paths are projections of paths in a bundle over $\mathcal{M}$ whose fiber at each point corresponds to a choice of tangent vector and a connection at that point of $T\mathcal{M}$. It is actually possible to introduce random dynamics in this bundle that is somehow expected to give birth to a conformally invariant class of random processes – tricky things are expected to happen with parametrization of paths. Would this picture happen be consistent, it would be extremelly interesting to relate the properties of this random process to the conformal properties of a given spacetime.

A Riemannian analogue of relativistic diffusion will be described in the next chapter. If the above picture about conformally invariant random dynamics is to make sense, its Riemannian analogue will also make sense. The interesting point here lies in dimension 2, where the metric conformal invariance notion is related to the holomorphic conformal invariance notion. In this setting, SLE curves were proved to have some universal properties amongst conformally invariant random processes. It would be deadly interesting to see what happens of the scaling limit of the potentially conformally invariant $\mathcal{C}^3$ random paths described above.

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Chapter 2

Subelliptic and hypoelliptic diffusions on manifolds

This chapter provides an introduction to two works **[P20,22]** on different classes of hypoelliptic and subelliptic diffusion processes. The first work provides a qualitative study of a family of Riemannian analogues of Dudley's diffusion, indexed by one parameter, and which was proved to interpolate between geodesic and Brownian flows. We were also able to determine their Poisson boundary, for a fixed value of the parameter, in the model setting of rotationally invariant manifolds. The question tackled in **[P21]** was to give a second order description of the bridges of subelliptic diffusions, when the travelling time tends to 0, under generic conditions on the end-points.

2.1 Kinetic Brownian motion

Together with J. Angst and C. Tardif, we introduced in the work **[P20]** the Riemannian counterpart of relativistic diffusions, under the form of a one parameter family of diffusion processes that model physical phenomena with a finite speed of propagation, collectively called **kinetic Brownian motion**. In the Euclidean space $\mathbb{R}^d$, kinetic Brownian motion with parameter σ, is simply described as a C^1 random path $(x_t)_{t\geqslant 0}$ with Brownian velocity on the unit sphere, run at speed σ^2, so

$$\frac{dx_t}{dt} = \dot{x}_t, \qquad \dot{x}_t = W_{\sigma^2 t}, \tag{2.1.1}$$

for some Brownian motion W on the unit sphere of $\mathbb{R}^d$. In contrast with Langevin process, whose $\mathbb{R}^d$-valued part can go arbitrarily far in an arbitrarily small amount of time, kinetic Brownian motion provides a bona fide model of random process with finite speed. Its definition on a Riemannian manifold follows the intuition provided by its $\mathbb{R}^d$ version, and can be obtained by rolling on $\mathcal{M}$ without slipping its Euclidean counterpart. Figure 2.1 illustrates the dynamics of kinetic Brownian motion on the torus, as time goes on.

We devoted most of our efforts in this work in relating the large noise and large time behaviour of the process to the geometry of the manifold. On the one hand, we showed that kinetic Brownian motion interpolates between geodesic and Brownian motions, as

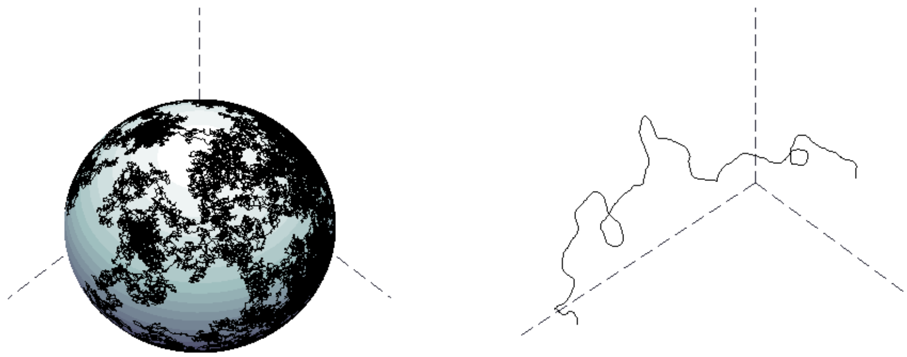

FIGURE 1. Kinetic Brownian motion in Euclidean space, velocity on the sphere on the left and the corresponding position on the right.

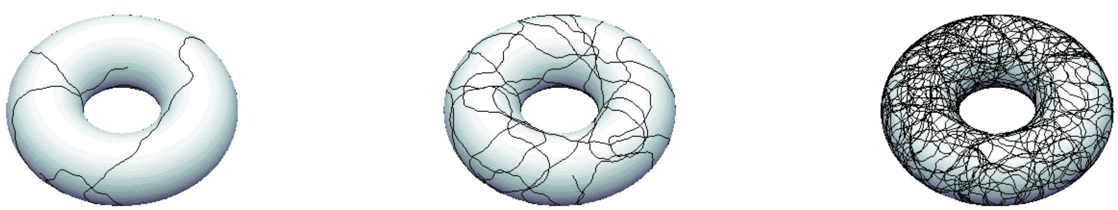

FIGURE 2. Simulations of kinetic Brownian motion on the torus over different time intervals.

σ ranges from 0 to ∞, leading to a kind of homogenization. Our use of rough paths theory for proving that fact may be of independent interest; it is described in section 3 of chapter 4. On the other hand, we were able to give a complete description of the Poisson boundary of kinetic Brownian motion when the underlying Riemannian manifold is sufficiently symmetric and σ is fixed. This is far from obvious as kinetic Brownian motion is a hypoelliptic diffusion which is non-subelliptic. We took advantage in this task of the powerful *dévissage method* that was introduced recently in [**1**] as a tool for the analysis of the Poisson boundaries of Markov processes on manifolds. Its typical range of application involves a diffusion $(z_t)_{t\geqslant 0}$ that admits a subdiffusion $(x_t)_{t\geqslant 0}$ whose Poisson boundary is known. If the remaining piece y_t of $z_t = (x_t, y_t)$ converges to some random variable y_∞, the dévissage method provides conditions that garantee that the invariant sigma field of $(z_t)_{t\geqslant 0}$ will be generated by y_∞ together with the invariant sigma field of $(x_t)_{t\geqslant 0}$. In the present situation, and somewhat like Brownian motion on model spaces, the Poisson boundary of kinetic Brownian motion is described by the asymptotic direction in which the process goes to infinity.

2.1.1. Definition and asymptotic behaviour We use in this section the same **notations** as in Chapter 1, with $(\mathcal{M}, g)$ standing here for a d-dimensional Riemannian manifold with orthonormal frame bundle $O\mathcal{M}$. The latter is equipped with the horizontal vector fields $H_1, \dots, H_d$, associated with the Levi-Civitta connection.

Definition. Given $z_0 \in O\mathcal{M}$, the **kinetic Brownian motion** with parameter σ, started from z_0, is the solution to the $O\mathcal{M}$-valued stochastic differential equation in Stratonovich form

$$dz_t = H_1(z_t)\,dt + \sigma V_i(z_t) \circ dB^i_t, \tag{2.1.2}$$

started from z_0. It is defined a priori up to its explosion time and has generator

$$\mathcal{L}_\sigma := H_1 + \frac{\sigma^2}{2} \sum_{j=2}^{d} V_j^2.$$

It is elementary to see that its canonical projection on $T^1\mathcal{M}$ is a diffusion on its own, also called *kinetic Brownian motion*. In the coordinate system $(x^i, \dot{x}^i) := (x^i, e^i_1)$ on $T^1\mathcal{M}$ induced by a local coordinate system on $\mathcal{M}$, kinetic Brownian motion satisfies the following stochastic differential equation in Itô form

$$\begin{cases} dx^i_t &= \dot{x}^i_t\,dt, \\ d\dot{x}^i_t &= -\Gamma^i_{jk}\,\dot{x}^j_t\dot{x}^k_t\,dt + \sigma dM^i_t - \dfrac{\sigma^2}{2}(d-1)\dot{x}^i_t\,dt, \end{cases} \tag{2.1.3}$$

where $1 \leqslant i \leqslant d$, and where M_t is an $\mathbb{R}^d$-valued local martingale with bracket

$$d\langle M^i, M^j\rangle_t = \Big(g^{ij}(x_t) - \dot{x}^i_t\dot{x}^j_t\Big)dt,$$

for any $1 \leqslant i, j \leqslant d$ and g^{ij} stand for the inverse of the matrix of the metric in the coordinates used here. It is not difficult to see that kinetic Brownian motion has almost surely an infinite lifetime if the manifold $(\mathcal{M}, g)$ is geodesically complete, which we shall assume in the sequel.

Our aim in the work **[P20]** was to try and understand the long time asymptotic behaviour of kinetic Brownian motion when t goes to infinity and σ is fixed. As for classical Brownian motion on a general Riemannian manifold, there is no hope to fully determine the asymptotic behaviour of kinetic Brownian motion on arbitrary Riemannian manifolds, as even for Brownian motion it is likely to depend on the base space in a very sensitive way; see for instance the work **[2]** of Arnaudon, Thalmaier and Ulsamer for the study of the asymptotic behaviour of Brownian motion on Cartan-Hadamard manifolds. Of crucial importance in the latter study is the fact that the distance to a fixed point defines a one-dimensional subdiffusion. The difficulties are actually a priori greater here as kinetic Brownian motion does not live on the base manifold, but on its unit tangent bundle, so that it is basically $(2d-1)-$dimensional when the base manifold have dimension d, and there is no general reason why it should have lower-dimensional subdiffusions.

The question of the long time asymptotic behaviour of a manifold-valued diffusion process is of course of different nature depending on whether or not the underlying manifold is compact. If it is compact, the question consists mainly in studying the trend to equilibrium of the process, and in relating the rate of convergence to the geometry of the manifold or to the parameters of the process. In our case, the infinitesimal generator of

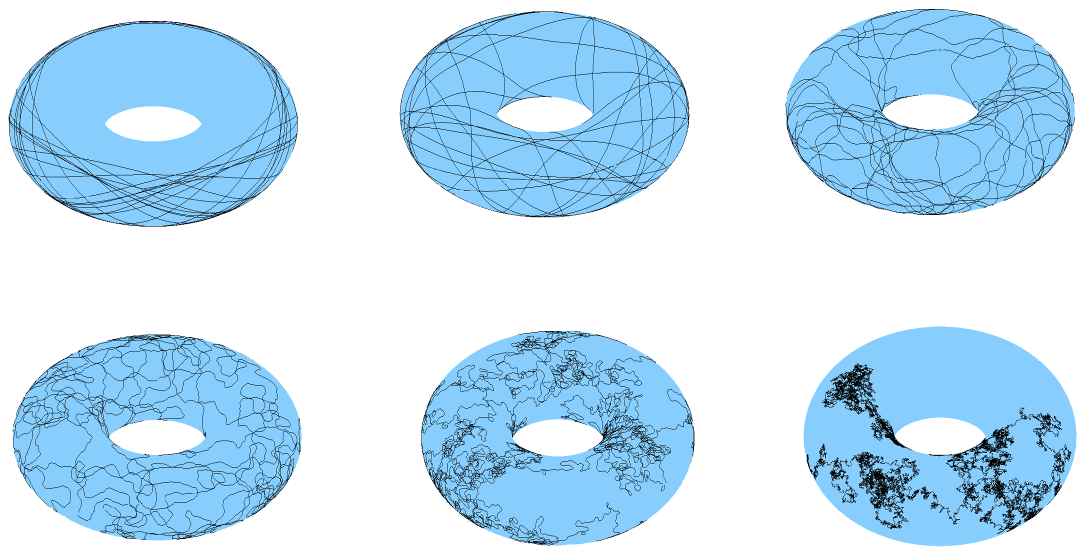

Figure 3. Kinetic Brownian motion on the flat torus for $\sigma = 10^{-2}, 10^{-1}, 1, 2, 4, 10$.

kinetic Brownian motion is hypoelliptic and $T^1\mathcal{M}$ is compact, if $\mathcal{M}$ is compact, and it is elementary to see that for all times t strictly greater than the diameter of $\mathcal{M}$, the density $p_t(\cdot,\cdot)$ of the process is uniformly bounded below by a positive time-dependent constant. This ensures that the law of the process converges exponentially fast to the equilibrium measure, which is the Liouville measure on the unitary tangent bundle here. The question of determining the exact rate of convergence to equilibrium, in terms of the geometry of $\mathcal{M}$ and the parameter σ is difficult and will be the object of another work by the authors.

If the base manifold is non-compact, the question of the long time asymptotics of a diffusion process amounts to find whether it is recurrent or transient, to exhibit some geometric asymptotic random variables associated to the sample paths, and, in the nicest situations, to determine the Poisson boundary of the process. In order to get some tractable and significant information on the asymptotic behaviour of kinetic Brownian motion, we restricted ourselves here to study the case of a rotationally invariant manifold $\mathcal{M}$. As in the case of classical Brownian motion, symmetries simplify greatly the study as they allow to exhibit some lower dimensional subdiffusions of the initial process. The class of rotationally invariant manifolds is nevertheless rich enough to have some good idea of the interplay between geometry, through curvature of the manifold, and the asymptotic behaviour of the process. The main information on curvature is contained in the sectional curvature of 2-planes containing the vector $g^\flat(dr)$; these curvatures are all equal to the quantity

$$K(r) := -\frac{f''(r)}{f(r)},$$

called the **radial curvature**. The importance of the convexity properties of the warping function f are clear on this formula. The following theorem provides a rather complete description of the long time asymptotics of kinetic Brownian motion on rotationally invariant manifolds.

Theorem 15 ([P20]). Let $(z_t)_{t\geqslant 0} := (x_t, \dot{x}_t)_{t\geqslant 0}$ be kinetic Brownian motion with values in the unit tangent bundle $T^1\mathcal{M}$ of a rotationally invariant manifold $(\mathcal{M}, g)$ parametrized by

$$\left((0,+\infty)\times \mathbb{S}^{d-1}, dr^2 + f^2(r)d\theta^2\right),$$

and let (r_t, θ_t) stand for the polar coordinates of x_t.

a. (1) If $K \leqslant 0$, then the process r_t is transient almost surely.
(2) If $K \geqslant 0$ the process r_t is transient almost surely if and only if

$$\int_1^{+\infty} f^{1-d}(r)dr < +\infty.$$

b. Suppose that the warping function f is $\log-$concave and satisfies

$$\int_1^{+\infty} f^{d-2}(r)\left(\int_r^{+\infty} f^{1-d}(\rho)d\rho\right)dr < +\infty.$$

Then, the angular process $(\theta_t)_{t\geqslant 0}$ converges almost surely to a random point θ_∞ on $\mathbb{S}^{d-1}$.

c. Under the assumptions of point **b**, the Poisson boundary of the whole process $(z_t)_{t\geqslant 0}$ coincides almost surely with $\sigma(\theta_\infty)$.

The first two points of theorem 15 are proved by a careful use of the equations of dynamics (2.1.3), written in polar coordinates, and using some ad hoc tricky arguments. The proof of point **c** about the Poisson boundary of kinetic Brownian motion rests on the powerful **dévissage method** introduced by J. Angst and C. Tardif in [**1**] as a tool for determining the Poisson boundary of Markov processes. Its typical range of application involves a two component diffusion (x_t, g_t), whose first component is a diffusion on its own, and whose second component converges. The dévissage theorem of Angst and Tardif gives conditions under which one can conclude that the invariant σ-algebra of $(x_\bullet, g_\bullet)$ is generated by the invariant σ-algebra of $x_\bullet$ and $\sigma(g_\infty)$. One can then use the same kind of reasoning to give an explicit description of the invariant σ-algebra of $x_\bullet$, and give a full description of the invariant σ-algebra of $(x_\bullet, g_\bullet)$ in a step by step procedure. Note that no kind of ellipticity condition is required at all to get the conclusion!

Intermezzo : Dévissage method

To be more specific, the dévissage method is fully described in the following statement.

Theorem 16 ([1]). Let $\mathcal{N}$ be a differentiable manifold and G be a finite dimensional connected Lie group. Let $(x_t, g_t)_{t\geqslant 0}$ be a diffusion process with values in $\mathcal{N}\times G$, starting from $(x,g)\in\mathcal{N}\times G$. Denote by $\mathbb{P}_{(x,g)}$ its law. Assume the following conditions are satisfied.

(1) **Dévissage condition.** The process $(x_t)_{t\geqslant 0}$ is itself a diffusion process with values in $\mathcal{N}$. Its own invariant sigma field $\mathrm{Inv}((x_t)_{t\geqslant 0})$ is either trivial or generated by a random variable ℓ_∞ with values in a separable measure space $(S, \mathcal{G}, \lambda)$ and the law of ℓ_∞ is absolutely continuous with respect to λ.

(2) **Convergence condition.** The second projection $(g_t)_{t\geqslant 0}$ converges almost surely to a random element g_∞ of G.
(3) **Equivariance condition.** The infinitesimal generator $\mathcal{L}$ of the diffusion $(x_t, g_t)_{t\geqslant 0}$ is equivariant under the action of G on $C^\infty(\mathcal{N} \times G, \mathbb{R})$, i.e. $\forall f \in C^\infty(\mathcal{N} \times G, \mathbb{R})$, we have for $(x, g, h) \in \mathcal{N} \times G \times G$:
$$\mathcal{L}(f(\cdot, g\cdot))(x, h) = (\mathcal{L}f)(x, gh).$$
(4) **Regularity condition.** All bounded $\mathcal{L}-$harmonic functions are continuous on the state space $\mathcal{N} \times G$.

Then, the σ-algebras $\mathrm{Inv}((x_t, g_t)_{t\geqslant 0})$ and $\mathrm{Inv}((x_t)_{t\geqslant 0}) \vee \sigma(g_\infty)$ coincide up to $\mathbb{P}_{(x,g)}-$negligeable sets.

This result can be generalized to the case where the second projection does not take values in a finite dimensional Lie group G but rather in a finite dimensional connected co-compact homogeneous space $Y := G/K$; see Theorem B of [**1**]. For practical use, this theorem requires as a first step to find good coordinates on (part of) the manifold under which the coordinates have the required behaviour. As a typical example, consider Brownian motion on a rotationally invariant Riemannian manifold isometric (up to removing a point) to $\mathbb{R}_*^+ \times \mathbb{S}^{d-1}$, equipped with the metric
$$g = dr^2 + f^2(r)d\theta^2,$$
where $d\theta^2$ stands for the usual metric on the sphere of constant curvature equal to 1. Explicit integrability conditions on f can be given [**3**], under which Brownian motion is transient, defined for all times, and its angular part θ_t converges almost surely to some limit random direction θ_∞. The radial part r_t of Brownian motion is a diffusion on its own, and an elementary shift coupling argument shows that its invariant σ-algebra is trivial, while the other conditions of theorem 16 hold for elementary reasons. We can thus apply the homogeneous version of the dévissage theorem, with $G = SO(d)$, and $K = SO(d-1)$, and conclude that the invariant σ-algebra of Brownian motion on this manifold is generated by the limit angle θ_∞; this is in particular the case of the hyperbolic space. The dévissage method was used in [**1**] to get back in a simple, short and elegant way the results on the Poisson boundary of Dudley's diffusion in Minkowski spacetime, as exposed in section 1.1.1, theorem 1. They were also used to find some explicit, geometric, description of the Poisson boundary of Dudley's diffusion in de Sitter, anti-de Sitter [**4**] and Robertson-Walker spacetimes [**5**].

Some non-trivial arguments were needed to implement the dévissage method in the setting of kinetic Brownian on a rotationally invariant manifold, due in particular to the 2-dimensional character of the radial subdiffusion and the fact that we had to lift first kinetic Brownian motion to a diffusion on $\mathbb{R}_*^+ \times [-1, 1] \times SO(d) \times SO(d-1)$, to which we could apply the dévissage method, as an intermediate step to describe the Poisson boundary of kinetic Brownian motion itself. We refer the reader to the original article **[P20]** for more details.

2.1.2. What next? The results presented in this section and section 3.2 about kinetic Brownian motion are just the beginning of a story. One of the basic questions left open in the work **[P20]** is that of the convergence to equilibrium in the case where the base manifold is compact. The hypoelliptic nature of the diffusion together with a basic coupling argument ensure that the convergence of the distribution of the process to Liouville measure on $T^1\mathcal{M}$ at an exponential rate. The question of determining an accurate rate of convergence to equilibrium, in terms of the geometry of $\mathcal{M}$ and the parameter σ is challenging. The semigroup method of Baudoin [**6**] can be used for that purpose, but it gives rates that vanish as the diffusivity parameter σ tends to 0. One knows on the other side that the geodesic flow converges at an exponential rate under some mild curvature assumptions, due to is contact preserving and Anosov properties [**7**]. Kinetic Brownian motion also happen to enjoy these properties, and I am currently investigating with S. Gouezel whether the analytic methods of Liverani can be used to prove some almost sure convergence result for the flow of kinetic Brownian motion in $T^1\mathcal{M}$, providing an unusual type of conclusion in this kind of stochastic problem.

In a different direction, I have started with J. Angst to investigate an infinite dimensional version of kinetic Brownian motion. This project is to be understood in the line of the works by Malliavin and co-workers on Brownian motion on the set of homeomorphisms of the circle introduced by Malliavin in [**8**]. Given the interpolation between geodesic flow and Brownian motion provided by kinetic Brownian motion in a finite dimensional setting, this point will be explained in section 3.2, it is very tempting to construct Malliavin Brownian motion from a similar point of view, as a limit of a time-rescaled kinetic Brownian motion, when the diffusivity parameter σ tends to ∞. On the other hand, the geodesic equation on the group of diffeomorphisms are nonlinear partial differential equations, depending on the metric considered; its may be the inviscid Burgers or the KdV equations. One thus expects kinetic Brownian motion to be given by the solution of a singular PDE. The paraproduct method developed in **[P20]**, or Hairer's regularity structures, may be used to investigate the well-posedness of these equations and the properties of their solutions.

2.2

Bridges of subelliptic diffusions

Together with J. Norris and L. Mesnager, I investigated in **[P21]** the small time fluctuations of bridges of sub-Riemannian diffusions on a manifold. The study of such questions in a Riemannian setting goes back to the wonderful work [**9**] of Molchanov, where he gave, amongst so many other things, a precise description of Brownian bridges as their traveling time goes to 0. On a complete and stochastically complete Riemannian manifold, if two points x and y are joined by finitely many geodesics along which they are non-conjugate, then the distribution of the Brownian bridge between x and y, in the time interval $[0,\varepsilon]$, converges weakly to a random Dirac mass on one of these geodesic. This gives a first order deterministic asymptotics. The second order asymptotics, given by the rescalled fluctuations around the deterministic limit, is a Gaussian process whose covariance involves the index form given by the second variation of the energy functional. While Molchanov only proved weak convergence of the finite dimensional marginals of the fluctuation process, the results proved in **[P21]** are much stronger and go far beyond the Riemannian setting. Heat kernel estimates are also proved along the way in a more general framework than was available up to now; we do not describe them here.

2.2.1. Bridges of degenerate diffusions Let $\mathcal{M}$ be a connected oriented smooth manifold of dimension d, and $\mathcal{L}$ be a smooth second order differential operator on $\mathcal{M}$. Assume that $\mathcal{L}1 = 0$ and that the principal symbol $\sigma : T^*\mathcal{M} \to T\mathcal{M}$ of L, is non-negative definite. Assume, for simplicity here, that there exists smooth vector fields $V_1, \dots, V_\ell$, such that

$$\sigma = \sum_{i=1}^{\ell} V_i \otimes V_i.$$

Absolutely continuous M-valued ***paths*** $(\gamma_t)_{0\leqslant t\leqslant 1}$ with $\dot\gamma_t \in \sigma\big(T^*_{\gamma_t}M\big)$, almost-everywhere, are said to be ***horizontal***, and we assume that *given any two points of M and any connected open set U containing them, there exists an U-valued horizontal path connecting them.* The semi-group $(P_t)_{t\geqslant 0}$ associated with $\mathcal{L}$ has in that case, by Hörmander's theorem, a smooth positive fundamental solution $p : \mathbb{R}_+^* \times M \times M \to \mathbb{R}_+^*$ with respect to any smooth measure Vol on M. Given x and y in $\mathcal{M}$, denote by $\mathbb{P}_\varepsilon^{x,y}$ the law of the diffusion process associated with ϵL, conditionned on having initial position x and position y at time 1, or equivalently, the law of the diffusion process associated with L, conditionned on having initial position x and position y at time ϵ.

The possible non-constant rank vector sub-bundle $\sigma(T^*\mathcal{M})$ of $T\mathcal{M}$ is endowed with a positive-definite scalar product defined without ambiguity by the formula $(q,q')_{\mathcal{L}} := p\big(\sigma(p')\big)$, for any $m \in \mathcal{M}$, $q,q' \in T_m\mathcal{M}$ and any $p,p' \in T_m^*\mathcal{M}$ such that $\sigma(p) = q$ and $\sigma(p') = q'$. Denote by $\mathfrak{H}^1$ the set of horizontal paths $(\gamma_t)_{0\leqslant t\leqslant 1}$ with finite energy $I(\gamma) := \int_0^1 (\dot\gamma_s, \dot\gamma_s)_{\mathcal{L}}\, ds$. We will say that a horizontal path γ from x to y is strongly minimal if there exist $\delta > 0$ and a relatively compact open set U in $\mathcal{M}$ such that $I(\omega) \geqslant I(\gamma)$, for all horizontal path ω from x to y, and $I(\omega) \geqslant I(\gamma) + \delta$, for all ω that leave U. The first order asymptotics of $\mathbb{P}_\epsilon^{x,y}$ is described as follows; we refer to **[P21]** for the definition of the 'sector condition' mentioned in the statement.

Theorem 17 ([P21] First order asymptotics). Let $\mathcal{M}$ be a connected smooth manifold. Let L be a second order differential operator on M of the form

$$Lf = \mathrm{div}(\sigma\nabla f) + \sigma(\beta, \nabla f)$$

where the diffusivity σ has a sub-Riemannian structure, where the divergence is taken with respect to a positive smooth locally invariant measure, and where β is a smooth 1-form satisfying some sector condition. If there is a unique strongly minimal path γ from x to y then the bridge measure $\mathbb{P}^{\epsilon}_{x,y}$ converges weakly to the Dirac mass on γ.

Theorem 18 below describes the second order asymptotics, and shows that the distribution of the centered and rescaled canonical process $\epsilon^{-1/2}\big(X_\bullet - \gamma^{x,y}_\bullet\big)$ converges weakly under $\mathbb{P}^{x,y}_\epsilon$ to a Gaussian process whose covariance is determined by the principal symbol σ of L, if x and y are joined by a unique minimizing path, say γ, and lie outside the sub-Riemannian cut locus. To achieve that description requires a deep investigation of the restriction of the energy functional I to the set $H^{x,y}$ of paths from x to y with finite energy, near the point γ where it is minimum. We constructed, under a regularity condition on γ, a vector space $T_\gamma H^{x,y}$ of absolutely continuous paths v in $T\mathcal{M}$, with $v_t \in T_{\gamma_t}\mathcal{M}$ and $v_0 = v_1 = 0$, along with an equivalence class of norms on $T_\gamma H^{x,y}$, each turning it into a Hilbert space. We further constuct a continuous non-negative quadratic form Q on $T_\gamma H^{x,y}$ such that $Q(v)$ is the minimal second variation of I in the directon v, in a precise sense. We showed that (x, y) lies outside the sub-Riemannian cut-locus if and only if our regularity condition on γ holds and Q is positive definite, giving as a consequence an intrinsic path-based characterization of the cut-locus in our general setting. One can then use Q to give $T_\gamma H^{x,y}$ an intrinsic Hilbert norm. Denote by $T_\gamma\Omega^{x,y}$ the closure of $T_\gamma H^{x,y}$ in uniform norm, this is a Banach space. We showed that there exists a unique zero-mean Gaussian measure $\mathbb{P}^{x,y}_\gamma$ on $T_\gamma\Omega^{x,y}$ with covariance determined by Q is some precise way. Theorem 18 show that the fluctuations of the canonical process converge, around γ, suitably rescalled, converge weakly in $T_\gamma\Omega^{x,y}$ to $\mathbb{P}^{x,y}_\gamma$.

2.2.2. Sub-Riemannian cut-locus Our sub-Riemannian setting is non-canonical, as the bundle $\sigma(T^*\mathcal{M})$ is not required to have constant rank. We recall here what the cut-locus is in that setting, under the assumption that the two end-points x and y are joined by a unique minimizing path that is the projection of a bicharacteristic associated with the Hamiltonian

$$\mathcal{H}(m, p) = \frac{1}{2}\big(\sigma_m(p)\big).$$

We define a smooth map from $H^1\big([0,1], \mathbb{R}^\ell\big)$ to $\mathcal{M}$ by sending a control g to the time 1 value γ^g_1 of the solution to the controlled ordinary differential equation

$$\dot\gamma^g_s = V_i\big(\gamma^g_s\big)\,\dot g^i_s,$$

with $\gamma^g_0 = x$, for $0 \leqslant s \leqslant 1$. Denote by h the unique control such that $\gamma^h = \gamma$, with minimal H^1-norm. One can show that under the above assumption on γ, the subspace $\mathrm{Im}\big(D_h\gamma_1\big)$ of $T_y\mathcal{M}$, depends only on the principal symbol a of L.

Assumption A. *We have* $\frac{\partial\gamma^g_1}{\partial g}\Big|_{g=h}(H^1_0) = T_yM$.

Denote by $p_0 \in T_x^*M$ the initial momentum of the bicharacteristic above γ, and by $p_1 \in T_y^*M$ its final momentum. For $0 \leqslant t \leqslant 1$, define linear maps

$$J_t : T_x^*M \to T_{\gamma_t^{x,y}}M, \qquad \text{and} \qquad K_{1-t} : T_y^*M \to T_{\gamma_t^{x,y}}M$$

setting for $p \in T_x^*M$

$$J_t(p) = \frac{d}{da}\Big|_{a=0} \big(\pi \circ \psi_t\big)(x, p_0 + ap),$$

and for $p' \in T_y^*M$

$$K_{1-t}(p') = \frac{d}{da}\Big|_{a=0} \big(\pi \circ \psi_{-(1-t)}\big)(y, p_1 + ap').$$

In a Riemannian setting, the vector fields $\big(J_t(p)\big)_{0\leqslant t\leqslant 1}$ and $\big(K_{1-t}(p')\big)_{0\leqslant t\leqslant 1}$ along γ are Jacobi fields. We say the x and y are **non-conjugate along** $\gamma^{x,y}$ if J_1 is invertible.

Assumption B. *The points x and y are non-conjugate along γ.*

2.2.3. Fluctuations of bridges Let $\mathfrak{f} : U \to \mathbb{R}^d$ be a chart on an open neighbourhood U of the support of γ in M. Extend it into a smooth $\mathbb{R}^d$-valued map defined on the whole of M. The second order asymptotics of $\mathbb{P}_\epsilon^{x,y}$ is given by the limit (if it exists) of the law of the process

$$Z_t^\epsilon = \big(D_{\gamma_t}\mathfrak{f}\big)^{-1}\left(\frac{\mathfrak{f}(X_t) - \mathfrak{f}(\gamma_t^{x,y})}{\sqrt{\epsilon}}\right),$$

for $0 \leqslant t \leqslant 1$, as ϵ goes to 0; each Z_t^ϵ belongs to $T_{\gamma_t^{x,y}}M$, it need not be horizontal. The following statement generalizes to our setting a result proved in a weaker form by Molchanov in [**9**], which is restricted to the Riemannian case.

Theorem 18 (**[P21]** Fluctuations for bridges of degenerate diffusion processes)**.** Assume the points x and y are joined by a unique minimizing path γ that is the projection of a bicharacteristic. Assume also that the non-degeneracy assumption $\mathbf{A}$ and the non-conjugacy assumption $\mathbf{B}$ hold. Denote by $T_\gamma^0 M$ the set of continous sections of $T\mathcal{M}$ over γ, with null ends.

(1) The map

$$0 \leqslant s \leqslant t \leqslant 1 : (s,t) \mapsto J_s J_1^{-1} K_{1-t}^* \in L\big(T_{\gamma_t}^*\mathcal{M}, T_{\gamma_s^{x,y}}\mathcal{M}\big)$$

is the covariance function of a unique zero-mean Gaussian measure $\mathbf{Q}^{x,y}$ on $T_\gamma^0\mathcal{M}$.

(2) The distribution of $Z_\bullet^\epsilon$ under $\mathbb{P}_\epsilon^{x,y}$ converges weakly to $\mathbf{Q}^{x,y}$ as ϵ goes to 0. It is in particular independent of the chart $\mathfrak{f}$ used to define $Z_\bullet^\epsilon$.

Riemannian case. The principal symbol σ of $\mathcal{L}$ defines a Riemannian metric g on $\mathcal{M}$ when it is positive-definite; the operator L is then the sum of the associated Laplace-Beltrami operator and a smooth vector field. Provided x and y are linked by a unique path of minimal energy, assumptions A and B hold automatically in that case, with $\gamma^{x,y} = \gamma$ the geodesic from x to y. One can give in that case a different description of the limit Gaussian measure $\mathbf{Q}^{x,y}$ in terms of the curvature tensor R of the Levi-Civita connection of the metric g. Define $R_t \in \mathrm{L}(T_{\gamma_t}\mathcal{M})$ by $R_t = R(\cdot, \dot\gamma_t)\dot\gamma_t$ and denote by $\mathfrak{I}_\gamma^0$ the set of sections

$q_\bullet$ of TM over γ with null ends, such that $\int_0^1 \mathsf{g}(q_s, q_s)ds < \infty$. Then the second variation of the energy functional I at γ defines a quadratic form on $\mathfrak{I}^0_\gamma$, given by

$$\mathfrak{Q}(q_\bullet) = \int_0^1 \left\{ \left|\nabla_{\gamma_s} q_s\right|^2 - \mathsf{g}(q_s, R_s q_s) \right\} ds.$$

For $0 \leqslant t \leqslant 1$, write τ_t for the parallel translation $T_x\mathcal{M} \to T_{\gamma_t}\mathcal{M}$ along γ, and denote by $\mathbf{Q}$ the law of $\left(\tau_t z_t\right)_{0\leqslant t\leqslant 1}$, where $(z_t)_{0\leqslant t\leqslant 1}$ is a Brownian bridge on $\mathbb{R}^d$ from 0 to 0 in time 1. Theorem 18 takes the following form in that setting; a weaker form of it was proved by Molchanov in [**9**]. Write $T^0_\gamma M$ for the set of continuous sections of $T\mathcal{M}$ above γ, with null ends.

Theorem 19 ([P21]). Assume the points x and y are the end points of a unique geodesic, along which they are non-conjugate. Then $\mathfrak{Q}$ is definite-positive and

(1) the distribution of $Z^\epsilon_\bullet$ under $\mathbb{P}^{x,y}_\epsilon$ converges weakly to a Gaussian measure $\mathbf{Q}^{x,y}$ on $T^0_\gamma\mathcal{M}$ with Cameron-Martin space $\left(\mathfrak{I}^0_\gamma, \mathfrak{Q}\right)$,

(2) $\mathbf{Q}^{x,y}$ is absolutely continuous with respect to $\mathbf{Q}$, with Radon-Nikodym derivative at point $q_\bullet \in T^0_\gamma\mathcal{M}$ proportional to $\exp\left(\int_0^1 \mathsf{g}(q_s, R_s q_s)ds\right)$.

Molchanov proves the weak convergence of the finite dimensional laws of $Z^\epsilon_\bullet$ under $\mathbb{P}^{x,y}_\epsilon$ towards those of a Gaussian process with covariance given by (1). In the case where $\mathcal{L}$ is the Laplace-Beltrami operator on a compact space of constant sectional curvature K, the Radon-Nikodym derivative of $\mathbf{Q}^{x,y}$ with respect to $\mathbf{Q}$ is proportional to

$$\exp\left(\frac{K d(x,y)^2}{2} \int_0^1 |q_t|^2 \, dt\right)$$

at $\mathbf{Q}$-almost every point $q \in \mathfrak{T}^0_\gamma\mathcal{M}$. Thus, on a sphere, the variance of the fluctuations is larger than in $\mathbb{R}^d$, whereas, in hyperbolic space it is less. This does not contradict the tendency of Brownian paths to separate quickly in hyperbolic space because we are conditioning on the endpoint. Thus we tend to see those paths which have never deviated far from the geodesic.

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Part 2

Rough paths theory and beyond

WHILE the first part of this memoir deals with stochastic differential geometry, the present part deals with my other main body of work, in rough paths theory and its extensions. As a matter of fact, a number of these works are motivated by geometrical applications, or involve some geometry; see **[P14,15,20,21]**. The works **[P17,14,15]** deal with some questions that are formulated in the classical setting of rough or controlled paths; they are described in chapter 3. The machinery of rough flows, introduced in **[P12,13]**, is described in chapter 4, together with its applications to stochastic flows **[P16]**. Chapter 5 is dedicated to two different works on rough or singular partial differential equations (PDEs) that give new settings for investigating linear rough PDEs on the one hand **[P18]**, and singular PDEs on manifolds or discrete structures on the other hand **[P20]**.

Chapter 3

Classical rough paths theory

We report in this chapter on four works **[P18,17,14,15]** that deal with different topics in Lyons' classical or in Gubinelli's controlled paths settings. Part of the work **[P18]** on kinetic Brownian motion gives a nice example of use of the continuity of the Itô-Lyons solution map to a rough differential equation, by proving that kinetic Brownian motion provides an interpolation between geodesic and Brownian flows. The work **[P17]**, joint with J. Diehl, tackles the following elementary inverse problem. Find conditions under which one can reconstruct a rough signal from the observation of a solution flow to a rough differential equation driven by that signal. A necessary and sufficient condition was found when the signal is a weak geometric α-Hölder rough path, with $\left(\frac{1}{3}, \frac{1}{2}\right]$, and illustrations and applications given.

I introduced in the work **[P14]** a new concept of rough integrator on a Banach manifold, that extends in that setting the notion of weak geometric Hölder rough path and can be used to solve rough differential equations with values in bundles over the manifold. I showed that these objects have a canonical form whenever the manifold is equipped with a regular enough linear connection.

Last, the work **[P15]** is dedicated to investigating the smoothness properties of the Itô-Lyons solution map in the setting of controlled paths. The regularity result obtained is used, amongst other things, to show the locally well-posed character of some evolution equations on path space over a finite dimensional manifold, such as Driver's equation.

I include an elementary introduction to rough paths theory in section 3.1 to invite the non-specialist reader to have a look at the different works described below, with sufficient background to get a hand on the problems investigated and some of the tools used to solve them. We only describe here the theory associated with weak geometric α-Hölder rough paths, with $\left(\frac{1}{3}, \frac{1}{2}\right]$, to make it more accessible.

3.1 A bird's-eye view on rough paths theory

3.1.1. From controlled differential equations (...) It is a classical topic in control theory to consider differential equations of the form

$$\dot{x}_t = \sum_{i=1}^{\ell} V_i(x_t)\dot{h}^i_t,$$

with nice enough vector fields $V_i : \mathbb{R}^d \to \mathbb{R}^d$, say bounded and Lipschitz for the moment, and the controls $\dot{h}^i : [0,T] \to \mathbb{R}, i = 1, \ldots, \ell$, are integrable functions that represent data that can usually be tuned on demand in a real-life problem. It will pay off to consider the controls $\dot{h}^i$ as the time-derivative of absolutely continuous functions h^i, and reformulate the above dynamics under the form

$$dx_t = V_i(x_t)dh^i_t, \tag{3.1.1}$$

with the usual convention for sums.

With a view to applications involving noisy signals such as Brownian paths, we are particularly interested in the setting where $h =: X$ is *not* absolutely continuous, but only α-Hölder continuous, for some $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, say. Although equation (3.1.1) seems non-sensical in this case, it is given meaning by the theory of rough paths, as invented by T. Lyons [**1**], and reformulated and enriched by numerous works since its introduction. We recall the basics of this theory in Section 3.1.2 so as to make it accessible to a large audience; it suffices for the moment to emphasize that driving signals in this theory do not consist only of $\mathbb{R}^\ell$-valued α-Hölder continuous paths X, rather they are pairs $(X, \mathbb{X})$, where the $\mathbb{R}^\ell \times \mathbb{R}^\ell$-valued second object $\mathbb{X}$, is to be thought of as the collection of "iterated integrals $\int X^j dX^k$". The point here is that the latter expression is a priori meaningless if we only know that X is α-Hölder, for $\alpha \leqslant \frac{1}{2}$, as one cannot make sense of such integrals without additional structure. So these iterated integrals have to be provided as *additional a priori data*. The enriched control $(X, \mathbb{X})$ is what we call a ***rough path***. Lyons' main breakthrough in his fundamental work [**1**] was to show that equation (3.1.1) can be given sense, and is well-posed, whenever X is understood as a rough path, provided the vector fields V_i are sufficiently regular. Moreover the solution map which associates to the starting point x_0, and the rough path $(X, \mathbb{X})$, the solution path $x_\bullet$ is continuous. This is in stark contrast with the fact that solutions of stochastic differential equations driven by Brownian motion are only measurable functions of their drivers, with no better dependence as a rule.

From a practical point of view, it may be argued that there are no physical examples of real-life observable paths which are truly α-Hölder continuous paths; rather, all paths can be seen as smooth, albeit with possibly very high fluctuations that make them appear rough on a macroscopic scale. However, the invisible microscopic fluctuations might lead to macroscopic effects when the path (the control) is acting on a dynamical system. A rough path somehow provides a mathematical abstraction of this fact by recording this microscopic scale effects that a highly-oscillatory smooth signal may have on a dynamical system in the second order object $\mathbb{X}$. One of the most basic and illustrative example of this phenomenon is associated with the following 2-dimensional spinning signal

$$h_t^n = \frac{1}{n}\big(\cos(n^2t), \sin(n^2t)\big), \quad 0 \leqslant t \leqslant 1.$$

Although h^n converges uniformly to 0, this path sweeps an area t in any time interval $[0,t]$, independently of n, suggesting that this sequence of paths may be really different from the zero signal. As a matter of fact, given two vector fields V_1, V_2 on $\mathbb{R}^2$, it is remarkable that the solution path $x_\bullet^n$ to the ordinary controlled differential equation

$$\dot{x}_t^n = \big(\dot{h}_t^n\big)^i V_i\big(x_t^n\big)$$

converges uniformly to the solution path $x_\bullet$ to the ordinary differential equation

$$\dot{x}_t = \big[V_1, V_2\big](x_t).$$

This is due to the fact that the canonical lift of $h_\bullet^n$ as a weak geometric Hölder p-rough path, for any $2 < p < 3$, converges in a rough paths sense to the pure area rough path with area process $(t-s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3.1.2. (...) to rough differential equations As a first step in rough paths theory, let us consider the controlled ordinary differential equation

$$dx_t = V_i(x_t)dh_t^i$$

driven by a smooth (or bounded variation) $\mathbb{R}^\ell$-valued control h. It is elementary to iterate the formula

$$V_i(x_t) = V_i(x_s) + \int_s^t \partial_j V_i(x_r) V_k^j(X_r) dh_r^k =: V_i(x_s) + \int_s^t V_k V_i(x_r) dh_r^k,$$

to obtain a kind of Taylor-Euler expansion of the solution path to the above equation, under the form

$$\begin{aligned} x_t = x_s &+ \sum_{n=1,\dots,N-1} \sum_{i_1,\dots,i_n=1,\dots,\ell} \left(\int_{s\leqslant r_1\leqslant\cdots\leqslant r_n\leqslant t} dh_{r_1}^{i_1} \dots dh_{r_n}^{i_n} \right) V_{i_1} \dots V_{i_n}(x_s) \\ &+ \sum_{i_1,\dots,i_N=1,\dots,\ell} \int_{s\leqslant r_1\leqslant\cdots\leqslant r_n\leqslant t} V_{i_1} \dots V_{i_N}\big(x_{r_1}\big)\, dh_{r_1}^{i_1} \dots dh_{r_N}^{i_n}, \end{aligned}$$

provided the vector fields are sufficiently regular for this expression to make sense. If V and its first N derivatives are bounded, the last term is of order $|t-s|^N$. So the following numerical scheme

(3.1.2)

$$\begin{cases} x_0^n & := x_0 \\ x_{t_i}^n & := x_{t_{i-1}}^n + \sum_{n=1,\dots,N-1} \sum_{i_1,\dots,i_n=1,\dots,\ell} V_{i_1} \dots V_{i_n}\big(x_{t_{n-1}}^n\big) \int_{s\leqslant r_1\leqslant\cdots\leqslant r_n\leqslant t} dX_{r_1}^{i_1} \dots dX_{r_n}^{i_n}, \end{cases}$$

written here for any partition $\tau = (t_i)$ of $[0,T]$, is of order $|\tau|^{N-1}$, where $|\tau|$ denotes the meshsize of the partition.

In the mid-90's T. Lyons [**1**] understood that one can actually make sense of the controlled differential equation (3.1.1) even if X is not of bounded variation, *if one provides a priori the values of sufficiently many iterated integrals* $\int \dots \int dX \dots dX$*, and one defines* *a solution path to the equation as a path for which the above scheme is exact up to a term of size* $\left|t_{i+1}-t_i\right|^a$*, for some constant* $a>1$*.*

Definition 20. Fix a finite time horizon T, and let $\alpha\in\left(\frac13,\frac12\right]$. An α-**Hölder rough path** $\mathbf{X}=(X,\mathbb{X})$ consists of a pair $(X,\mathbb{X})$, made up of an α-Hölder continuous path $X:[0,T]\rightarrow\mathbb{R}^\ell$, and a two parameter function $\mathbb{X}:\{(t,s)\,;\,0\leqslant s\leqslant t\leqslant T\}\rightarrow(\mathbb{R}^\ell)^{\otimes 2}\simeq\mathbb{R}^{\ell\times\ell}$, such that the inequality

$$|\mathbb{X}_{s,t}|\leq C|t-s|^{2\alpha}$$

hold for some positive constant C, and all $0\leqslant s\leqslant t\leqslant T$, and we have Chen's relation

$$\mathbb{X}^{jk}_{ts}-\mathbb{X}^{jk}_{tu}-\mathbb{X}^{jk}_{us}=X^j_{s,u}X^k_{u,t},\tag{3.1.3}$$

for all $0\leqslant s\leqslant u\leqslant t\leqslant T$. The rough path $\mathbf{X}$ is said to be **weakly geometric** if the symmetric part of $\mathbb{X}_{ts}$ is given in terms of X_{ts} by the relation

$$\mathbb{X}^{jk}_{s,t}+\mathbb{X}^{kj}_{s,t}=X^j_{s,t}\,X^k_{s,t}.$$

We define a norm on the set of α-Hölder rough path setting

$$\|\mathbf{X}\|:=\|X\|_\infty+\|X\|_\alpha+\|\mathbb{X}\|_{2\alpha},$$

where $\|\cdot\|_\gamma$ stands for the γ-Hölder norm of a 1 or 2-index map, for any $0<\gamma<1$.

To make sense of these conditions, think of $\mathbb{X}^{jk}_{ts}$ as $\int_s^t(X_r-X_s)^j\,dX^k_r$, even though this integral does not make sense in our setting. (Young integration theory can only make sense of such integrals if X is α-Hölder, with $\alpha>\frac12$.) When X is smooth its increments have size $(t-s)$, and the increments $\mathbb{X}_{ts}$ have size $(t-s)^2$. Convince yourself that relation (3.1.3) comes in that model setting from Chasles' relation $\int_s^u+\int_u^t=\int_s^t$. The symmetry condition satisfied by weak geometric rough paths is satisfied by the rough paths lift of any smooth path. We invite the reader to check that if B stands for a Brownian motion and $\frac13\leqslant\alpha<\frac12$, we define an α-Hölder rough path setting

$$\mathbf{B}_{ts}=\left(B_t-B_s,\int_s^t(B_r-B_s)\otimes dB_r\right),$$

where the above integral is an Itô integral. This rough path is not weakly geometric however, while we would define a weakly geometric rough path by using a Stratonovich integral in the above definition of $\mathbf{B}$.

Denote by V the collection of some vector fields $\left(V_1,\dots,V_\ell\right)$ on $\mathbb{R}^d$, and identify below a vector field to a first order differential operator, so $V_if:=(Df)(V_i)$.

Definition. Fix a finite time horizon T, and let $\mathbf{X}$ be a weak geometric α-Hölder rough path, with $\alpha\in\left(\frac13,\frac12\right]$. A **path** $x_\bullet:[0,T]\rightarrow\mathbb{R}^d$ is said to **solve the rough differential equation**

$$dx_t=\mathsf{V}(x_t)\,\mathbf{X}(dt),\tag{3.1.4}$$

if for any function $f\in\mathcal{C}^3_b$, we have the following Taylor-Euler expansion

$$f(x_t)=f(x_s)+X^i_{ts}\big(V_if\big)(x_s)+\mathbb{X}^{jk}_{ts}\big(V_jV_kf\big)(x_s)+O\big(|t-s|^{3\alpha}\big),\tag{3.1.5}$$

with

$$\Big|O\big(|t-s|^{3\alpha}\big)\Big|\leqslant c(\mathbf{X},V)\,\|f\|_{\mathcal{C}^3}\,|t-s|^{3\alpha},$$

for some positive constant $c(\mathbf{X},\mathsf{V})$ depending only on $\mathbf{X}$ and the Lip^3 norm of the V.

It can be proved that if $(X, \mathbb{X})$ is the Brownian rough path introduced above, but with a Stratonovich integral rather than an Itô integral, then a solution path to the above rough differential equation is a solution path to the Stratonovich differential equation

$$dx_t = V_i(x_t) \circ dB_t^i.$$

This is what makes rough paths theory so appealing for applications to stochastic calculus!

Theorem 21 is the cornerstone of the theory of rough differential equations. It was first proved in a different form by Lyons [**1**], and was named '*Lyons' universal limit theorem*' by Malliavin. A vector field of class $\mathcal{C}_b^2$,with Lipschitz second derivative is said to be Lip^3 in the sense of Stein.

Theorem 21 (Lyons' universal limit theorem)**.** Let $\mathbf{X}$ be an α-Hölder rough path, $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$. Let $\mathsf{V} = (V_i)_{i=1,\dots,\ell}$ be a collection of Lip^3 vector fields on $\mathbb{R}^d$.

(1) Given any starting point $x \in \mathbb{R}^d$, there exists a unique solution path $x_\bullet$ to the rough differential equation (3.1.4).

(2) The solution path $x_\bullet \in \Big(\mathcal{C}\big([0,T], \mathbb{R}^d\big), \|\cdot\|_\infty\Big)$ depends *continuously* on $\mathbf{X}$.

The crucial point in the above statement is the continuous dependence of the solution path as a function of the rough signal $\mathbf{X}$, in stark contrast with the fact that solutions of stochastic differential equations are only measurable functionals of the Brownian path, while rough differential equations can be used to solve Stratonovich differential equations. The twist here is that the only purely measurable operation that is done here is in defining the iterated integrals $\int_s^t (B_r - B_s) \otimes \circ dB_r$; once this is done, the machinery for solving the rough differential equation (3.1.4) is continuous with respect to the Brownian rough path. This continuity result was used for instance to give streamlined proofs of deep results in stochastic analysis such as Stroock-Varadhan support theorem for diffusion processes, or the basics of Freidlin-Wentzell theory of large deviations for diffusion processes.

3.2

From geodesic to Brownian flow

As a further illustration of the use of the continuity of Itô-Lyons solution map, I will describe here the interpolation result on kinetic Brownian motion proved in **[P18]**. Recall for that purpose that kinetic Brownian motion is a one parameter family of diffusion processes with values in the orthonormal frame bundle of a d-dimensional Riemannian manifold $(\mathcal{M}, g)$, whose generators are given by the formula

$$\mathcal{L}_\sigma = H_1 + \frac{\sigma^2}{2} \sum_{j=2}^{d} V_j^2,$$

with $\sigma \geqslant 0$, and the notations of section 2.1. Denote kinetic Brownian motion by $z_t^\sigma = \big(x_t^\sigma, e_t^\sigma\big) \in O\mathcal{M}$, to emphasize its dependence on the parameter σ. We proved in **[P18]** that the family of laws of $x_\cdot^\sigma$ provides a kind of interpolation between geodesic and Brownian motions, as σ ranges from zero to infinity, as expressed in Theorem 22 below and illustrated in Fig. 3.2 below when the underlying manifold is the 2-dimensional flat torus.

To fix the setting, add a cemetary point ∂ to $\mathcal{M}$, and endow the union $\mathcal{M} \sqcup \{\partial\}$ with its usual one-point compactification topology. That being done, denote by Ω_0 the set of continuous paths $\gamma : [0,1] \to \mathcal{M} \sqcup \{\partial\}$, that start at some reference point x_0 and that stay

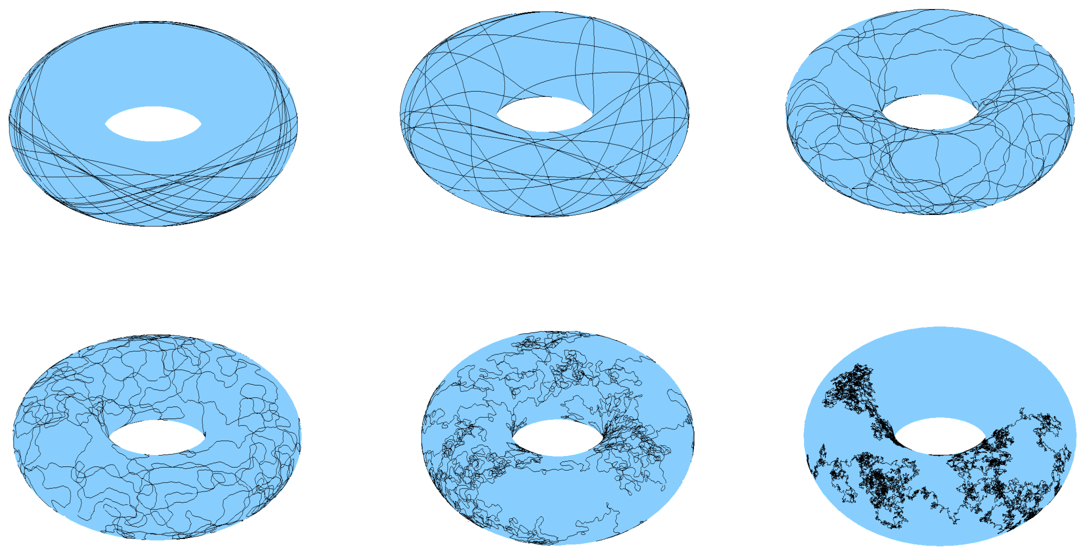

FIGURE 1. Kinetic Brownian motion on the flat torus for $\sigma = 10^{-2}, 10^{-1}, 1, 2, 4, 10$.

at point ∂ if they exit the manifold $\mathcal{M}$. Let $\mathcal{F} := \bigvee_{t\in[0,1]} \mathcal{F}_t$ where $(\mathcal{F}_t)_{0\leqslant t\leqslant 1}$ stands for the filtration generated by the canonical coordinate process. Denote by B_R the geodesic open ball with center x_0 and radius R, for any $R > 0$. The first exit time from B_R is denoted by τ_R, and used to define a measurable map

$$T_R : \Omega_0 \to C\big([0,1], \bar{B}_R\big),$$

which associates to any path $(\gamma_t)_{0\leqslant t\leqslant 1} \in \Omega_0$ the path which coincides with γ on the time interval $\big[0, \tau_R\big]$, and which is constant, equal to γ_{τ_R}, on the time interval $\big[\tau_R, 1\big]$. The following definition then provides a convenient setting for dealing with sequences of random process whose limit may explode.

Definition. A sequence $(\mathbb{P}_n)_{n\geqslant 0}$ of probability measures on $\big(\Omega_0, \mathcal{F}\big)$ is said to converge locally weakly to some limit probability $\mathbb{P}$ on $\big(\Omega_0, \mathcal{F}\big)$ if the sequence $\mathbb{P}_n \circ T_R^{-1}$ of probability measures on $C([0,1], \bar{B}_R)$ converges weakly to $\mathbb{P} \circ T_R^{-1}$, for every $R > 0$.

Equipped with this definition, we can give a precise sense to the above interpolation between geodesic and Brownian motions provided by kinetic Brownian motion.

Theorem 22 ([P18] Interpolation). Assume the Riemannian manifold $(\mathcal{M}, g)$ is complete. Given $z_0 = \big(x_0, e_0\big) \in O\mathcal{M}$ we have the two following asymptotics behaviours.

- The law of the *rescaled* process $\big(x^{\sigma}_{\sigma^2 t}\big)_{0\leqslant t\leqslant 1}$ converges locally weakly under $\mathbb{P}_{z_0}$ to Brownian motion on $\mathcal{M}$, run at speed $\frac{4}{d(d-1)}$ over the time interval $[0,1]$, as σ goes to infinity.

- The law of the *non-rescaled* process $\left(x_t^\sigma\right)_{0\leqslant t\leqslant 1}$ converges locally weakly under $\mathbb{P}_{z_0}$, as σ goes to zero, to a Dirac mass on the geodesic started from x_0 in the direction of the first vector of the basis e_0.

Implicit in the above statement concerning the small noise asymptotics is the fact that the geodesic curve is only run over the time interval $[0,1]$; no stochastic completeness assumption is made above. The proof of the interpolation theorem 22 and Corollary 23 is split into three steps. We first prove the result stated in the first item by elementary means in the model Euclidean case. Using the tools of rough paths analysis, elementary moment estimates allow a strengthening of this weak convergence result into the weak convergence of the rough path lift of the Euclidean kinetic Brownian motion to the (Stratonovich) Brownian rough path. To link kinetic Brownian motion in Euclidean space to its Riemannian analogue, we use the fact that the latter can be constructed by rolling on $\mathcal{M}$ without slipping the former, using Cartan's development map. This means, from a practical point of view, that one can construct the $\mathcal{M}$-valued part of kinetic Brownian motion as the solution of an $\mathcal{M}$-valued controlled differential equation in which the Euclidean kinetic Brownian motion plays the role of the control. This key fact enables us to use the continuity of the Itô-Lyons map associated with Cartan's development map, and transfer the weak convergence result proved for the Euclidean kinetic Brownian motion to the curved setting of *any* complete Riemannian manifold. Note that the above theorem only involves local weak convergence; it can be strengthened under a very mild and essentially optimal natural assumption.

Corollary 23 ([P18]). If the manifold $(\mathcal{M}, g)$ is complete and stochastically complete then the *rescaled* process $\left(x_{\sigma^2 t}^\sigma\right)_{0\leqslant t\leqslant 1}$ converges in law under $\mathbb{P}_{z_0}$, as σ goes to infinity, to Brownian motion run at speed $\frac{4}{d(d-1)}$ over the time interval $[0,1]$.

X.-M. Li proved recently in [**7**] an interpolation theorem similar to Corollary 23, under stronger geometric assumptions on the base manifold $\mathcal{M}$, requiring a positive injectivity radius and a control on the norm of the Hessian of the distance function on some geodesic ball. Her proof rests on a formulation of the weak convergence result in terms of a martingale problem, builds on ideas from homogenization theory, and uses tightness techniques. It is likely that the very robust nature of our proof, based on the rough paths machinery, offers a convenient setting for proving more general homogenization results at a low cost. As a basic illustration, notice for example that our proof works verbatim with the Levi-Civita connection replaced by any other affine metric preserving connection $\mathrm{H} : T\mathcal{M} \to TO\mathcal{M}$. The limit process in $O\mathcal{M}$ is not in that case the lift to $O\mathcal{M}$ of a Brownian motion on $\mathcal{M}$ anymore, but it is still described as the solution to the Stratonovich differential equation

$$dz_t = \mathrm{H}(z_t) \circ dB_t,$$

where B is an $\mathbb{R}^d$-valued Brownian motion.

The idea of using rough paths theory for proving elementary homogenization results as in theorem 22 was first tested in the work [**6**] of Friz, Gassiat and Lyons, in their study of the so-called *physical Brownian motion in a magnetic field.* That random process is described as a C^1 path $(x_t)_{0\leqslant t\leqslant 1}$ in $\mathbb{R}^d$ modeling the motion of an object of mass m, with momentum $p_\cdot = m\dot{x}_\cdot$, subject to a damping force and a magnetic field. Its momentum satisfies a stochastic differential equation of Ornstein-Uhlenbeck form

$$dp_t = -\frac{1}{m}\,Mp_t dt + dB_t,$$

for some matrix M whose eigenvalues all have positive real parts, and B is a d-dimensional Brownian motion. While the process $(Mx_t)_{0\leqslant t\leqslant 1}$ is easily seen to converge to a Brownian motion, its rough path lift is shown to converge in a rough paths sense in $\mathbb{L}^q$, for any $q\geqslant 2$, to a random rough path *different from* the Brownian rough path.

What next? The above reasoning should actually have a scope that goes farther than the proof of the results of this section; I hope it should be useful to prove a number of other homogenization results on a class of fast-slow systems of which the following example is a typical element. Consider a fast-slow system in $\mathbb{R}^d\times\mathbb{R}^n$, of the form

$$\begin{aligned} dy_t &= g(x_t,y_t)dt + \sigma\, h(x_t,y_t)\circ dB_t,\\ dx_t &= f(x_t,y_t)dt, \end{aligned}$$

with $z_t := (x_t,y_t)\in\mathbb{R}^d\times\mathbb{R}^n$ and B an ℓ-dimensional Brownian motion. It is an interesting question to look for conditions on the vector field (f,g) and h on $\mathbb{R}^d\times\mathbb{R}^n$ that ensure that the above system for z_t reads

$$dz_t = H_1(z_t)dt + \sigma V\circ dB_t$$

for some connection on the bundle $\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}^d$, for which the vector field (f,g) is the lift by the connection of the first vector of the canonical basis of $\mathbb{R}^d$, and $V\circ dB_t$ can be interpreted as something similar to $V_j\circ dB_t^j$ in the above setting.

3.3

Inverse problem for rough differential equations

In a joint work with J. Diehl, I tackled in **[P17]** the inverse problem of finding back a rough signal from the observation of the solution to a rough differential equation driven by that signal. To be more specific, our problem read as follows. Given some sufficiently regular vector-field valued 1-form $\mathsf{V}=(V_1,\dots,V_\ell)$ on $\mathbb{R}^\ell$, and a weak geometric α-Hölder rough path $\mathbf{X}$ over $\mathbb{R}^\ell$, with $\alpha\in\left(\frac13,\frac12\right]$, defined on the time interval $[0,1]$ say, denote by $\left(\varphi_{ts}\right)_{0\leqslant s\leqslant t\leqslant 1}$ the solution flow to the rough differential equation

$$dx_r = \mathsf{V}(x_r)\,\mathbf{X}(dr) \tag{3.3.1}$$

in $\mathbb{R}^d$. Assume we observe the increments $\varphi_{ts}(x_i)$ of the different solution paths, started from c distinct points $x_i, i=1,\dots,c$ at time s; that is, we have access to the data

$$z_{ts}^{x_1,\dots,x_c} := \Big(\varphi_{ts}(x_1),\dots,\varphi_{ts}(x_c)\Big).$$

As the counter-example of constant vector fields shows, our ability to find back $\mathbf{X}$ from the solution flow to equation (5.2.8) depends on the 1-form V.

Definition. The 1-form V is said to have the **reconstruction property** if one can find an integer $c\geq 1$, points $x_1,\dots,x_c\in\mathbb{R}^d$, a constant $a>1$, and a function $\mathcal{X}:\mathbb{R}^{cd}\to T^2(\mathbb{R}^\ell)\cong\mathbb{R}^d\oplus(\mathbb{R}^d)^{\otimes 2}$, with components $\mathcal{X}^1$ and $\mathcal{X}^2$, such that one can associate to every positive constant M another positive constant C_M such that the inequalities

$$\left|\mathcal{X}^1\big(z_{ts}^{x_1,\dots,x_c}\big) - X_{ts}\right| \leqslant C_M|t-s|^a,\quad \left|\mathcal{X}^2\big(z_{ts}^{x_1,\dots,x_c}\big) - \mathbb{X}_{ts}\right| \leqslant C_M|t-s|^a \tag{3.3.2}$$

hold for *all* weak geometric α-Hölder rough paths $\mathbf{X}$ with $\|\mathbf{X}\| \leq M$, for all times $0 \leqslant s \leqslant t \leqslant 1$ sufficiently close.

These inequalities ensure that the $T^2(\mathbb{R}^\ell)$-valued functional $\mathcal{X}\big(z_{ts}^{x_1,\dots,x_c}\big)$ is almost-multiplicative [**2**], with associated multiplicative functional $\mathbf{X}$. Hence, by a fundamental result of Lyons on almost multiplicative functionals, one can - in principle - completely reconstruct $\mathbf{X}_{ts}$ from the knowledge of the $\mathcal{X}\big(z_{ba}^{x_1,\dots,x_c}\big)$, with $s \leqslant a \leqslant b \leqslant t$.

Our main result takes the form of a sufficient and necessary condition on the 1-form V for equation (3.3.1) to have the reconstruction property. Only brackets of the form $\big[V_j, V_k\big]$, with $j < k$, appear in the matrix below.

Theorem 24 ([P17] Reconstruction**).** Let $\mathsf{V} = (V_1, \dots, V_\ell)$ be a $Lip^3(\mathbb{R}^d)$-valued 1-form on $\mathbb{R}^\ell$.Set

$$m := \frac{\ell(\ell+1)}{2}.$$

Then equation (3.3.1) has the reconstruction property if and only if there exists an integer c and points $x_1, \dots, x_c$ in $\mathbb{R}^d$ such that the $(cd \times m)$ matrix

$$\mathsf{M} = \begin{pmatrix} V_1(x_1) & \cdots & V_\ell(x_1) & [V_1, V_2](x_1) & \cdots & [V_{\ell-1}, V_\ell](x_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ V_1(x_c) & \cdots & V_\ell(x_c) & [V_1, V_2](x_c) & \cdots & [V_{\ell-1}, V_\ell](x_c) \end{pmatrix}$$

has rank m. We call M the **reconstruction matrix.**

Note that the above condition on the reconstruction matrix is unrelated to Hörmander's bracket condition, and that there is no need of any kind of ellipticity or hypoellipticity for Theorem 24 to apply. The above rank condition holds true for instance if $\ell = 2$ and $\big(V_1, V_2, [V_1, V_2]\big)$ forms a free family at some point – for which we need $d \geq 3$.

Rather than giving some details on our method of proof, it is best illustrated with the example of the rolling ball. This equation describes the motion of a ball with unit radius rolled on a table without slipping. The position of the ball at time t is determined by the orthogonal projection $x_t \in \mathbb{R}^2$ of the center of the ball on the table (i.e. the point touching the table, with the latter identified with $\mathbb{R}^2$), and by a (3×3) orthonormal matrix $M_t \in O\mathbb{R}^3$ giving the orientation of the ball. Set

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We define right invariant vector fields V_1, V_2 on $O\mathbb{R}^3$ by the formula

$$V_1(M) = A_1 M, \quad V_2(M) = A_2 M,$$

for any $M \in O\mathbb{R}^3$. The non-slipping assumption on the motion of the ball relates the evolution of the path $x_\bullet$ to that of $M_\bullet$, when the path $x_\bullet$ is $\mathcal{C}^1$, as follows

$$dM_t = V_1(M_t)\, dx_t^1 + V_2(M_t)\, dx_t^2. \tag{3.3.3}$$

This equation makes perfect sense when $x_\bullet$ is replaced by a rough path $\mathbf{X}$ and the equation is understood in a rough path sense. Set $\mathsf{V} = (V_1, V_2)$. Working with invariant vector fields, the solution flow $\varphi_\bullet$ to the rough differential equation

$$dM_t = \mathsf{V}(M_t)\, \mathbf{X}(dt) \tag{3.3.4}$$

is given by the map

$$\varphi_\bullet : M \in O\mathbb{R}^3 \mapsto M^0_\bullet M,$$

where $M^0_\bullet$ is the solution path to the rough differential equation (3.3.4) started from the identity. We know from the work of Strichartz [**4**] on the Baker-Campbell-Dynkin-Hausdorff formula that the solution to the time-inhomogeneous ordinary differential equation (3.3.3) is formally given by the time-1 map of a *time-homogeneous* ordinary differential equation involving a vector field explicitly computable in terms of V_1, V_2 and their brackets, and the iterated integrals of the signal $x_\bullet$, under the form of an infinite series. Truncating this series provides an approximate solution whose accuracy can be quantified precisely under some mild conditions on the driving vector fields. This picture makes perfect sense in the rough path setting of equation (3.3.4) and forms the basis of the flow method put forward in **[P12]** presented in details in section 4.1.2. In the present setting, given a 2-dimensional rough path $\mathbf{X}$, with Lévy area process $\mathbb{A}_\bullet$, and given $0 \leqslant s \leqslant t \leqslant 1$, denote by ψ_{ts} the time-1 value of the solution path to the ordinary differential equation

$$dz_u = X^1_{ts} V_1(z_u) + X^2_{ts} V_2(z_u) + \mathbb{A}_{ts}\big[V_1, V_2\big](z_u), \quad 0 \leqslant u \leqslant 1,$$

in $\mathbb{O}(\mathbb{R}^3)$ started from the identity; that is

$$\psi_{ts} = \exp\Big(X^1_{ts} A_1 + X^2_{ts} A_2 + \mathbb{A}_{ts}\big[A_2, A_1\big]\Big).$$

Then it follows from the results in **[P12]**, to be described below, that there exists some positive constant c_1 such that the inequality

$$\big\|\varphi_{ts} - \psi_{ts}\big\|_\infty \leqslant c_1 |t-s|^{3\alpha} \tag{3.3.5}$$

holds for all $0 \leqslant s \leqslant t \leqslant 1$. Since the vectors $A_1, A_2, \big[A_2, A_1\big]$ form a basis of the vector space $\mathcal{A}_3$ of anti-symmetric (3×3) matrices, and the exponential map is a local diffeomorphism between a neighbourhood of 0 in $\mathcal{A}_3$ and a neighbourhood of the identity in $O\mathbb{R}^3$, we get back the coefficient X^1_{ts}, X^2_{ts} and $\mathbb{A}_{ts}$ from the knowledge of φ_{ts} and relation (3.3.5), up to an accuracy of order $|t-s|^{3\alpha}$. This shows that one can reconstruct $\mathbf{X}$ from $\varphi_\bullet$, in the sense of Definition 3.3, as the symmetric part of $\mathbb{X}_{ts}$ are given in terms of X_{ts} One could argue that perfect knowledge of φ_{ts} may seem unrealistic from a practical point of view. Note that the above proof makes it clear that it is sufficient to know φ_{ts} up to an accuracy of order $|t-s|^{3\alpha}$ to get the reconstruction result.

3.4 Rough integrators on Banach manifolds

The elementary example of a spinning 2-dimensional signal given at the end of section 3.1.1 makes it clear that, when understood as controls, smooth paths may have a richer structure than expected at first sight, as the above constant path with non-trivial area process shows. Smooth paths have canonical lifts as rough paths; a smooth path in $\mathbb{R}^\ell$ whose rough path lift is non-canonical will be called a *spinning path*. While this may not be obvious, it makes perfect sense to talk of a spinning manifold-valued path, as an example of a manifold valued 'rough path'. One should keep in mind, however, that the classical notion of rough path only becomes really interesting when understood as a control in some differential equation. Does it make sense to think of a manifold-valued smooth path as a control? Yes. It is indeed common, in physics and differential geometry, to be given a bundle $\mathcal{B}$ over some manifold $\mathcal{M}$, together with a connection, given by a $T\mathcal{B}$-valued 1-form

H on $\mathcal{B}$. Lifting a smooth $\mathcal{M}$-valued path $\gamma_\bullet$ into a smooth horizontal $\mathcal{B}$-valued smooth path $z_\bullet$ is a basic procedure in which γ is used as a control in the ordinary differential equation

$$\dot{z}_t = \mathrm{H}(z_t)\dot{\gamma}_t$$

defining the path $z_\bullet$. What would happen if γ were a kind of spinning smooth $\mathcal{M}$-valued path? What about *spinning geodesics* in a Riemannian setting? The notion of p-rough integrator introduced below, and the results proved, will enable us to answer these questions.

So far there has been only a few works dealing with rough paths in a geometrical setting, starting with the work [**9**] of Lyons and Qian. This seminal work investigated the well-posedness problem for the ordinary differential equation on the path space of a compact manifold generated by Itô vector fields, with an eye on probabilistic applications related to path space versions of the Cameron-Martin theorem and Driver's flow equation. It was enriched by the work [**8**] of Li and Lyons, showing that the Itô-Lyons solution map to a Young differential equation is Fréchet regular under appropriate conditions, when the driving signal has finite p-variation, with $1 < p < 2$, and by my work **[P15]** providing a general regularity result for the Itô-Lyons solution map, for any $p \geqslant 2$, in the setting of controlled paths. Another work [**10**] of Lyons and Qian addressed well-posedness issues for ordinary differential equations on path space associated with Itô vector fields obtained by varying the driving rough signal.

These works only use rough paths as an ingredient to construct some dynamics in a geometric configuration space. Cass, Litterer and Lyons made a step further in putting rough paths theory in a geometrical setting and proposed in [**11**] a notion of rough path on a manifold extending the classical notion defined in a linear setting. In the same way as a vector field on a manifold $\mathcal{M}$ can be understood in an analytic/algebraic setting as a differentiation in the ring of smooth functions on the manifold $\mathcal{M}$, a rough path is abstractly defined as a linear form on the space of sufficiently regular 1-forms on $\mathcal{M}$, which is required to have some continuity property; call it an *integrator*. This functional analytic definition rests on a basic chain rule which eventually enables to understand their notion of rough path on a manifold as an equivalence class of classical rough paths, related by some chain rule under change of coordinates. This situation is exactly similar to representing a tangent vector on a d-dimensional manifold as an equivalence class of vectors in $\mathbb{R}^d$, indexed by local diffeomorphisms of a neighbourhood of 0 (that is local changes of coordinates), and related by a change of coordinate rule which exactly balances the changes in the numerical representation of a given 1-form α on $\mathcal{M}$ associated with local coordinates, so the quantity $\alpha(u)$ is independent of any choice of coordinates used to compute it. Their approach rests however on a notion of Lip-γ manifold which prevents its easy use even with non-compact finite dimensional manifolds, not to speak about infinite dimensional manifolds.

The ideas of [**11**] have been reloaded in a different and more accessible form in the recent work [**12**] by Cass, Driver and Litterer, in which they define a weak geometric Hölder p-rough path on a finite dimensional compact embedded submanifold of $\mathbb{R}^d$ as an integrator, obtained by "projection" of a weak geometric Hölder p-rough path in the ambiant Euclidean space, for $2 \leqslant p < 3$ only. Nothing is lost in working with submanifolds, and this notion is eventually shown to be independent of the embedding of the manifold in its environment, while these objects can be used to define and solve uniquely rough differential equations driven by weak geometric Hölder p-rough paths on compact manifolds. They can construct parallel transport along manifold-valued rough paths in their sense, and use it to show a one-to-one correspondence between rough paths on a finite d-dimensional manifold and rough paths on d-dimensional Euclidean space.

We proposed in the work **[P14]** an elementary and flexible notion of weak geometric Hölder p-rough path on Banach manifolds that goes far beyond the previous works. Roughly speaking, a weak geometric Hölder p-rough path on a manifold $\mathcal{M}$ is a triple made up of a vector field valued 1-form F defined on some Banach space U, a weak geometric Hölder p-rough path $\mathbf{X}$ over U, and the maximal solution to the rough differential on $\mathcal{M}$ constructed from F and $\mathbf{X}$. The data of these three objects is sufficient to define and solve uniquely rough differential equations driven by "manifold-valued" rough paths, or better, by *p-rough integrators*. The above-mentionned spinning smooth paths on $\mathcal{M}$ are precisely those p-rough integrators whose associated $\mathcal{M}$-valued paths are smooth. Nothing else than the (smooth) manifold structure is needed to make sense of a p-rough integrator. I proved that these objects have a canonical representation if the tangent bundle of $\mathcal{M}$ is equipped with a connection, in which case one can always choose for Banach space U in the preceding description of a p-rough integrator the Banach space on which the manifold is modelled.

The interest of working with Banach manifolds comes from the fact that they naturally pop up in a number of geometric situations, as path or loop spaces over some finite dimensional manifold, as in the works [**13, 14**] of Brzezniak, Carroll and Elworthy, or the works [**15, 16**] of Inahama and Kawabi, or as manifolds of maps of a given finite dimensional manifold, as in the works [**17, 18**] of Elworthy and Brzezniak, to mention but a few works from the probability community. Note however that many interesting infinite dimensional manifolds are Fréchet manifolds, for which no theory of rough paths is presently available.

Notations. We gather here a few notations that used below.

- We shall denote by U a generic Banach space; the notation $T^{[p]}(\mathrm{U})$ will be used for the truncated tensor product of order $[p]$, completed for some choice of tensor norms. The letter $\mathbf{X}$ will stand for a weak geometric Hölder p-rough path over U, for some $p \geqslant 2$, and for the above choice of tensor norm on $T^{[p]}(\mathrm{U})$.
- We shall denote by $\mathcal{M}$ a Banach manifold modelled on some Banach space E. The set of continuous linear homomorphisms of E will be denoted by L(E), and the set of continuous linear isomorphisms of E will be denoted by GL(E).

Intermezzo : RDEs in an infinite dimensional setting

We adopt here the definition of a solution path to an $\mathcal{M}$-valued rough differential equation given in **[P13]** in a Banach space setting, as it is perfectly suited for our needs. It essentially amounts to requiring from a solution path that it satisfies some uniform Taylor-Euler expansion formulas, in the line of Davie' seminal work [**1**]. Let U be a Banach space, and F be a 1-form on U with values in the space of vector fields on a Banach manifold $\mathcal{M}$, of class $\mathcal{C}^{[p]+1}$. Given $u \in \mathrm{U}$, we identify the vector field $\mathrm{F}(\cdot\,; u)$ on $\mathcal{M}$ with its associated first order differential operator $\mathrm{F}^{\otimes}(u)$, and extend the definition of $\mathrm{F}^{\otimes}$ to $T^{[p]}(\mathrm{U})$ setting $\mathrm{F}^{\otimes}(1) = \mathrm{Id}$, and

$$\mathrm{F}^{\otimes}(u_1 \otimes \cdots \otimes u_k) = \mathrm{F}^{\otimes}(u_1) \cdots \mathrm{F}^{\otimes}(u_k),$$

for $1 \leqslant k \leqslant [p]$, and $u_i \in \mathrm{U}$, and by linearity; so $\mathrm{F}^{\otimes}(u_1 \otimes \cdots \otimes u_k)$ is a differential operator of order k. Given a weak geometric Hölder p-rough path $\mathbf{X}$ over U, recall that an $\mathcal{M}$-valued continuous path $(x_t)_{0 \leqslant t < \zeta}$ is said to solve the rough differential equation

$$dx_t = \mathrm{F}^{\otimes}\big(x_t\,;\mathbf{X}(dt)\big) \tag{3.4.1}$$

if there exists a constant $a > 1$ such that, for any $0 \leqslant s < \zeta$, there exists an open neighbourhood V_s of x_s such that we have the Taylor-Euler expansion

$$f(x_t) = \big(\mathrm{F}^{\otimes}\big(\mathbf{X}_{ts}\big)f\big)(x_s) + O\big(|t-s|^a\big)$$

for all t close enough to s for x_t to belong to V_s, and any function f of class $\mathcal{C}^{[p]+1}$ defined on V_s, where it has bounded derivatives. The results of **[P13]** show that such a rough differential equation has a unique maximal solution started from any given point, as awaited. It also follows from the results of **[P13]**, or other classical works, that the solution path $x_\bullet$ depends continuously on the driving signal $\mathbf{X}$ in the following sense. Fix $T < \zeta$ and cover the compact support of the path $(x_t)_{0 \leqslant t \leqslant T}$ by finitely many local chart domains $(O'_i)_{1 \leqslant i \leqslant N}$ and $(O_i)_{1 \leqslant i \leqslant N}$, with $O'_i \subset O_i$ for all $1 \leqslant i \leqslant N$. Then, there exists a positive constant δ such that for $\mathbf{Y}$ δ-close to $\mathbf{X}$ in the Hölder rough path distance, the solution path $y_\bullet$ to the rough differential equation

$$dy_t = \mathrm{F}^{\otimes}\big(y_t\,;\mathbf{Y}(dt)\big)$$

started from x_0, will be well-defined on the time interval $[0, T]$ and will remain in the open neighbourhood $\bigcup_{i=1}^N O_i$ of the support of $(x_t)_{0 \leqslant t \leqslant T}$, with y_t in O_i whenever x_t is in O'_i.

Note here that the rough path setting allows to work with any Banach manifold, without any restriction on the Banach space on which it is modelled, unlike what needs to be done in a stochastic setting where a robust theory of stochastic integration is only available for some special types of Banach spaces, the so-called M-type 2 and UMD spaces.

3.4.1. Rough integrators on Banach manifolds

The definition of a p-rough integrator given below is best understood in the light of the following example. Given a 1-form α on a d-dimensional manifold $\mathcal{M}$ equipped with a connection, how can one define the integral $\int_0^1 \alpha(\circ dx_t)$, of α along an $\mathcal{M}$-valued continuous semimartingale $(x_t)_{0 \leqslant t \leqslant 1}$? The classical answer consists in showing that $x_\bullet$ can always be constructed as the projection in $\mathcal{M}$ of a $\mathrm{GL}(\mathcal{M})$-valued path $(z_t)_{0 \leqslant t \leqslant 1}$ obtained by solving a stochastic differential equation in $\mathrm{GL}(\mathcal{M})$ involving the horizontal vector fields associated with the connection and some $\mathbb{R}^d$-valued semimartingale $(w_t)_{0 \leqslant t \leqslant 1}$. One then defines $\int_0^1 \alpha(\circ dx_t)$ in terms of w only, setting

$$\int_0^1 \alpha(\circ dx_t) = \int_0^1 \big(\alpha \circ z_t^{-1}\big) \circ dw_t.$$

So the datum of w and the horizontal vector fields are all we need to define $x_\bullet$ and use it as a control. The next definition adopts a similar point of view in the present setting.

Definition. Let $p \geqslant 2$ be given. A **weak geometric Hölder p-rough path on $\mathcal{M}$** is a a triple $\Theta = \big((x_t)_{0 \leqslant t < \zeta}, \mathrm{F}, \mathbf{X}\big)$, where

- $\mathbf{X}$ a weak geometric Hölder p-rough path over some Banach space U, defined on the time interval $[0, \zeta)$,

- F is a continuous linear map from U to the space of vector fields on $\mathcal{M}$ of class $\mathcal{C}^{[p]+1}$,
- the path $(x_t)_{0\leqslant t<\zeta}$ solves the rough differential equation

$$dx_t = \mathrm{F}^{\otimes}\big(x_t\,;\mathbf{X}(dt)\big). \tag{3.4.2}$$

We also call Θ a **basic p-rough integrator**.

Given a p-rough integrator Θ as above, the triple $\big((x_t)_{0\leqslant t\leqslant T}, \mathrm{F}, \mathbf{X}\big)$, with $T<\zeta$, is also called a basic p-rough integrator. Let us insist on the fact that the Banach space U in the above definition is not fixed a priori, and depends on the basic p-rough integrator Θ on $\mathcal{M}$. So any weak geometric Hölder p-rough path $\mathbf{Y}$ over some Banach space V can be canonically seen as a weak geometric Hölder p-rough path in the above sense by choosing $\mathrm{U}=\mathrm{V}$ and $\mathbf{X}=\mathbf{Y}$, the identity map for F, and the first level of the rough path $\mathbf{X}_t$ for x_t. Note that we do not endow the set of basic p-rough integrators with a topology of a distance as this is somewhat subtle and we do not need it in the sequel.

3.4.2. Weak geometric Hölder p-rough paths as integrators

Recall the construction of a line integral along a continuous semimartingale on a manifold described above. The next proposition shows that a weak geometric Hölder p-rough path on $\mathcal{M}$ can be used as an integrator in a rough differential equation, which justifies calling it a *p-rough integrator*.

Definition. Let Θ be a weak geometric Hölder p-rough path on $\mathcal{M}$ in the sense of definition 3.4.1. Let also $\pi:\mathcal{B}\mapsto\mathcal{M}$ be a smooth fiber bundle over $\mathcal{M}$, equipped with a connection, of class $\mathcal{C}^{[p]+1}$, given under the form of a smooth $T\mathcal{B}$-valued *1*-form H on $\mathcal{M}$ such that the restriction to $\mathrm{H}(T\mathcal{M})$ of $\pi_*:T\mathcal{B}\to T\mathcal{M}$ is an isomorphism. Given a rough integrator $\Theta=\big((x_t)_{0\leqslant t<\zeta},\mathrm{F},\mathbf{X}\big)$ in $\mathcal{M}$, and $z_0\in\mathcal{B}$ with $\pi(z_0)=x_0$, a path $(z_t)_{0\leqslant t<\zeta'}$ in $\mathcal{B}$ is said to be a lift of $(x_t)_{0\leqslant t<\zeta'}$, if $\zeta'\leqslant\zeta$ and it is the solution path to the rough differential equation

$$dz_t = \mathrm{H}\Big(\mathrm{F}\big(\pi(z);\mathbf{X}(dt)\big)\Big) \tag{3.4.3}$$

started from z_0. This equation ensures in particular that $\pi(z_t)=x_t$. The p-rough integrator $\big((z_t)_{0\leqslant t<\zeta'},\mathrm{H}\circ\mathrm{F}\circ\pi,\mathbf{X}\big)$ is then said to be the ***lift of Θ to $\mathcal{B}$.***

The results on rough differential equations recalled in the introduction of this section apply and show that

Proposition 25 ([P14]). The rough differential equation (3.4.3) has a unique maximal solution started from any given point.

Solving successively some rough differential equations of the form (3.4.3) with some basic p-rough integrators $\Theta^{(1)}=\Big(\big(x_t^{(1)}\big)_{0\leqslant t\leqslant t_1},\mathrm{F}^{(1)},\mathbf{X}^{(1)}\Big),\dots,\Theta^{(k)}$, with $x_0^{(j)}=x_{t_{j-1}}^{(j-1)}$, $y_0^{(j)}=y_{t_{j-1}}^{(j-1)}$ and $t_{j-1}<\zeta'_{j-1}\leqslant\zeta_{j-1}$, for $2\leqslant j\leqslant k$, defines, whenever this makes sense, the *concatenation $\Theta^{(k)}\cdots\Theta^{(1)}$ of the basic p-rough integrators $\Theta^{(i)}$*. We call such an object a **p-rough integrator**.

3.4.3. Canonical representation of rough integrators

We show in this section that p-rough integrators have a canonical representation when the tangent bundle of $\mathcal{M}$ is equipped with a connection. This representation is the analogue of the representation of a regular path γ on $\mathcal{M}$ by a regular path in $T_{\gamma_0}\mathcal{M}$ using Cartan's development map, or moving frame method. The interest of Cartan's method is that it somehow provides a dimensionally-optimal coding of a path in $\mathcal{M}$ in terms of vector space-valued paths. Here is a trivial illustration of this fact. Let $u_\bullet$ be an $\mathbb{R}^{10}$-valued path and $\gamma_t = \sum_{i=1}^{10} u_t^i$ be an $\mathbb{R}$-valued path. The coding of $\gamma_\bullet$ by the 10 components of $u_\bullet$ is optimized by coding it with its real value at each time. Replacing $\mathbb{R}^{10}$ and $\mathbb{R}$ by an infinite dimensional spaces emphasizes the importance of such parcimonious representations.

Recall we work with a manifold $\mathcal{M}$ modelled on some Banach space E. One can actually give a parcimonious description of weak geometric Hölder p-rough paths on Banach manifolds similar to the above one, providing a description of these objects in terms of weak geometric Hölder p-rough paths in the tensor space $T^{[p]}(\mathrm{E})$, as opposed to the a priori unrelated Banach space U involved in the primary definition of a weak geometric Hölder p-rough path, see definition 3.4.1. As in Cartan's moving frame method, this requires the tangent bundle $T\mathcal{M} \to \mathcal{M}$ to be equipped with a connection; this is a non-trivial assumption in an infinite dimensional setting, linked to the fact that there exists some Banach spaces that do not even admit a smooth partition of unity. No such pathology happens in finite dimension, and finite dimensional manifolds can always be endowed with an arbitrary connection; so the results below always hold for rough integrators on finite dimensional manifolds.

The frame bundle $\mathrm{GL}(\mathcal{M})$ of $\mathcal{M}$ will play a crucial role in that play. This is the collection of all isomorphisms from E to $T_m\mathcal{M}$, for $m \in \mathcal{M}$; it has a natural manifold structure modelled on $\mathrm{GL(E)} \times \mathrm{E}$. We shall denote by z a generic element of $\mathrm{GL}(\mathcal{M})$, and by $\mathcal{H}\mathrm{GL}(\mathcal{M})$ the vertical sub-bundle of $T\mathrm{GL}(\mathcal{M})$, canonically identified with the Lie algebra $\mathrm{gl(E)} = \mathrm{L(E)}$ of $\mathrm{GL(E)}$. The connection ∇ on the bundle $T\mathcal{M} \to \mathcal{M}$ is naturally lifted into a connection on the bundle $\pi : \mathrm{GL}(\mathcal{M}) \to \mathcal{M}$, still denoted by the same symbol. Remark that π_* is an isomorphism between the horizontal sub-bundle $\mathcal{H}^\nabla\mathrm{GL}(\mathcal{M})$ in $T\mathrm{GL}(\mathcal{M})$ and $T\mathcal{M}$. The horizontal distribution in $T\mathrm{GL}(\mathcal{M})$ can be used to define a continuous linear map $\mathfrak{H}^\nabla$ from E to the space of horizontal vector fields on $T\mathrm{GL}(\mathcal{M})$, defined by the requirement that $\mathfrak{H}^\nabla(z\,;a) \in T_z\mathrm{GL}(\mathcal{M})$ is horizontal and corresponds to $z(a) \in T\mathcal{M}$, for any $a \in \mathrm{E}$ and $z \in \mathrm{GL}(\mathcal{M})$. This vector field valued 1-form $\mathfrak{H}^\nabla$ on E is called the **canonical horizontal 1-form**. (Once again, we refer the reader to the nice books [**19**] and [**20**] for the basics of differential geometry on Banach manifolds.)

Theorem 26 below gives a canonical and parcimonious representation of a given weak geometric Hölder p-rough path, seen as a rough integrator. We assume for that purpose that the manifold $\mathcal{M}$ is endowed with a connection ∇. Given a 1-form F on U with values in the space of vector fields on $\mathcal{M}$ as in the above definition of a weak geometric Hölder p-rough path on $\mathcal{M}$, we denote by $\mathfrak{F}$ its lift to a 1-form with values in the space of ∇-horizontal vector fields on $\mathrm{GL}(\mathcal{M})$. Parallel translation along a weak geometric Hölder p-rough path $\Theta = \big((x_t)_{0\leqslant t<\zeta}, \mathrm{F}, \mathbf{X}\big)$ is defined as the solution path to the rough differential equation in $\mathrm{GL}(\mathcal{M})$

$$dz_t = \mathfrak{F}\big(z_t, \mathbf{X}(dt)\big)$$

started from any given frame $z_0 \in \mathrm{GL}(\mathcal{M})$ above x_0. It is elementary to see that the paths $z_\bullet$ and $x_\bullet$ are defined on the same maximal interval.

Given another Banach manifold $\mathcal{N}$, and a 1-form $\mathrm{G} : \mathcal{N} \times T\mathcal{M} \to T\mathcal{N}$, as used above to write down a rough differential equation in definition 3.4.2, define $\mathfrak{S}$ on $\mathrm{GL}(\mathcal{M}) \times \mathcal{N} \times \mathrm{U}$ by the formula

$$\mathfrak{S}(z, y\,; u) = \Big(\mathfrak{F}(z\,; u),\ \mathrm{G}\big(y, \pi(z)\,; \pi_* \mathfrak{F}(z\,; u)\big)\Big),$$

and $\overline{\mathfrak{S}}$ on $\mathrm{GL}(\mathcal{M}) \times \mathcal{N} \times \mathrm{E}$ by the formula

$$\overline{\mathfrak{S}}(z, y\,; a) = \Big(\mathfrak{H}^\nabla(z\,; a),\ \mathrm{G}\big(y, \pi(z)\,; \pi_* \mathfrak{H}^\nabla(z\,; a)\big)\Big).$$

The next theorem shows that one can always understand the solution of a rough differential equation driven by a p-rough integrator Θ as the solution to another rough differential equation driven by a p-rough integrator involving a weak geometric Hölder p-rough path over the model space E, and the canonical horizontal 1-form $\mathfrak{H}^\nabla$.

Theorem 26 (**[P14]** Canonical representation of rough integrators)**.** Let $\mathcal{M}$ be a Banach manifold endowed with a connection ∇, and $\Theta = \big((x_t)_{0\leqslant t<\zeta}, \mathrm{F}, \mathbf{X}\big)$ be a basic p-rough integrator on $\mathcal{M}$.

(1) One defines a weak geometric Hölder p-rough path $\mathbf{Z}$ over E, on the time interval $[0,\zeta)$, by solving the rough differential equation

$$\begin{aligned} dz_t &= \mathfrak{F}\big(z_t\,; \mathbf{X}(dt)\big), \\ d\mathbf{Z}_t &= \mathbf{Z}_t \otimes z_t^{-1} \mathrm{F}\big(\pi(z_t)\,; \mathbf{X}(dt)\big). \end{aligned} \tag{3.4.4}$$

in $\mathrm{GL}(\mathcal{M}) \times T^{[p]}(\mathrm{E})$, started from (z_0, Id).

(2) The solution paths (z_t, y_t) and $(\overline{z}_t, \overline{y}_t)$ in $\mathrm{GL}(\mathcal{M}) \times \mathcal{N}$ to the rough differential equations

$$\begin{aligned} d(z_t, y_t) &= \mathfrak{S}\big(z_t, y_t\,; \mathbf{X}(dt)\big), \\ d(\overline{z}_t, \overline{y}_t) &= \overline{\mathfrak{S}}\big(\overline{z}_t, \overline{y}_t\,; \mathbf{Z}(dt)\big), \end{aligned} \tag{3.4.5}$$

coincide if they start from the same initial condition.

Given z_0 above x_0, the "$\mathrm{GL}(\mathcal{M})$-valued" p-rough integrator $\big((z_t)_{0\leqslant t<\zeta}, \mathfrak{H}^\nabla, \mathbf{Z}\big)$ is said to be the **canonical representation of** Θ.

3.5 Regularity of the Itô-Lyons solution map

It is probably fair to say that, from a probabilist's point of view, the main success of the theory of rough paths, as developped originally by T. Lyons [**1**], was to provide a framework in which a notion of integral can be defined as a *continuous* function of both its integrand and its integrator, while extending the Itô-Stratonovich integral when both theories apply. This continuity result is in striking contrast with the fact that the stochastic integration map defines only a measurable function of its integrator, with no hope for a better dependence on it as a rule. The continuity of the rough integration map provides a very clean way of understanding differential equations driven by rough signals, and their approximation theory. However, the highly nonlinear setting of rough paths and its purely metric topology prevent the use of the classical Banach space calculus in this setting; so, as a consequence, one cannot hope for a better statement than the following one for instance. Under some conditions to be made precise, the solution to a rough differential equation is a locally Lipschitz continuous function of the driving rough path.

It is fortunate that Gubinelli developped in **[21]** an alternative approach to rough differential equations based on the *Banach space* setting of paths controlled by a fixed rough path, and which somehow allows to linearize many considerations. We recall here the basics, using here and below the following notation. Given a Banach space H and $\gamma > 0$, we denote by $\mathrm{Lip}_\gamma(\mathrm{U},\mathrm{H})$, resp. $\mathrm{Lip}_\gamma(\mathrm{U})$, the set of H-valued, resp. real-valued, maps on U that are γ-Lipschitz in the sense of Stein; these maps are in particular bounded. Write $\|\cdot\|_\gamma$ for the γ-Lipschitz norm. Given $k \geqslant 3$, a continuous linear map F from U to the space of Lip_k vector fields on V is called a Lip_k **V-valued 1-form on U**. The natural operator norm on the space $\mathrm{L}\big(\mathrm{U}, \mathrm{Lip}_k(\mathrm{V},\mathrm{V})\big)$ of all such maps turns this space into a Banach space.

Intermezzo : Controlled paths

Given $2 \leqslant p < 3$ and any Hölder p-rough path $\mathbf{X}$ over E, defined on the time interval $[0,1]$ say, a V-valued **path** $(z_t)_{0\leqslant t\leqslant 1}$ is said to be **controlled by X** if its increments $\delta z_{ts} := z_t - z_s$, satisfy

$$\delta z_{ts} = z'_s X_{ts} + R_{ts},$$

for all $0 \leqslant s \leqslant t \leqslant 1$, for some L(E,V)-valued $\frac{1}{p}$-Lipschitz map $z'_\bullet$, and some V-valued $\frac{2}{p}$-Lipschitz 2-index map R. One should more properly speak of a controlled path as the pair (z, z') rather than just z. One endows the set $\mathcal{C}_{\mathbf{X}}(\mathrm{V})$ of such paths with a Banach space structure setting

$$\big\|(z,z')\big\| := \|z'\|_{\frac{1}{p}} + \|R\|_{\frac{2}{p}} + \big|z_0\big|.$$

Given $(z,z') \in \mathcal{C}_{\mathbf{X}}$ and $(y,y') \in \mathcal{C}_{\mathbf{X}}$, one defines the rough integral

$$\int_0^\bullet \mathrm{F}(z_u)\, dy_u$$

as the unique additive functional uniquely associated with the almost-additive map

$$\mathrm{F}(z_s) y_{ts} + \big(D_{z_s}\mathrm{F}\big)\big(Z'_s \otimes y'_s\big)\, \mathbb{X}_{ts}$$

by the sewing lemma – see **[21]** or **[2]**, or go and see the intermezzo in the beginning of section 4.1.1.

3.5.1. Differentiability of the Itô-Lyons map

I proved in the note **[P15]** that the Itô-Lyons solution map that associates to some Banach space-valued controlled path $y_\bullet$ the solution to a rough differential equation driven by $y_\bullet$ is actually a Fréchet regular map of both the controlled path and the vector fields in the equation. This regularity result provides a straightforward approach to investigating the dependence of the solution to a parameter-dependent rough differential equation with respect to this parameter, as in the works of Inahama and Kawabi **[15]**, and to constructing some dynamics on some path spaces in a geometrical setting, such as Driver's flow.

Denote by $\mathfrak{I}(\mathrm{F}, y)$ the Itô map that associates to any Lip_k V-valued 1-form F on U, and any path $y_\bullet$ in U controlled by $\mathbf{X}$, the unique solution of the rough differential equation

$$dx_t = \mathrm{F}(x_t)\, dy_t, \tag{3.5.1}$$

with given initial condition $x_0 \in \mathrm{V}$. It is defined on the product Banach space

$$\mathrm{L}\big(\mathrm{U}, \mathrm{Lip}_k(\mathrm{V}, \mathrm{V})\big) \times \mathcal{C}_{\mathbf{X}}(\mathrm{U})$$

and takes values in the affine subspace $\mathcal{C}^0_{\mathbf{X}}(\mathrm{V})$ of $\mathcal{C}_{\mathbf{X}}(\mathrm{V})$, made up of V-valued paths controlled by $\mathbf{X}$ started from x_0; so there is no difficulty in defining Fréchet differentiability in that setting. A simple application of the inverse function theorem gives the following regularity result.

Theorem 27 ([P15]). The Itô-Lyons solution map

$$\mathfrak{I} : \mathrm{L}\big(\mathrm{U}, \mathrm{Lip}_k(\mathrm{V}, \mathrm{V})\big) \times \mathcal{C}_{\mathbf{X}}(\mathrm{U}) \to \mathcal{C}^0_{\mathbf{X}}(\mathrm{V})$$

is $\mathcal{C}^{[k]-2}$ Fréchet-differentiable.

In particular, if $\mathbf{X}$ is a piece of a "higher dimensional" rough path $\widehat{\mathbf{X}}$, one can work in $\mathcal{C}_{\widehat{\mathbf{X}}}(\mathrm{U}) \supset \mathcal{C}_{\mathbf{X}}(\mathrm{U})$ and perturbe the original rough differential equation (3.5.1) with rough signals non-controlled by $\mathbf{X}$, but rather controlled by $\widehat{\mathbf{X}}$, that is controlled by other "pieces" of $\widehat{\mathbf{X}}$. Note that one cannot hope for boundedness of the derivative of $\mathfrak{I}$ in the above level of generality; so the Itô-Lyons solution map is a priori only locally Lipschitz.

A similar regularity result has been proved by Li and Lyons in [**8**] for the solution map to a Young differential equation with fixed F, corresponding to the case $1 < p < 2$. Their result is more general than the statement corresponding in that setting to theorem 27, for the Itô map they consider is defined on a much bigger space than $\mathcal{C}_{\mathbf{X}}(\mathrm{U})$; this is the reason why the proof of their result requires non-trivial arguments. On the other hand, the non-linear nature of the space of rough paths, for $2 < p < 3$, makes the extension of their approach to that setting hard, as illustrated by the work [**23**] of Qian and Tudor, in which they introduce an appropriate notion of tangent space to the space of p-rough paths which is of delicate use. The present use of the Banach setting of controlled paths somehow linearizes many considerations, while providing results that are still widely applicable.

3.5.2. Dynamics on path space

Assume here U=V is a given Banach space and F is a fixed Lip_3-1-form on U; fix an initial condition $x_0 \in \mathrm{U}$. As the Itô-Lyons map

$$\mathfrak{I}(\mathrm{F}, \cdot) : \mathcal{C}^0_{\mathbf{X}}(\mathrm{U}) \to \mathcal{C}^0_{\mathbf{X}}(\mathrm{U})$$

is locally Lipschitz, one can apply the classical Cauchy-Lipschitz theorem in $\mathcal{C}^0_{\mathbf{X}}(\mathrm{V})$ and solve uniquely the ordinary differential equation

$$\frac{dy^t_\bullet}{dt} = \mathfrak{I}\big(\mathrm{F}, y^t_\bullet\big)$$

for a given initial condition $y_\bullet$ in $\mathcal{C}^0_{\mathbf{X}}(\mathrm{V})$, on a maximal interval of definition. This simple remark can be used in a number of geometrical situations to define some natural evolutions on the path space over some given manifold. (Note that this evolution on path space keeps the starting point fixed.) Here is an illustration.

Let $\mathcal{M}$ be a compact smooth n-dimensional submanifold of $\mathbb{R}^d$, endowed with the Riemannian structure inherited from the ambiant space. Nash's theorem ensures there is no loss of generality in considering that setting. Let $\nabla : \mathbb{R}^d \times \mathcal{C}^\infty\big(\mathbb{R}^d, \mathbb{R}^d\big) \to \mathcal{C}^\infty\big(\mathbb{R}^d, \mathbb{R}^d\big)$, be a compactly supported smooth extension to $\mathbb{R}^d$ of the Levi-Civita covariant differentiation operator on $T\mathcal{M}$. This $\mathbb{R}^d$-dependent operator has a natural extension to sections of $\mathrm{L}(\mathbb{R}^d)$, still denoted by the same symbol.

Denote by $\mathcal{C}_{\mathbf{X}}(\mathcal{M})$ the subset of $\mathcal{C}_{\mathbf{X}}(\mathbb{R}^d)$ made up of M-valued paths $y_\bullet$ controlled by $\mathbf{X}$. Given $T_0 \in \mathrm{L}(\mathbb{R}^d)$, the solution to the rough differential equation in $\mathrm{L}(\mathbb{R}^d)$

$$dT_s = \nabla(T_s)\, dy_s, \tag{3.5.2}$$

started from T_0, defines a path in $\mathrm{L}(\mathbb{R}^d)$ above $y_\bullet$. If the path $y_\bullet$ takes values in $\mathcal{M}$ and T_0 is orthonormal and has its first n columns forming an orthonormal basis of $T_{y_0}\mathcal{M}$, then the first n columns of T_s form an orthonormal basis of $T_{y_s}\mathcal{M}$ at any time $0 \leqslant s \leqslant 1$. Also, the restriction of $\big(T_s\big)_{0\leqslant s\leqslant 1}$ to $\mathbb{R}^n$ does not depend on $T_0\big|_{(\mathbb{R}^n)^\perp}$, and f_h depends only on the restriction to $\mathcal{M}$ of ∇; see for instance the recent work **[12]** of Cass, Driver and Litterer on constrained rough paths, section 5. The map T_s transports parallelly $T_{y_0}\mathcal{M}$ to $T_{y_s}\mathcal{M}$ as an isometry. In that case, and given any controlled path $h_\bullet$ in $\mathcal{C}_{\mathbf{X}}(\mathbb{R}^n)$, with $\mathbb{R}^n$ seen as a subset of $\mathbb{R}^d$, we define a Lipschitz continuous map from $\mathcal{C}_{\mathbf{X}}(\mathcal{M})$ to $\mathcal{C}_{\mathbf{X}}(\mathbb{R}^d)$ setting

$$\mathfrak{F}_h(y_\bullet)_s = T_s h_s,$$

for $0 \leqslant s \leqslant 1$. The vector $\mathfrak{F}_h(y)_s$ belongs to $T_{y_s}\mathcal{M}$ at any time $0 \leqslant s \leqslant 1$. It is elementary to see that if $y_\bullet \in \mathcal{C}_{\mathbf{X}}(\mathcal{M})$, then the solution in $\mathcal{C}_{\mathbf{X}}(\mathbb{R}^d)$ to the ordinary differential equation

$$\frac{dy^t_\bullet}{dt} = \mathfrak{F}_h\big(y^t_\bullet\big) \tag{3.5.3}$$

started from $y_\bullet$ takes values in $\mathcal{C}_{\mathbf{X}}(\mathcal{M})$. The local flow associated with the above dynamics is precisely Driver's flow when $h \in \mathrm{H}^1(\mathbb{R}^n)$ and $\mathbf{X}$ is a typical realization of the n-dimensional Brownian rough path. As Driver's flow is known to be almost-surely defined for all times, the local flow defined by equation (3.5.3) is actually almost-surely globally defined. This construction of Driver's flow simplifies the corresponding construction by Lyons and Qian given in **[9]**, where a local flow on the space of $\mathcal{M}$-valued paths with finite 1-variation is constructed and then extended to a rough path space by a continuity argument. The present straightforward approach illustrates the interest of working in the Banach setting of controlled paths, as opposed to the non-linear setting of rough paths.

What next? It should be interesting to see whether the above regularity result can be used, together with the approximate flow-to-flow machinery described in Chapter 4, to investigate quasi-invariance of Wiener measure on $\mathcal{M}$ by the flow of the vector field $\mathfrak{F}_h$, from a fresh point of view. This would in particular be relevant if one is interested in replacing Wiener measure by the image measure of a Gaussian process on $\mathbb{R}^\ell$ by the Itô-Lyons map associated with Cartan's moving frame dynamics on $O\mathcal{M}$. An integration by parts formula for such a measure on path space of a Riemannian manifold should be within reach, with potential applications to Logarithmic-Sobolev or isoperimetric inequalities on path space under this measure.

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Chapter 4

Rough flows

An elementary construction recipe of flows was recently introduced in **[P12]** and used there to get back the core results of Lyons' theory of rough differential equations in a very short and elementary way **[P12,13]**. These works emphasizes the fact that it may be worth considering flows of maps as the primary objects from which the individual trajectories can be built, as opposed to the classical point of view that constructs a flow from an uncountable collection of individual trajectories. (Probabilists know how tricky it can be to deal with uncountably many null sets.) As is well-recognized now, the main success of Lyons' theory was to disentangle probability from pure dynamics in the study of stochastic differential equations by showing that the dynamics is a deterministic and continuous function of an enriched signal that is constructed from the noise in the equation by purely probabilistic means. This very clean picture led to new proofs and extensions of foundational results in the theory of stochastic differential equations, such as Stroock and Varadhan support theorem or the basics of Freidlin and Ventzel theory of large deviations for diffusions.

It was realized in the late 70's that stochastic differential equations not only define individual trajectories, they also define flows of regular homeomorphisms, depending on the regularity of the vector fields involved in the dynamics. This opened the door to the study of stochastic flows of maps for themselves, and it did not took long time before Le Jan and Watanabe clarified definitely the situation by showing that, in a semimartingale setting, there is a one-to-one correspondence between flows of diffeomorphisms and time-varying stochastic velocity fields, under proper regularity conditions on the objects involved. We offered in the work **[P16]** an embedding of the theory of semimartingale stochastic flows into the theory of rough flows similar to the embedding of the theory of stochastic differential equations into the theory of rough differential equations.

It is based on the "approximate flow-to-flow" machinery introduced in **[P12]**, which gives body to the following fact. To a 2-index family $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$ of maps which falls short from being a flow, in a quantitative way, one can associate a unique flow $(\varphi_{ts})_{0\leqslant s\leqslant t\leqslant T}$ close to $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$; moreover the flow φ depends continuously on the approximate flow μ. More details in section 4.1. The point about such a machinery is that approximate flows appear naturally in a number of situations as simplified descriptions of complex evolutions, often under the form of Taylor-like expansions of complicated dynamics. The model situation is given by a controlled ordinary differential equation

$$\dot{x}_t = \sum_{i=1}^{\ell} V_i(x_t)\,\dot{h}^i_t, \tag{4.0.1}$$

in $\mathbb{R}^d$, driven by an $\mathbb{R}^\ell$-valued $\mathcal{C}^1$ control h. The Euler scheme

$$\mu_{ts}(x) = x + \big(h^i_t - h^i_s\big)V_i(x)$$

defines, under proper regularity conditions on the vector fields, an approximate flow whose associated flow is the flow generated by equation (4.0.1). One step farther, if we are given a weak geometric Hölder p-rough path $\mathbf{X}$, with $2 \leqslant p < 3$, and sufficiently regular vector fields $\mathrm{F} = \big(V_1, \dots, V_\ell\big)$ on $\mathbb{R}^d$, one can associate to the rough differential equation

$$dx_t = \mathrm{F}(x_t)\mathbf{X}(dt), \tag{4.0.2}$$

some maps μ_{ts} defined, for each $0 \leqslant s \leqslant t \leqslant T$, as the time 1 map of an ordinary differential equation involving $\mathbf{X}_{ts}$, the V_i and their brackets, that have the same Taylor expansion as the awaited Taylor expansion of a solution flow to equation (4.0.2). They happen to define an approximate flow whose associated flow is the solution flow to equation (4.0.2). See section 4.1.2 to see how the core results of rough paths theory can be recovered from that point of view, and section 4.1.3 for an application of my approach to the study of stochastic mean field rough differential equations.

A similar approach can be used to deal with a general class of stochastic time-dependent velocity fields. We introduced for that purpose in **[P16]** a notion of rough driver, that is an enriched version of a time-dependent vector field, that will be given by the additional datum of a time-dependent second order differential operator satisfying some algebraic and analytic conditions. A notion of solution to a differential equation driven by a rough driver will be given, in the line of what was done in **[P12]** for rough differential equations, and the approximate flow-to-flow machinery will be seen to lead to a clean and simple well-posedness result for such equations. As awaited from the above discussion, the main point of this result is that the Itô map, that associates to a rough driver the solution flow to its associated equation, is continuous. This continuity result is the key to deep results in the theory of stochastic flows.

We proved indeed in **[P16]** that reasonable semimartingale velocity fields can be lifted to rough drivers under some mild boundedness and regularity conditions, and that the solution flow associated to the 'semimartingale' rough driver coincides almost surely with the solution flow to the Kunita-type Stratonovich differential equation driven by the velocity field. As a consequence of the continuity of the Itô map, a Wong-Zakai theorem was proved for a general class of semimartingale velocity fields, together with sharp support and large deviation theorems for Brownian flows.

One can actually weaken the notion of rough driver to a notion of approximate rough driver and still get an integration theory for such drivers, with a continuous Itô-Lyons map in the right topologies. This fact is used in a critical way in a coming joint work on homogenization of fast-slow dynamics, joint with R. Catellier. We showed that the framework of approximate drivers and flows provides a setting that unifies a number of important results in the domain and offers the possibility to investigate new situations as well.

4.1 Flows, approximate flows and rough differential equations

We start this chapter by a description of the approximate flow-to-flow machinery that was introduced in **[P12]** and by showing how it can be used to get back in a very clean and simple way the core results of rough paths theory, in their full force.

4.1.1. Flows and approximate flows The workhorse of my approach to rough differential equations and rough flows is a simple non-commutative extension of Feyel-de la Pradelle' sewing lemma that can be considered as a far reaching generalization of Lyons' main technical tool in his original formulation of the theory – the existence of a unique rough path close to an almost rough path. Roughly speaking, the approximate flow-to-flow machinery says that if a 2-index family $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$ of maps which falls short from being a flow, in a quantitative way, one can associate a unique flow $(\varphi_{ts})_{0\leqslant s\leqslant t\leqslant T}$ close to $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$; moreover the flow φ depends continuously on the approximate flow μ. This phenomenon is best illustrated by Feyel-de la Pradelle' sewing lemma.

Intermezzo : Feyel-de la Pradelle' sewing lemma

This lemma [**2, 3**] provides the essential tool for a quick access to Lyons' original formulation of rough paths theory via the rough integral; we present here its commutative version and denote here by E a Banach space.

Theorem 28 (Feyel-de la Pradelle [**2**])**.** Let T be a finite time horizon, and $\big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ be an E-valued 2-index family of elements of E depending continuously on (s,t). If there exists a positive constant c_1 such that the inequality

$$\big|(\mu_{tu}+\mu_{us})-\mu_{ts}\big| \leqslant c_1\,|t-s|^a \tag{4.1.1}$$

holds for some exponent $a>1$, and all times $0\leqslant s\leqslant u\leqslant t\leqslant T$, then there exists a positive constant δ, and a unique E-valued path φ whose increments $\varphi_{ts}:=\varphi_t-\varphi_s$ satisfy

$$\big|\varphi_{ts}-\mu_{ts}\big| \leqslant |t-s|^a, \tag{4.1.2}$$

for all $0\leqslant s\leqslant t\leqslant T$, with $t-s\leqslant\delta$; furthermore, defining $\mu_{\pi_{ts}}$ as below for a partition π_{ts} of an interval (s,t), we have

$$\big\|\varphi_{ts}-\mu_{\pi_{ts}}\big\|_\infty \leqslant 2\,c_1\,T\,\big|\pi_{ts}\big|^{a-1} \tag{4.1.3}$$

for any partition π_{ts} of any interval $(s,t)\subset[0,T]$, of mesh $\big|\pi_{ts}\big|\leqslant\delta$.

Note as a start that the increments φ_{ts} of φ satisfy the identity

$$\varphi_{ts}=\varphi_{tu}+\varphi_{us} \tag{4.1.4}$$

for all times $0\leqslant s\leqslant u\leqslant t\leqslant T$, and that this additivity identity characterizes increments of 1-index functions amongst 2-index functions. So inequality (4.1.1) quantifies the default of μ from being an increment of 1-index function.

The construction of φ is actually simple. Given a partition $\pi=\big\{0<t_1<\cdots<1\big\}$ of $[0,1]$ and $0\leqslant s\leqslant t\leqslant 1$, set

$$\mu_{\pi_{ts}} := \sum_{s\leqslant t_i<t_{i+1}\leqslant t}\mu_{t_{i+1}t_i}.$$

This map is almost additive as we have

$$\mu_{\pi_{tu}} + \mu_{\pi_{us}} = \mu_{\pi_{ts}} - \mu_{u^+u^-} = \mu_{\pi_{ts}} + o_{|\pi|}(1),$$

for all $0 \leqslant s \leqslant u \leqslant t \leqslant 1$, where u^-, u^+ are the elements of π such that $u^- \leqslant u < u^+$, and $|\pi| = \max\{t_{i+1} - t_i\}$ stands for the mesh of the partition. So we expect to find a solution φ to our problem under the form μ_π, for a partition of $[0,1]$ of infinitesimal mesh, that is as a limit of μ_π's, say along a sequence of refined partitions π^n where π^{n+1} has only one more point than π^n, say u_n. To fix further the setting, let us consider partitions π_n of $[0,1]$ by dyadic times, where we exhaust first all the dyadic times multiples of 2^{-k}, in any order, before taking in the partition points multiples of $2^{-(k+1)}$. Two dyadic times s and t being given, both multiples of 2^{-k_0}, take n big enough for them to be points of π^n. Then, denoting by u_n^-, u_n^+ the two points of π_n such that $u_n^- < u_n < u_n^+$, the quantity $\mu_{\pi_{ts}^{n+1}} - \mu_{\pi_{ts}^n}$ will either be null if $u_n \notin [s,t]$, or

$$\mu_{\pi_{ts}^{n+1}} - \mu_{\pi_{ts}^n} = \big(\mu_{u_n^+ u_n} + \mu_{u_n u_n^-}\big) - \mu_{u_n^+ u_n^-}, \tag{4.1.5}$$

otherwise. So we have

$$\Big|\mu_{\pi_{ts}^{n+1}} - \mu_{\pi_{ts}^n}\Big| \leqslant c_1\, 2^{-am},$$

where $\big|\pi^{n+1}\big| = 2^{-m}$. There will be 2^m such terms in the formal series $\sum_{n\geqslant 0}\big(\mu_{\pi_{ts}^{n+1}} - \mu_{\pi_{ts}^n}\big)$, giving a total contribution for these terms of size $2^{-(a-1)m}$, summable in m. So this sum converges to some quantity φ_{ts} which satisfies the additivity condition (4.1.4) by construction (on dyadic times only, as defined as above). Note that commutativity of the addition operation was used implicitly to write down equation (4.1.5). It is then elementary to see that φ enjoys the approximation properties (4.1.2) and (4.1.3) by a telescopic sum argument.

Let me emphasize that the non-commutative extension provided in [**3**], that suffices to construct the rough integral, cannot be used to get the results on flows given below in theorem 29. However, and somewhat surprisingly, the above approach also works in the non-commutative setting of maps from V to itself under a condition which essentially amounts to replacing the addition operation and the norm $|\cdot|$ in condition (4.1.1) by the composition operation and the $\mathcal{C}^1$ norm.

Given a family of maps $(\mu_{ts})_{0\leqslant s\leqslant t\leqslant T}$ from E to itself, and a partition $\pi_{ts} = \{s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t\}$ of (s,t), set

$$\mu_{\pi_{ts}} = \mu_{s_n s_{n-1}} \circ \cdots \circ \mu_{s_1 s_0}.$$

Theorem 29 ([P12]). Let $\big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ be a family of $\mathcal{C}^1$ maps from E to itself, depending continuously on (s,t) in the uniform topology, and enjoying the following two properties.

- **Perturbation of the identity** – There exists two positive constants α and ρ, with

$$0 < 1 - \rho < \alpha < 1,$$

such that the maps μ_{ts} are $(1+\rho)$-Lipschitz, and one has

$$D_x\mu_{ts} = \mathrm{Id} + A_x^{ts} + B_x^{ts}, \tag{4.1.6}$$

for all $x \in E$, for some L(E)-valued ρ-Lipschitz maps A^{ts} on E, with ρ-Lipschitz norm bounded above by $c|t-s|^\alpha$, and some L(E)-valued $\mathcal{C}^1$ bounded maps B^{ts} on E, with $\mathcal{C}^1$-norm bounded above by $o_{t-s}(1)$.

• $\mathcal{C}^1$**-approximate flow property** – There exists a positive constant c_1 and $a>1$, such that one has

$$\big\|\mu_{tu}\circ\mu_{us}-\mu_{ts}\big\|_{\mathcal{C}^1}\leqslant c_1|t-s|^a \tag{4.1.7}$$

for all $0\leqslant s\leqslant u\leqslant t\leqslant T$.

Under these conditions, there exist a positive constant δ and a unique flow of maps $\big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ on E such that

$$\big\|\varphi_{ts}-\mu_{ts}\big\|_\infty\leqslant|t-s|^a \tag{4.1.8}$$

holds for all $0\leqslant s\leqslant u\leqslant t\leqslant T$, with $t-s\leqslant\delta$; furthermore, we have

$$\big\|\varphi_{ts}-\mu_{\pi_{ts}}\big\|_\infty\leqslant 2\,c_1\,T\,\big|\pi_{ts}\big|^{a-1} \tag{4.1.9}$$

for any partition π_{ts} of any interval $(s,t)\subset[0,T]$, of mesh $\big|\pi_{ts}\big|\leqslant\delta$.

The crucial point in the above statement is the fact that if μ depends continuously in $\mathcal{C}^1$ on some parameter then φ also happens to depend continuously on that parameter, in $\mathcal{C}^0$, as a direct consequence of estimate (4.1.9). As explained in the next section and **[P12]**, theorem 29 can be seen as the cornerstone of the theory of rough differential equations, with the continuity of the Itô-Lyons solution map given as a consequence of the aforementioned continuity of φ on a parameter.

4.1.2. Flows driven by rough paths The point about the machinery of $\mathcal{C}^1$-approximate flows is that they actually pop up naturally in a number of situations, under the form of a local in time description of the dynamics under study; nothing else than a kind of Taylor expansion. This was quite clear in the example of the ordinary controlled differential equation

$$dx_t=V_i(x_t)\,dh_t^i,$$

with $\mathcal{C}^1$ real-valued controls $h^1,\dots,h^\ell$ and $\mathcal{C}_b^2$ vector fields $V_1,\dots,V_\ell$ in $\mathbb{R}^d$. The 1-step Euler scheme

$$\mu_{ts}(x)=x+\big(h_t^i-h_s^i\big)V_i(x)$$

defines in that case a $\mathcal{C}^1$-approximate flow which has the awaited Taylor-type expansion, in the sense that one has

$$f\big(\mu_{ts}(x)\big)=f(x)+\big(h_t^i-h_s^i\big)\big(V_if\big)(x)+O\big(|t-s|^{>1}\big)$$

for any function f of class $\mathcal{C}_b^2$; but μ fails to be a flow. Its associated flow (by our machinery) is not only a flow, it also satisfies equation (4.1.8) as a consequence of identity (4.1.2).

We shall proceed in a very similar way to give some meaning and solve the rough differential equation on flows

$$d\varphi=V\,dt+\mathrm{F}\mathbf{X}(dt), \tag{4.1.10}$$

where V is a Lipschitz continuous vector field on E and $\mathrm{F}=\big(V_1,\dots,V_\ell\big)$ is a collection of sufficiently regular vector fields on E, and $\mathbf{X}$ is a Hölder p-rough path over $\mathbb{R}^\ell$. A **solution flow** to equation (4.1.10) will be defined as a flow on E with a **uniform Taylor-Euler expansion** of the form

$$f\big(\varphi_{ts}(x)\big)=f(x)+\sum_{|I|\leqslant[p]}X_{ts}^I\big(V_If\big)(x)+O\big(|t-s|^{>1}\big), \tag{4.1.11}$$

where $I = (i_1, \dots, i_k) \in [\![1, \ell]\!]^k$ is a multi-index with size $k \leqslant [p]$, and X^I_{ts} stands for the coordinates of $\mathbf{X}_{ts}$ in the canonical basis of the truncated tensor product space $T_\ell^{([p]),1}$ over $\mathbb{R}^\ell$. The vector field V_i is seen here as a 1^{st}-order differential operator, and $V_I = V_{i_1} \cdots V_{i_k}$ as the k^{th}-order differential operator obtained by applying successively the operators V_{i_n}.

For $V = 0$ and $\mathbf{X}$ the weak geometric Hölder p-rough path canonically associated with an $\mathbb{R}^\ell$-valued $\mathcal{C}^1$ control h, with $2 \leqslant p < 3$, equation (4.1.11) becomes

$$f\big(\varphi_{ts}(x)\big) = f(x) + \big(h^i_t - h^i_s\big)\big(V_i f\big)(x) + \left(\int_s^t \int_s^r dh^j_u\, dh^k_r\right) \big(V_j V_k f\big)(x) + O\big(|t-s|^{>2}\big)$$

which is nothing else than Taylor formula at order 2 for the solution to the ordinary differential equation (4.1.2) started at x at time s. Condition (4.1.11) is a natural analogue of (4.1.2) and its higher order analogues.

There is actually a simple way of constructing a map μ_{ts} which satisfies the Euler expansion (4.1.11). It can be defined as the time 1 map associated with an ordinary differential equation constructed form the V_i and their brackets, and where $\mathbf{X}_{ts}$ appears as a parameter under the form of its logarithm. That these maps μ_{ts} form a $\mathcal{C}^1$-approximate flow will eventually appear as a consequence of the fact that the time 1 map of a differential equation formally behaves as an exponential map, in some algebraic sense. In the sequel, p and γ will always denote real numbers such that

$$1 \leqslant p < \gamma \leqslant [p] + 1.$$

Given a bounded Lipschitz continuous vector field V, and some $[\gamma]$-Lipschitz vector fields $V_1, \dots, V_\ell$ on E, let μ_{ts} be the well-defined time 1 map associated with the ordinary differential equation

$$\dot{y}_u = (t-s)V(y_u) + \sum_{r=1}^{[p]} \sum_{I \in [\![1,\ell]\!]^r} \Lambda^{r,I}_{ts} V_{[I]}(y_u), \quad 0 \leqslant u \leqslant 1. \tag{4.1.12}$$

For $2 \leqslant p < 3$, and $\mathbf{X} = (X, \mathbb{X})$, equation (4.1.12) reads

$$\dot{y}_u = (t-s)V(y_u) + X^i_{ts} V_i(y_u) + \frac{1}{2}\Big\{\mathbb{X}_{ts} + \frac{1}{2} X_{ts} \otimes X_{ts}\Big\}^{jk} [V_j, V_k](y_u).$$

As the matrix $X_{ts} \otimes X_{ts}$ is symmetric, we have $X^j_{ts} X^k_{ts}\, [V_j, V_k] = 0$, so (4.1.12) simplifies into

$$\dot{y}_u = (t-s)V(y_u) + X^i_{ts} V_i(y_u) + \frac{1}{2}\mathbb{X}^{jk}_{ts} [V_j, V_k](y_u).$$

Guided by the insightfull work of Davie [**1**], we adopt the following definition of a solution flow to a rough differential equation, in terms of uniform Euler-Taylor expansion property.

Definition. Let $2 \leqslant p < \gamma < [p] + 1$ be given. Let $V_1, \dots, V_\ell$ be $\mathcal{C}^{[p]}_b$-Lipschitz vector fields on E, and $\mathbf{X}$ be a Hölder weak geometric p-rough path. Write F for $(V_1, \dots, V_\ell)$. Let V be a bounded Lipschitz continuous vector field on E. With the above notations, a ***flow*** $(\varphi_{ts}\,;\, 0 \leqslant s \leqslant t \leqslant T)$ is said to **solve the rough differential equation**

$$d\varphi = V\,dt + \mathrm{F}\,\mathbf{X}(dt)$$

if there exist a constant $a > 1$ independent of $\mathbf{X}$ and a possibly $\mathbf{X}$-dependent positive constant δ such that

$$\big\|\varphi_{ts} - \mu_{ts}\big\|_\infty \leqslant |t-s|^a$$

holds for all $0 \leqslant s \leqslant t \leqslant T$ with $t - s \leqslant \delta$.

The well-posed character of the rough differential equation (4.1.2) is then a direct consequence of theorem 29 on the approximate flows and the following result.

Theorem 30 ([P12]). Suppose V is $(1+\rho)$-Lipschitz, for some $\rho > \frac{p-[p]}{p}$, and the V_i are γ-Lipschitz, then $\big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ is a $\mathcal{C}^1$-approximate flow.

It provides for free a strong version of Lyons' universal limit theorem.

Corollary 31 ([P12]). Suppose V is $(1+\rho)$-Lipschitz, for some $\rho > \frac{p-[p]}{p}$, and the V_i are γ-Lipschitz. Then the rough differential equation

$$d\varphi = V dt + \mathrm{F}\,\mathbf{X}(dt)$$

has a unique solution flow; it takes values in the space of uniformly Lipschitz continuous homeomorphisms of E with uniformly Lipschitz continuous inverses, and depends continuously on $\mathbf{X}$.

The crucial continuous dependence of the solution flow with respect to the driving rough path $\mathbf{X}$ comes as a direct consequence of identity (4.1.9) giving φ as a limit of compositions of μ_{ba}'s. As shown in **[P13]**, this approach works as efficiently with Banach space-valued rough paths as drivers, as with finite dimensional rough paths, as described here. The present approach has the advantage to give for free some higher order Taylor expansion of a solution flow to a rough differential equation, if ever the driving vector fields are more regular than the minimal regularity required in the above statements. The machinery of $\mathcal{C}^1$-approximate flows can also be refined to deal in applications with dynamics with exponential growth, such as linear dynamics, or dynamics of derivative flows. This requires the introduction of the notion of local $\mathcal{C}^1$-approximate flow with exponential growth introduced in **[P12]**, and used there to get a well-posedness result for rough differential equations whose driving vector fields with linear growth and bounded derivatives. In any case, the flow-based approach to rough differential equations gives back the classical results on E-valued paths solutions to such equations, under almost minimal regularity assumption on the vector fields.

4.1.3. Mean field rough differential equations We illustrate in this section how our point of view on rough differential can be used to study some simple mean field stochastic rough differential equations, such as done in **[P12]**. This kind of dynamics pops in naturally in the study of the large population limit of some classes of interacting random evolutions. The interaction holds through the dependence of the local characteristics of the random motion of each particle on the empirical measure of the whole family of particles. In a diffusion setting, each particle i would satisfy a stochastic differential equation of the form

$$dx_t^{(i)} = b\Big(x_t^{(i)}, \mu_t^N\Big)dt + \sigma\Big(x_t^{(i)}, \mu_t^N\Big)dB_t^{(i)},$$

where $\mu_t^N = \frac{1}{N}\sum_{k=1}^N \delta_{x_t^{(k)}}$. A large industry has been devoted to showing that the limit distribution in paths space of a typical particle of the system when N tends to infinity has a dynamics of the form

$$dx_t = b\big(x_t, \mathcal{L}(x_t)\big)dt + \sigma\big(x_t, \mathcal{L}(x_t)\big)dB_t, \tag{4.1.13}$$

where $\mathcal{L}(x_t)$ stands for the law of x_t. Theorem 32 provides a well-posedness result for such a limit equation, in the context of rough differential equations. As emphasized in [**4**], almost all the works in this area are set in the framework of a filtered probability space and rely crucially on some martingale arguments. On the other hand, the increasing importance of non-semi-martingale processes, like fractional Brownian motion, makes it desirable to have some more flexible tools to investigate equation (4.1.13) in such contexts. The flow approach to the theory of rough differential equations provides a nice framework for that.

A few notations are needed to set the probem. Given $2 \leqslant p < \gamma \leqslant [p]+1$, we equip the set $\mathcal{M}_1\big(\mathbb{R}^d\big)$ of probability measures on $\mathbb{R}^d$ with the metric induced by its embedding in the dual of $\mathcal{C}^\gamma\big(\mathbb{R}^d\big)$:

$$\mathrm{d}(\mu,\nu) = \sup\Big\{ (g,\mu) - (g,\nu)\,;\, g \in \mathcal{C}^\gamma\big(\mathbb{R}^d\big),\ \|g\|_\gamma \leqslant 1 \Big\}.$$

This metric topology is stronger than the weak convergence topology. Given any positive constant m, note that the set $\mathrm{Lip}(m)$ of Lipschitz continuous paths from $[0,T]$ to $\big(\mathcal{M}_1\big(\mathbb{R}^d\big),\mathrm{d}\big)$, with Lipschitz constant no greater than m, is closed under the norm of uniform convergence, so it is a Banach space.

Fix a finite positive time horizon T, and suppose $\mathbf{X}$ is a random variable defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, with values in the set of Hölder weak geometric p-rough paths over $\mathbb{R}^\ell$, on the time interval $[0,T]$; write $\mathbf{X} = 1 \oplus X^1 \oplus \cdots \oplus X^{[p]}$, and $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ for the filtration generated by $\mathbf{X}$, with $\mathcal{F}_t = \sigma\big(\mathbf{X}_{vu}\,;\, 0 \leqslant u \leqslant v \leqslant t\big)$. Let $V, V_1, \dots, V_\ell : \mathbb{R}^d \times \mathcal{M}_1\big(\mathbb{R}^d\big) \to \mathbb{R}^d$ be measure-dependent vector fields on $\mathbb{R}^d$, and F stands for the collection $(V_1,\dots,V_\ell)$ – or the associated $\mathbb{R}^d$-valued one-form on $\mathbb{R}^\ell$. Given a Lipschitz continuous path $\mathcal{P} = (P_t)_{0\leqslant t\leqslant T}$ in $\big(\mathcal{M}_1\big(\mathbb{R}^d\big),\mathrm{d}\big)$ and $\omega \in \Omega$, denote by $x_\bullet(\omega)$ the unique solution (under appropriate conditions) to the rough differential equation on paths

$$dx_t = V(x_t,P_t)dt + \mathrm{F}(x_t,P_t)\,\mathbf{X}(dt), \tag{4.1.14}$$

where x_0 may be an integrable random variable independent of $\mathbf{X}$. Denote by $\Phi(\mathcal{P})_t$ the law of x_t.

Definition. A solution $x_\bullet$ to equation (4.1.14) for which

$$\Phi(\mathcal{P})_t = P_t$$

for all $0 \leqslant t \leqslant T$, is said to be a solution of the **mean field stochastic rough differential equation**

$$dx_t = V\big(x_t,\mathcal{L}(x_t)\big)dt + \mathrm{F}\big(x_t,\mathcal{L}(x_t)\big)\,\mathbf{X}(dt).$$

Theorem 32 below provides conditions on the vector fields V, F and the rough path $\mathbf{X}$ under which existence and uniqueness for solutions of this equation can be proved.

Theorem 32 ([P12]). Given any $P \in \mathcal{M}_1\big(\mathbb{R}^d\big)$, we assume that $V(\cdot,P)$ is of class $\mathcal{C}^{2+[p]}$, with associated norm no greated than λ, and that the vector fields $V_i(\cdot,P)$ are of class $\mathcal{C}^{2[p]+1}$, with associated norms uniformly bounded with respect to P, and that they satisfy the inequalities

$$\max_{i=1..\ell} \big\|V_i(\cdot,P) - V_i(\cdot,Q)\big\|_\infty \vee \big\|V(\cdot,P) - V(\cdot,Q)\big\|_\infty \leqslant \lambda\, d(P,Q), \tag{4.1.15}$$

for all $P,Q \in \mathcal{M}_1\big(\mathbb{R}^d\big)$.

(1) Assume the vector fields $V_i(\cdot, P) = V_i(\cdot)$ do not depend on their $\mathcal{M}_1(\mathbb{R}^d)$-component, and the polynomial moment condition

$$\mathbb{E}\Big[\|\mathbf{X}\|^{([p]+1)^2}\Big] < \infty. \tag{4.1.16}$$

Then the map Φ has a unique fixed point in $\mathrm{Lip}(m)$, for any positive constant m; it depends continuously on the law of $\mathbf{X}$.

(2) In the general case where the vector fields $V_i(\cdot, P)$ are allowed to depend on their $\mathcal{M}_1(\mathbb{R}^d)$-component, assume that the random variables $\big(X_{ba}^{r,I}\big)_{r=1..[p],\, I\in[\![1,\ell]\!]^r}$ are integrable and that there exists for each $0 \leqslant a \leqslant T$, a positive random variable C_a, such that each of them satisfies the inequality

$$\Big|\mathbb{E}\Big[X_{ba}^{r,I}\,\Big|\,\mathcal{F}_a\Big]\Big| \leqslant C_a(b-a), \tag{4.1.17}$$

for all $0 \leqslant a \leqslant b \leqslant T$, with $\sup_{0\leqslant a\leqslant T} \mathbb{E}[C_a] < \infty$. Then one can choose m big enough so that the map Φ has a fixed point in $\mathrm{Lip}(m)$.

Assumption (4.1.17) holds for instance for the Brownian rough path, the rough path above an Ornstein-Uhlenbeck process, or fractional Brownian motion with Hurst index no smaller than $\frac{1}{2}$. The above regularity assumptions on V and the V_i ensure that the solution flow to the rough differential equation (4.1.14) is of class $\mathcal{C}^{[p]}$, with polynomial bounds on the size of its derivatives, in terms of $\|\mathbf{X}\|$. The moment condition (4.1.16) ensures the integrability of all these bounds.

So far there has been only one other work dealing with mean field stochastic rough differential equations, by Cass and Lyons [**4**]. They prove a well-posedness result under more restrictive assumptions on the vector fields and the driving rough path, asking for a *linear* mean field interaction in the drift and no mean field interaction in the V_i's, and requiring exponential moments for the accumulated local variation of $\mathbf{X}$. As they work with Wassersein distance on $\mathcal{M}_1\big(\mathbb{R}^d\big)$, our results and theirs are not directly comparable, but the assumptions of theorem 32 are significantly weaker when both settings apply. This covers in particular the case of Gaussian p-rough paths, with $2 \leqslant p < 4$.

4.2 Rough and stochastic flows

4.2.1. Rough drivers and rough flows Let $2 \leqslant p < 3$ be given, and $\big(V(\cdot,t)\big)_{0\leqslant t\leqslant T}$ be a time dependent vector field on E, with time increments

$$V_{ts}(\cdot) := V(\cdot,t) - V(\cdot,s).$$

To get a hand on the definition of a weak geometric p-rough driver given below, think of V_{ts} as given by the formula

$$V_{ts} = V X_{ts}, \tag{4.2.1}$$

where $V(x) \in \mathrm{L}(\mathbb{R}^\ell, \mathbb{R}^d)$, for all $x \in \mathbb{R}^d$, and $\mathbf{X} = (X, \mathbb{X})$ is a p-rough path over $\mathbb{R}^\ell$. Write V_i for the image by V of the i^{th} vector in the canonical basis of $\mathbb{R}^\ell$. A solution path $x_\bullet$ to the rough differential equation

$$dx_t = V(x_t)\,\mathbf{X}(dt)$$

can be characterized as a path satisfying some uniform Euler-Taylor expansion of the form

$$f(x_t) = f(x_s) + X^i_{ts}\big(V_i f\big)(x_s) + \mathbb{X}^{jk}_{ts}\big(V_j V_k f\big)(x_s) + O\Big(|t-s|^{\frac{3}{p}}\Big)$$

for all sufficiently regular real-valued function f on $\mathbb{R}^d$. The present section will make it clear that the operators $X^i_{ts}V_i = VX_{ts}$ and $\mathbb{X}^{jk}_{ts}V_jV_k = (DV)V\,\mathbb{X}_{ts}$ are all we need in this formula to run the theory, with no need to separate their spatial part, given by V and $(DV)V$, from their temporal part $\mathbf{X}_{ts}$.

Definition. Let $2 \leqslant p < 3$ be given. A weak geometric p-**rough driver** is a family $\big(\mathbf{V}_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$, with

$$\mathbf{V}_{ts} := \big(V_{ts}, \mathbb{V}_{ts}\big),$$

and $\mathbb{V}_{ts}$ a second order differential operator, such that

(i) the vector fields V_{ts} are $\mathcal{C}^3$, with

$$\sup_{0\leqslant s<t\leqslant T} \frac{\big\|V_{ts}\big\|_{\mathcal{C}^3}}{|t-s|^{\frac{1}{p}}} < \infty,$$

(ii) the second order differential operators

$$W_{ts} := \mathbb{V}_{ts} - \frac{1}{2}V_{ts}V_{ts},$$

are actually a vector fields, and

$$\sup_{0\leqslant s<t\leqslant T} \frac{\big\|W_{ts}\big\|_{\mathcal{C}^2}}{|t-s|^{\frac{2}{p}}} < \infty,$$

(iii) we have

$$\mathbb{V}_{ts} = \mathbb{V}_{tu} + V_{us}V_{tu} + \mathbb{V}_{us},$$

for any $0 \leqslant s \leqslant u \leqslant t \leqslant T$.

With in mind the model weak geometric p-rough driver given by formula (4.2.1), the first order condition on the operators W_{ts} justifies that we call $\mathbf{V}$ a *weak geometric* p-rough driver, and condition **(iii)** stands for an analogue of *Chen's relation.* We shall freely talk about rough drivers rather than weak geometric p-rough drivers in the sequel.

Definition. We define the norm of $\mathbf{V}$ to be

$$\|\mathbf{V}\| := \sup_{0\leqslant s<t\leqslant T} \left\{ \frac{\big\|V_{ts}\big\|_{\mathcal{C}^3}}{|t-s|^{\frac{1}{p}}} \vee \frac{\big\|W_{ts}\big\|_{\mathcal{C}^2}}{|t-s|^{\frac{2}{p}}} \right\},$$

and define an associated metric on the set $\mathcal{D}_{p,\rho}$ of weak geometric (p,ρ)-rough drivers setting

$$d(\mathbf{V}, \mathbf{V}') = \big\|\mathbf{V} - \mathbf{V}'\big\|.$$

Like the space of rough paths the space of rough drivers is not a linear space.

We shall adopt below a definition of a solution flow to the equation

$$d\varphi = \mathbf{V}(\varphi\,; dt)$$

similar to the above definition of a solution path to a rough differential equation. A solution flow will be required to satisfy some uniform Euler-Taylor expansion of the form

$$f \circ \varphi_{ts} - \big\{ f + V_{ts} f + \mathbb{V}_{ts} \big\} = O\Big(|t-s|^{\frac{3}{p}} \Big),$$

for all sufficiently regular real-valued functions f on $\mathbb{R}^d$. It is actually elementary to construct a family of maps $\big(\mu_{ts}\big)_{0 \leqslant s \leqslant t \leqslant T}$ which enjoys the above Euler-Taylor expansion property. The key point is that this family will turn out to be a $\mathcal{C}^1$-approximate flow, so we shall get the existence and uniqueness of a solution flow from its very definition and theorem 29.

Given $0 \leqslant s \leqslant t \leqslant T$, consider the ordinary differential equation

$$\dot{y}_u = V_{ts}(y_u) + W_{ts}(y_u), \quad 0 \leqslant u \leqslant 1, \tag{4.2.2}$$

with well-defined time 1 map μ_{ts} associating to $x \in \mathrm{E}$ the value at time 1 of the solution of (4.2.2) started from x.

Theorem 33 ([P16]). The family μ enjoys the following two fundamental properties.

- **Euler-Taylor expansion.** We have
$$\Big\| f \circ \mu_{ts} - \big\{ f + V_{ts} f + \mathbb{V}_{ts} f \big\} \Big\|_{\infty} \leqslant c \, \|f\|_{\mathcal{C}^3} |t-s|^{\frac{3}{p}},$$
for any $f \in \mathcal{C}^3$ and any $0 \leqslant s \leqslant t \leqslant T$.
- **$\mathcal{C}^1$-approximate flow.** The family of maps $\big(\mu_{ts}\big)_{0 \leqslant s \leqslant t \leqslant T}$ is a $\mathcal{C}^1$-approximate flow which depends continuously on $\big((s,t), \mathbf{V}\big)$ in $\mathcal{C}^0$ topology.

Guided by the first item in the above theorem, we give the following definition of a solution to an equation driven by a rough driver.

Definition. A **flow** $\big(\varphi_{ts}\big)_{0 \leqslant s \leqslant t \leqslant T}$ is said to **solve the rough differential equation**

$$d\varphi = \mathbf{V}(\varphi\,; dt) \tag{4.2.3}$$

if there exists a possibly $\mathbf{V}$-dependent positive constant δ such that the inequality

$$\big\| \varphi_{ts} - \mu_{ts} \big\|_{\infty} \leqslant |t-s|^{\frac{3}{p}}$$

holds for all $0 \leqslant s \leqslant t \leqslant T$ with $t - s \leqslant \delta$. Flows solving a differential equation of the form (4.2.3) are called **rough flows**. If equation (4.2.3) is well-posed, the map which associates to a rough driver $\mathbf{V}$ the solution flow to equation (4.2.3) is called the **Itô map**.

The following well-posedness result comes as a direct consequence of theorem 29 on approximate flows and the above theorem 33.

Theorem 34 ([P16]). The differential equation on flows

$$d\varphi = \mathbf{V}(\varphi\,; dt)$$

has a unique solution flow; it takes values in the space of uniformly Lipschitz homeomorphisms of E, with uniformly Lipschitz inverses, and depends continuously on $\mathbf{V}$ in the topology of uniform convergence.

Despite its simplicity, the setting of rough drivers and rough flows appears as extremely useful, as the next two sections will make it clear. We shall indeed see that one can embed the theory of semimartingale stochastic flows of maps into the theory of rough flows, with great benefits; it also provides an ideal environment to investigate a number of problems about homogenization for fast-slow systems.

4.2.2. Semimartingale stochastic flows of maps The theory of stochastic flows grew out of the pioneering works of the Russian school [**5, 6**] on the dependence of solutions to stochastic differential equations with respect to parameters and the proof by Bismut [**7**] and Kunita [**8**] that stochastic differential equations generate continuous flows of diffeomorphisms under proper regularity conditions on the driving vector fields. The Brownian character of these random flows, that is the fact that they are continuous with stationnary and independent increments, was inherited from the Brownian character of their driving noise. The next natural step consisted in the study of Brownian flows for themselves. After the works of Harris [**9**], Baxendale [**10**] and Le Jan [**11**], they appeared to be generated by stochastic differential equations driven by infinitely many Brownian motions, or better, to be in one-to-one correspondence with vector field-valued Brownian motions. A probabilistic integration theory of such random time-varying velocity fields was developed to establish that correspondence, and it was extended by Le Jan and Watanabe [**12**] to a large class of continuous semimartingale flows and continuous semimartingale velocity fields. Kunita [**13, 14**] studied the problem of convergence of stochastic flows, with applications to averaging and homogenization results, and promoted the use of stochastic flows to implement a version of the characteristic method in the setting of first of second order stochastic partial differential equations, notably those coming from the nonlinear filtering theory.

Semimartingale stochastic flows of maps are random flows of maps whose 2-point motions are semimartingales whose bounded variation part and bracket satisfy some reasonnable regularity conditions. Their infinitesimal counterpart is given by the notion of semimartingale velocity field, which is basically a field of semimartingales indexed by (a domain of) $\mathbb{R}^d$, say, which is regular enough as a function of $x \in \mathbb{R}^d$, with a reasonnable bracket between the processes based at two different points. An Itô and Stratonovich integration theory of such velocity fields was constructed in the 80's and a one-to-one correspondence between velocity fields and flows was proved. In the joint work with S. Riedel **[P16]**, we proved that (a nice class of) semimartingale velocity fields can be lifted to p-rough drivers. The benefits of such a fact are similar to what happens in the setting of rough differential equations. Semimartingale stochastic flows of maps can be constructed in a two step process: In a first and purely probabilistic step, one lifts a semimartingale velocity field to a p-rough driver, while the dynamics itself is constructed in a second and purely deterministic step, with the crucial continuity of the Itô map in hands. As we were able to prove that rough driver lift of the continuous, piecewise linear time-interpolation of a (semi)martingale velocity field V converge in the lift $\mathbf{V}$ of V to a rough driver, the continuity of the Itô map gives for free a **Wong-Zakai theorem for semimartingale stochastic flows**, giving them as limits (in probability) of flows associated with ordinary differential equations. **Support** and **large deviation theorems for Brownian flows** of maps were also proved in that work as consequences of support and large deviation results for the rough driver lift of Brownian velocity fields. We refer the reader to **[P16]** for more details.

What next? The work **[P16]** opens a number of very interesting perspectives that I am currently investigating with S. Riedel. First, following the development of rough paths theory, we are building an integration theory for Gaussian rough drivers that will extend the theory of stochastic flows beyond its present (semimartingale) limits, with support and large deviation theorems without much efforts. In a different direction, our work offers the possibility to look at the solution flow of a rough differential equation driven by a rough driver as a random dynamical system, in the line of L. Arnold' school [**15**]. We are currently investigating the associated theory of invariant manifolds. Our results on semimartingale stochastic flows of maps already give a straightforward road to deep results of Mohammed and Scheutzow [**16**] on that question.

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Chapter 5

Beyond rough paths theory

STIMULATED by the recent breakthrough of Hairer [**2**] and Gubinelli-Imkeller-Perkowski [**3**], and motived by some applications to problems in infinite dimensional stochastic differential geometry to be investigated soon, I have recently turned to the study of singular partial differential equations (PDEs), along two lines of attack. In **[P18]**, together with Massimiliano Gubinelli, we started a program of developing a general theory of rough PDEs aiming at extending classical PDE tools such as weak solutions, a priori estimates, compactness results, duality. This is quite an unexplored territory where few tools are available, so as a start, we only paid attention to linear symmetric hyperbolic systems of the form

$$\partial_t f + a\nabla f = 0, \tag{5.0.1}$$

where f is an $\mathbb{R}^N$-valued space-time distribution on $\mathbb{R}_+ \times \mathbb{R}^d$, and $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathrm{L}(\mathbb{R}^d, \mathbb{R}^{N\times N})$ is a $N \times N$ matrix-valued family of time–dependent vector fields in $\mathbb{R}^d$. This setting includes as a particular case scalar transport equations. At a heuristic level, note that a formal integration of the weak formulation (5.0.1) over any time interval $[s,t]$, gives an equation of the form

$$f_t = f_s + \int_s^t V_r f_r dr,$$

where $V_r = -a_r\nabla$ is a matrix-valued vector-field and $f_r(x) = f(r,x)$, is a convenient notation of the distribution f evaluated at time r, assuming this make sense. An expansion for the time evolution of f is obtained by iterating the above equation, and reads

$$f_t = f_s + A^1_{ts} f_s + A^2_{ts} f_s + R_{ts}, \tag{5.0.2}$$

where

$$A^1_{ts} = \int_s^t V_r dr, \qquad \text{and} \qquad A^2_{ts} = \int_s^t \int_s^r V_r V_{r'} dr' dr,$$

are respectively a first order differential operator (that is a vector field) and a second order differential operator, for each $s \leqslant t$, and R_{ts} is a remainder term characterized by its Hölder size. As a function of (s,t), they satisfy formally *Chen's relation*

$$A^2_{ts} = A^2_{tu} + A^2_{us} + A^1_{tu} A^1_{us} \tag{5.0.3}$$

for all $0 \le s \le u \le t$. We recognize exactly here some objects which are dual to the rough drivers introduced in the work **[P16]**. There is a complete theory of such equations in the case where the equation is set in a Banach algebra and the operators A^1_{ts} and A^2_{ts} are given by left multiplication by some elements of the algebra. It is however natural, in a PDE setting, to consider also unbounded operators A^1, A^2, which makes the use of rough paths ideas non-trivial, unless we work in the analytic category or in similar topologies.

We layed out in **[P18]** a theory of such rough linear equations driven by unbounded drivers $\mathbf{A}$, and obtained some a priori estimates that were used to study the well–posedness of some classes of linear symmetric systems in $\mathbb{L}^2$ and of the rough transport equation in $\mathbb{L}^\infty$. A major difficulty which had to be overcome was the lack of a Grönwall lemma for rough equations and the main contribution to this paper was to develop suitable a priori estimates on weak controlled solutions that replace the use of Grönwall lemma in classical proofs. Another major difficulty was to adapt Lions-di Perna's method of renormalized solutions to that setting.

In another direction, together with F. Bernicot, we turned our attention to the method of paracontrolled distributions introduced in [**3**] for studying a class of parabolic singular PDEs on the torus or $\mathbb{R}^d$, of the form

$$\big(\partial_t + L\big)u = \mathrm{F}(u, \xi)$$

for a nonlinear term $\mathrm{F}(u, \xi)$. We introduced in the work **[P20]** a functional analytic setting which allows to extend the paracontrolled approach on potentially unbounded, Riemannian or even sub-Riemannian, manifolds or graphs. This is a priori far from obvious as the main analytic tool used in the paracontrolled approach in the torus involves some tools from Fourier analysis that do not make sense on manifolds or graphs. We developed to that end a functional calculus adapted to the heat semigroup associated with the operator $\big(\partial_t+L\big)$, which was used to define a paraproduct enjoying the same regularity properties as its Euclidean analogue. The irregular character of the noises ξ involved in the motivating equations from SPDEs required us to a newsemigroup based paraproduct for which one can deal with Hölder functions with a negative exponent. Building on these tools, we were able to set up, as in [**3**], a framework where to investigate the well-posed character of a whole class of parabolic singular PDEs. It is especially nice that all the objects in our framework are defined uniquely in terms of semigroups, unlike the notions of Hölder spaces used in the theory of regularity structures that involve a metric structure unrelated to the equation under study. We illustrated our machinery on the study of the (generalized) parabolic Anderson model equation (gPAM)

$$\big(\partial_t + L\big)u = \mathrm{F}(u)\,\xi,$$

with a time-independent white noise ξ, by proving a local/global in time well-posedness result in a manifold setting, for a rather general class of operators L.

5.1 Rough drivers

The work **[P18]** on rough drivers provides a dual point of view on the work **[P16]** on rough flows. It takes as a starting point the description of an evolution through the datum of an evolution equation on a function space of observables, as in the classical Eulerian-Lagrangian duality. In that setting, a rough driver is specified by a pair $\mathbf{A} = \big(A^1, A^2\big)$ of time-dependent operators on some function spaces, and a solution $(f_s)_{0\leqslant s\leqslant T}$ to the linear equation

$$df_s = \mathbf{A}(ds) f_s$$

is defined by duality as a path with values in some specified space, whose increments $\delta f_{ts} := f_t - f_s$, satisfy

$$\delta f_{ts}(\varphi) = f_s\Big(\big\{(A^1_{ts})^* + (A^2_{ts})^*\big\}\varphi\Big) + f^\sharp_{ts}(\varphi)$$

for a class of test functions φ, and a remainder which needs to be small in some Hölder size sense. The challenge in that setting is to set up a functional setting weak enough to have existence of solutions while retaining their unique character. This is especially easy in the setting of bounded rough drivers.

5.1.1. Linear differential equations with bounded rough drivers Let $\big(\mathcal{A}, |\cdot|\big)$ be a Banach algebra with unit $\mathbf{1}_\mathcal{A}$; one may think for instance to the space of continuous linear maps from some Hilbert space to itself, or to the truncated tensor algebra over some Banach space, equipped with a tensor norm and completed for that norm. We shall measure the size of an $\mathcal{A}$-valued path $f_\bullet$ by its Picard's norm

$$(\!| f_\bullet |\!) := \sup_{t\geqslant 0} e^{-\lambda^{-1}t}\,\big|f(t)\big|,$$

which depends implicitly on a positive parameter that can be tuned on demand.

Definition. Let $\frac{1}{3} < \gamma \leqslant \frac{1}{2}$. A ***bounded γ-rough driver in*** $\mathcal{A}$ is a pair $\mathbf{A} = \big(A^1, A^2\big)$ of $\mathcal{A}$-valued 2-index maps satisfying *Chen's relations*

$$\delta A^1 = 0, \qquad \text{and} \qquad \delta A^2_{tus} = A^1_{tu}A^1_{us}, \tag{5.1.1}$$

and such that A^1 is γ-Hölder and A^2 is 2γ-Hölder. The norm of $\mathbf{A}$ is defined by the formula

$$\|\mathbf{A}\| := \sup_{0\leqslant s<t<T\,;\,t-s\leqslant 1} \frac{\big|A^1_{ts}\big|}{|t-s|^\gamma} \vee \frac{\big|A^2_{ts}\big|}{|t-s|^{2\gamma}}.$$

As an elementary example, think of $\mathcal{A}$ as the truncated tensor algebra $\bigoplus_{i=0}^N (\mathbb{R}^\ell)^{\otimes i}$ over $\mathbb{R}^\ell$, for $N \geqslant 2$, and consider a weak geometric γ-Hölder rough path $\mathbf{X}_{ts} = 1 \oplus X_{ts} \oplus \mathbb{X}_{ts} \in \bigoplus_{i=0}^N (\mathbb{R}^\ell)^{\otimes i}$, with $\frac{1}{3} < \gamma \leqslant \frac{1}{2}$. Left multiplication by X_{ts} and $\mathbb{X}_{ts}$ define operators A^1 and A^2 that are the components of a rough driver.

Theorem 35 (Integration of bounded rough drivers **[P18]**)**.** Given any initial condition $f_0 \in \mathcal{A}$, there exists a unique γ-Hölder path $f_\bullet$ starting from f_0, and such that the formula

$$\delta f_{t,s} - \Big(A^1_{t,s} + A^2_{t,s}\Big) f_s$$

defines a 3γ-Hölder 2-index map $f^\sharp$. Moreover, the following estimate holds

$$(\!| f_\bullet |\!) \leqslant 2\,\big|f_0\big| \tag{5.1.2}$$

for all λ greater than some λ_0 depending only on $\|\mathbf{A}\|$ and γ. When $f_0 = \mathbf{1}_\mathcal{A}$, we will use the notation $f_t = e^{\mathbf{A}}_{t,0}$; the flow property

$$e^{\mathbf{A}}_{t,s} = e^{\mathbf{A}}_{t,u} e^{\mathbf{A}}_{u,s},$$

holds for all $0 \leqslant s \leqslant u \leqslant t < T$.

Applied to the above example of rough driver in the truncated tensor product space, this well–posedness result provides a proof of Lyons' extension theorem and of the fact that there is a unique rough path associated to an almost rough path [**1**], two of the core results of rough paths theory in his original formulation. This result applies in the particular case where $\mathcal{A}$ is the Banach algebra of bounded operators on an Hilbert space H. The main object of the work **[P18]** was to study the integration problem

$$\delta f_{ts} = \Big(A^1_{ts} + A^2_{ts}\Big) f_s + f^\sharp_{ts}, \tag{5.1.3}$$

for a particular class of drivers $\mathbf{A}$ associated to a class of *unbounded* operators on H, or other Banach spaces, with in mind the model case of the **rough transport equation**

$$\delta f(\varphi)_{ts} = X_{ts}\, f_s\big(V^*\varphi\big) + \mathbb{X}_{ts}\, f_s\big(V^*V^*\varphi\big) + f^\sharp_{ts}(\varphi),$$

where $\mathbf{X} = (X, \mathbb{X})$ is an ℓ-dimensional γ-Hölder rough path and $V = \big(V_1, \dots, V_\ell\big)$ is a collection of ℓ vector fields on $\mathbb{R}^d$.

5.1.2. Unbounded rough drivers and rough differential equations To make sense of equation (5.1.3), one needs to complete the funtional setting by the datum of a scale of Banach spaces $\big(E_n, |\cdot|_n\big)_{n\geqslant 0}$, with E_{n+1} continuously embedded in E_n. For $n \geqslant 0$, we shall denote by $E_{-n} = E_n^*$ the dual space of E_n, equipped with its natural norm,

$$|e|_{-n} := \sup_{\varphi\in E_n,\, |\varphi|_n\leqslant 1} (\varphi, e), \qquad e \in E_{-n}.$$

We require that the following continuous inclusions

$$E_n \subset \cdots \subset E_2 \subset E_1 \subset E_0$$

hold for all $n \geqslant 2$. One can think of n as quantifying the 'regularity' of elements of some test functions, with the elements of E_n being more regular with n increasing.

Definition. Let $\frac{1}{3} < \gamma \leqslant \frac{1}{2}$ be given. An ***unbounded γ-rough driver on the scales*** $\big(E_n, |\cdot|_n\big)_{n\geqslant 0}$, is a pair $\mathbf{A} = \big(A^1, A^2\big)$ of 2-index maps, with

$$\begin{aligned} A^1_{ts} &\in L\big(E_n, E_{n-1}\big), \text{ for } n \in \{-0, -2\}, \\ A^2_{ts} &\in L\big(E_n, E_{n-2}\big), \text{ for } n \in \{-0, -1\}, \end{aligned} \tag{5.1.4}$$

for all $0 \leqslant s \leqslant t < T$, which satisfy Chen's relations (5.1.1), and such that A^1 is γ-Hölder and A^2 is 2γ-Hölder.

Denote by $\big\|B\big\|_{\gamma\,;(-i,-j)}$ the γ-Hölder norm of a time-dependent continuous operator from E_{-i} to E_{-j}, and set

$$C_0 := \big\|A^1\big\|_{\gamma\,;(-0,-1)} + \big\|A^2\big\|_{2\gamma\,;(-0,-2)} + \big\|A^1\big\|_{\gamma\,;(-2,-3)} + \big\|A^2\big\|_{2\gamma\,;(-1,-3)} < \infty. \tag{5.1.5}$$

Equipped with the above definition of a rough driver, one can make precise what we mean by a solution to equation (5.1.3).

Definition. An E_{-0}-valued ***path $f_\bullet$*** is said to ***solve the linear rough differential equation***

$$df_s = \mathbf{A}(ds) f_s \tag{5.1.6}$$

if there exists an E_{-2}-valued 2-index map $f^\sharp$ such that one has

$$\delta f_{ts}(\varphi) = f_s\Big(\big\{(A^1_{ts})^* + (A^2_{ts})^*\big\}\varphi\Big) + f^\sharp_{ts}(\varphi) \tag{5.1.7}$$

for all $0 \leqslant s \leqslant t < T$ and all $\varphi \in E_2$, and the map $f^\sharp_{ts}(\varphi)$ is 3γ-Hölder, for each $\varphi \in E_2$.

Our workhorse for investigating existence and well-posedness for the linear rough differential equation (5.1.6) was the following a priori estimate, the proof of which requires an additional ingredient in the functional setting, the existence of a 1-parameter family of smoothing operators satisfying classical size requirements in terms of the parameter. Set, for some Banach space valued path a,

$$(5.1.8) \qquad \llparenthesis\, a \,\rrparenthesis_\gamma := \sup_{0 \leqslant s < t < T} e^{-\lambda^{-1} t} \, \frac{|a_{ts}|}{|t-s|^\gamma_\lambda},$$

and $\llparenthesis\, a \,\rrparenthesis_{\gamma\,;-i}$, when the Banach space is E_{-i}.

Theorem 36 ([P18]). Let $f_\bullet$ be a solution to the rough linear equation (5.1.6). For any $\lambda \leq 1$ such that $\lambda^\gamma C_0 \lesssim 1$, we have the a priori bounds

$$(5.1.9) \qquad \max\Big\{ \llparenthesis\, \delta f \,\rrparenthesis_{\gamma\,;-1}, \llparenthesis\, f^\flat \,\rrparenthesis_{2\gamma\,;-2}, \llparenthesis\, f^\sharp \,\rrparenthesis_{3\gamma\,;-3} \Big\} \lesssim_\gamma 2\, C_0\, \lambda^{-2\gamma}\, \llparenthesis\, f \,\rrparenthesis_{-0}.$$

This a priori bound was used to get some existence result for equations with a conservative rough driver, via an approximation and compactness result, the latter based on estimate (5.1.9). This situation corresponds in the model setting of the rough transport equation

$$\delta f_{ts} = \big(X_{ts} V + \mathbb{X}_{ts} VV \big) f_s + f^\sharp_{ts},$$

with a weak geometric Hölder p-rough path $\mathbf{X}$, and $2 \leqslant p < 3$, and a one-form $V = \big(V_1, \dots, V_\ell\big)$ given by some divergence-free vector fields V_i on $\mathbb{R}^d$. To get an existence and uniqueness result for more general classes of rough drivers, called **symmetric** rough drivers, we proved a crucial renormalization lemma that basically says that, under some additional reasonnable condition on the rough driver, if $f_\bullet$ is a solution to the rough differential equation (5.2.8), then the 2-variable function $f_\bullet^{\otimes 2}(x,y) = f_\bullet(x) f_\bullet(y)$, is a solution to another linear rough differential equation. One can conclude from the latter that $f_\bullet^2$ is the solution to an equation of the form

$$\delta f^2(\phi)_{ts} = \Big\langle f_s, \big(B^1_{ts}(\phi) + B^2_{ts}(\phi) \big) f_s \Big\rangle + O\Big(\|\phi\|_{W^{3,\infty}} |t-s|^{3\gamma} \Big),$$

for some operators

$$\begin{aligned} &\textbf{(i)} \quad B^1_{ts}(\phi) = (A^1_{ts})^* M_\phi + M_\phi A^1_{ts}, \\ &\textbf{(ii)} \quad B^2_{ts}(\phi) = M_\phi (A^2_{ts})^* + A^2_{ts} M_\phi + A^1_{ts} M_\phi (A^1_{ts})^*, \end{aligned}$$

where M_ϕ stands for the multiplication operator by ϕ. It follows for instance as a direct consequence that if $\mathbf{A}$ is conservative, that is if $B^1_{ts}(\mathbf{1}) = B^2_{ts}(\mathbf{1}) = 0$, then any solution to equation (5.2.8) has a preserved 0-norm, hence it is unique. Rough drivers $\mathbf{A}$ for which the operators B^1 and B^2 are associated to another rough driver $\mathbf{B} = \big(B^1, B^2\big)$ via the formula

$$\begin{aligned} B^1_{ts}(\phi) &= \big(B^1_{ts}\big)^* \phi, \\ B^2_{ts}(\phi) &= \big(B^2_{ts}\big)^* \phi, \end{aligned}$$

are called **closed**. Our second main contribution in **[P18]** was to prove that solution to linear rough differential equations driven by symmetric closed rough drivers, one has a substitute of a consequence of Grönwall's lemma in classical settings, under the form of an a priori estimate on the 0-size of f_t in terms only of the 0-size of f_0, the associated rough driver $\mathbf{B}$ and time. This is a crucial point as we are definitely missing any kind of

Grönwall-type lemma in the classical setting of rough paths theory. These results were used to investigate the well-posed character of the rough transport equation in an L^2 and L^∞ setting. Stability results were also proved, as awaited from the classical Lions-di Perna theory of transport equations.

5.2 Heat semigroups and singular PDEs

Following the recent breakthrough of Hairer [**2**] and Gubinelli, Imkeller, Perkowski [**3**], there has been recently a tremendous activity in the study of parabolic singular partial differential equations (PDEs), such as the KPZ equation

$$\big(\partial_t - \partial_x^2\big)u = \big(\partial_x u\big)^2 + \xi,$$

the stochastic quantization equation

$$\big(\partial_t - \Delta\big)u = -u^3 + \xi,$$

or the Parabolic Anderson Model equation

$$\big(\partial_t - \Delta\big)u = \mathrm{F}(u)\xi$$

in all of which ξ stands for a space or space-time white noise. Each of these equations involves, under the form of a product, a term which does not make sense a priori, given the expected regularity of the solution in terms of the regularity of the noise ξ. Hairer's theory of regularity structures is built on the insights of earlier works [**4, 5, 6**] on $(1+1)$-dimensional space-time problems where he used the framework of rough paths theory, under the form of Gubinelli's controlled paths, to make sense of previously ill-posed singular PDEs and give a meaningful solution theory. Rough paths theory was used in this approach as a framework for studying the properties in the 1-dimensional *space variable* of potential solutions. However, the very notion of a rough path is intimately linked with the 1-dimensional time axis that parametrizes paths.

To by-pass this barrier, both the theory of regularity structures and the paracontrolled approach developed in [**3**] take as a departure point the fact that, like in rough paths theory, to make sense of the equation, one needs to enrich the noise ξ into a finite collection of objects/distributions, and that one should try and describe the potential solution of a singular PDE in terms of that enriched noise. The latter depends on the equation under study and plays in the theory of regularity structures the role played by polynomials in the usual C^k world to give local descriptions of functions under the form of Taylor expansions at every space-time point. The description of a solution in the paracontrolled approach is of a different nature and rests on a global comparison with the solution to a linear equation ($\big(\partial_t - \Delta\big)u = \xi$, in the above examples) via the use of Bony's paraproduct. In both approaches, the use of an ansatz for the solution space allows for fixed point arguments to give a robust solution theory where the solution becomes a continuous function of all the parameters of the problem.

The paracontrolled approach to singular PDEs presents the advantage over the theory of regularity structures to require a minimum of new tools, for a working probabilist, and that one can borrow on a corpus of knowledge already available, about paraproduct analysis. It inherits from its strength a major drawback in so far as the use of Fourier analysis, at the heart of the paraproduct decomposition, imposes strong conditions on the geometric background in which one sets the singular PDE; it can be, at best, a homogeneous space, on which some kind of non-commutative Fourier analysis could be used. This prevents for instance the systematic study of discretization of singular PDEs from the controlled paraproduct approach if we are to keep this setting.

We developed in **[P20]** a Fourier-free approach to controlled paraproducts based on semigroup technics, developed in the geometrical setting of a Dirichlet space $\big(L^2(M,\mu),\mathcal{E}\big)$, where (M,d) is a metric space and μ is some Radon measure with the volume doubling property and the Dirichlet form $\mathcal{E}$ is strongly local and regular. We assume that the heat kernel of its associated semigroup satisfies some Gaussian upper bound and a similar Lipschitz continuity property. Large classes of Markov chains on graphs, or second order differential operators on Riemannian, or subRiemannian, manifolds satisfy these conditions.

5.2.1. Functional calculus adapted to a semigroup

The starting point of the classical paraproduct calculus is the decomposition of a distribution f as a sum of smooth functions, each of them defined in terms of Fourier multipliers supported on dyadic annuli, so

$$f = \sum_{j\geqslant 0} \Delta_j f, \tag{5.2.1}$$

with $\widehat{\Delta_j f}(\xi)$ supported on $\big\{|\xi|\simeq 2^j\big\}$. This decomposition in frequency space can be used to characterize classical Hölder spaces in terms of Besov norm of f, involving the speed of convergence to 0 of the L^p norm of $\Delta_j f$. The point is that this extended setting also makes sense with negative "Hölder" exponents. Now, if we are given two distributions f and g, we have the formal product

$$fg = \sum_{j,k\geqslant 0} \Delta_j f\,\Delta_k g = \sum_{|j-k|\geqslant 2} \Delta_j f\,\Delta_k g + \sum_{|j-k|\leqslant 1} \Delta_j f\,\Delta_k g.$$

Bony's crucial observation in his groundbreaking work **[7]** is that the first sum, over $\big\{|j-k|\geqslant 2\big\}$, always makes sense in some Hölder-Besov space if f and g are themselves Hölder-Besov regular, say in $\mathcal{C}^\alpha$ and $\mathcal{C}^\beta$ respectively. The resonant term $\sum_{|j-k|\leqslant 1}\Delta_j f\,\Delta_k g$, even make sense if $\alpha+\beta>0$. Together with a clever commutator estimate and a paralinearisation formula, this laid the foundations of an efficient calculus on distributions, where to make sense of product of two distributions and solve nonlinear equations in a clearly defined setting.

What can we do without any Fourier analysis?! The heuristic behind the semigroup approach to singular PDEs put forward in **[P20]** is that the semigroup P_t associated with the Dirichlet form provides us with a whole family of objects that can be used as Fourier multiplier. To make it clear, let us work on the torus with its usual Laplace operator. Then $\widehat{P_t f}$ is given by the multiplication with the Gaussian kernel $e^{-t\,|\xi|^2}$, and $\widehat{(tL)^a\,P_t f}$ is given, for any integer a, by the multiplication with the kernel $\big(t|\xi|^2\big)^a\,e^{-t\,|\xi|^2}$, with the latter function localized near frequencies $t^{-\frac{1}{2}}$. This is the function that plays the role in our setting of the 'perfect' Fourier multipliers used in the Paley-Littlewood theory. Note this time that it is no longer true that multipliers corresponding to well-separated frequencies have a null product. The latter fact is actually used in a crucial way to get such fundamental properties of Paley-Littlewood decomposition. Given an integer a, set

$$Q_t^{(a)} := (tL)^a e^{-tL}$$

and

$$P_t^{(a)} := \phi_a(tL), \qquad \text{where} \qquad \phi_a(x) := \frac{1}{\gamma_a}\int_x^\infty s^a e^{-s}\,\frac{ds}{s}, \quad x \geqslant 0,$$

for every $t > 0$. So we have for instance $P_t^{(1)} = e^{-tL}$, and $Q_t^{(1)} = tLe^{-tL}$. The two families of operators $\big(P_t^{(a)}\big)_{t>0}$ and $\big(Q_t^{(a)}\big)_{t>0}$ are defined so as to have

$$t\partial_t P_t^{(a)} = tL\phi_a'(tL) = -\gamma_a^{-1}\, Q_t^{(a)},$$

so $Q_t^{(a)} = (-1)^a t^a \partial_t^a e^{-tL}$, and $P_t^{(a)} = p_a(tL)e^{-tL}$, for some polynomial p_a of degree $a-1$, with $p_a(0) = 1$. Our starting point is the following analogue of the Paley-Littlewood decomposition (5.2.1), where we can understand the measure $\frac{dt}{t}$ as the analogue of the counting measure on the integers, i.e. a measure that roughly assigns a unit mass to any dyadic interval $\big[2^{-k-1}, 2^{-k}\big]$.

Proposition 37 (Calderón reproducing formula)**.** Given $p \in (1,+\infty)$ and $f \in L^p(M,\mu)$, we have

$$\lim_{t\to 0^+} P_t^{(a)} f = f \quad \text{ in } L^p(M,\mu)$$

for every positive integer a, and so

$$f = \gamma_a^{-1}\int_0^1 Q_t^{(a)} f\,\frac{dt}{t} + P_1^{(a)}(f). \tag{5.2.2}$$

Besov spaces $B^\sigma_{p,q}$ defined in terms of the operators $Q_t^{(a)}$ can be introduced, and give back the classical Hölder spaces for exponents in $(0,1)$. A Besov embedding $B^\sigma_{p,p} \hookrightarrow B^\sigma_{p,\infty} \hookrightarrow B^{\sigma-\frac{\nu}{p}}_{\infty,\infty} = \mathcal{C}^{\sigma-\frac{\nu}{p}}$, can even be proved under the additional mild assumption that the metric measure space (M,d,μ) is Ahlfors regular, with positive exponent ν. Rather than starting from identity (5.2.2) and writing the formal product of two distributions f and g as a double integral, we write

(5.2.3)

$$\begin{aligned} fg &= \lim_{t\to 0} P_t^{(b)}\Big(P_t^{(b)} f \cdot P_t^{(b)} g\Big) = -\int_0^1 t\partial_t\,\Big\{P_t^{(b)}\Big(P_t^{(b)} f\cdot P_t^{(b)} g\Big)\Big\}\,\frac{dt}{t} + \Delta_{-1}(f,g) \\ &= \frac{1}{\gamma_b}\int_0^\infty \Big\{P_t^{(b)}\Big(Q_t^{(b)} f\cdot P_t^{(b)} g\Big) + P_t^{(b)}\Big(P_t^{(b)} f\cdot Q_t^{(b)} g\Big) + Q_t^{(b)}\Big(P_t^{(b)} f\cdot P_t^{(b)} g\Big)\Big\}\,\frac{dt}{t} + \Delta_{-1}(f,g), \end{aligned}$$

where

$$\Delta_{-1}(f,g) := P_1^{(b)}\Big(P_1^{(b)} f\cdot P_1^{(b)} g\Big)$$

stands for the 'low-frequency part' of the product of f and g, and where we implicitly make the necessary assumptions on f and g for the above formula to make sense. Guided by the above heuristic argument about the role of the operators $P_t^{(a)}, Q_t^{(a)}$, etc. as frequency projectors, one can use the carré du champ operator to write down the product fg as a sum of the form

$$fg = \Pi_g^{(b)}(f) + \Pi_f^{(b)}(g) + \Pi^{(b)}(f,g) + \Delta_{-1}(f,g),$$

where

$$\begin{aligned}\Pi_g^{(b)}(f) &:= \frac{1}{\gamma_b}\int_0^1 \Big\{A_g(f) + B_g(f)\Big\}\,\frac{dt}{t} \\ &= \frac{1}{\gamma_b}\int_0^1 \Big\{(tL)P_t^{(b)}\Big(Q_t^{(b-1)}f\cdot P_t^{(b)}g\Big) + Q_t^{(b-1)}\Big((tL)P_t^{(b)}f\cdot P_t^{(b)}g\Big)\Big\}\,\frac{dt}{t},\end{aligned}$$

defines the paraproduct of f and g, or rather the modulation of g by f – this is not a symmetric formula, hence this non-symmetric name. The 'resonnant' term is then given by the seemingly unfriendly formula

$$\begin{aligned}\Pi^{(b)}(f,g) &= \frac{1}{\gamma_b}\int_0^1 \Big\{S(f,g) + S(g,f) + R(f,g)\Big\}\,\frac{dt}{t} \\ &= \frac{1}{\gamma_b}\int_0^1 \Big\{-P_t^{(b)}\Big(Q_t^{(b-1)}f\cdot (tL)P_t^{(b)}g\Big) + 2P_t^{(b)}\Gamma\Big(\sqrt{t}\,Q_t^{(b-1)}f, \sqrt{t}\,P_t^{(b)}g\Big)\Big\}\,\frac{dt}{t} \\ &\quad + \frac{1}{\gamma_b}\int_0^1 \Big\{-P_t^{(b)}\Big((tL)P_t^{(b)}f\cdot Q_t^{(b-1)}g\Big) + 2P_t^{(b)}\Gamma\Big(\sqrt{t}\,P_t^{(b)}f, \sqrt{t}\,Q_t^{(b-1)}g\Big)\Big\}\,\frac{dt}{t} \\ &\quad - \frac{1}{\gamma_b}\int_0^1 2Q_t^{(b-1)}\Gamma\Big(\sqrt{t}P_t^{(b)}f, \sqrt{t}P_t^{(b)}g\Big)\,\frac{dt}{t}.\end{aligned}$$

5.2.2. Paraproduct and commutator estimates and paracontolled calculus The crucial point about these definitions is that they are taylor-made to get the following paraproduct and commutator estimates that are the core of Bony's theory, together with the above-mentioned paralinearization statement.

Proposition 38 (Continuity estimates **[P20]**). We have the following continuity results for the paraproduct and the resonant term.

- Fix an integer $b \geqslant 2$. For any $\alpha, \beta \in \mathbb{R}$ and every $\gamma > 0$ we have for every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$

$$\big\|\Delta_{-1}(f,g)\big\|_{\mathcal{C}^\gamma} \lesssim \|f\|_{\mathcal{C}^\alpha}\|g\|_{\mathcal{C}^\beta}. \tag{5.2.4}$$

- Fix an integer $b \geqslant 2$. For any $\alpha \in (-2,1)$ and $f \in \mathcal{C}^\alpha$, we have
  - for every $g \in L^\infty$

$$\Big\|\Pi_g^{(b)}(f)\Big\|_{\mathcal{C}^\alpha} \lesssim \|g\|_\infty\|f\|_{\mathcal{C}^\alpha} \tag{5.2.5}$$

  - for every $g \in \mathcal{C}^\beta$ with $\beta < 0$ and $\alpha + \beta \in (-2,1)$

$$\Big\|\Pi_g^{(b)}(f)\Big\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|g\|_{\mathcal{C}^\beta}\|f\|_{\mathcal{C}^\alpha}. \tag{5.2.6}$$

- Fix an integer $b > 2$. For any $\alpha, \beta \in (-\infty, 1)$ with $\alpha + \beta > 0$, for every $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, we have the continuity estimate

$$\Big\|\Pi^{(b)}(f,g)\Big\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha}\|g\|_{\mathcal{C}^\beta}.$$

These results are complemented by the following commutator estimate, which takes a form similar to its Euclidean counterpart.

Proposition 39 (**[P20]**)**.** Consider the a priori unbounded trilinear operator

$$C(f,g,h) := \Pi^{(b)}\Big(\Pi_g^{(b)}(f), h\Big) - g\Pi^{(b)}(f,h),$$

on some distribution space. Let α,β,γ be Hölder regularity exponents with $\alpha \in (-1,1), \beta \in (0,1)$ and $\gamma \in (-\infty, 1]$. If

$$0 < \alpha+\beta+\gamma < 1 \qquad \text{and} \qquad \alpha+\gamma < 0$$

then, setting $\delta := (\alpha+\beta)\wedge 1 + \gamma$, we have

$$\big\|C(f,g,h)\big\|_{\mathcal{C}^\delta} \lesssim \|f\|_{\mathcal{C}^\alpha}\,\|g\|_{\mathcal{C}^\beta}\,\|h\|_{\mathcal{C}^\gamma}, \tag{5.2.7}$$

for every $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$ and $h \in \mathcal{C}^\gamma$; so the commutator defines a trilinear map from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{C}^\delta$.

Two ingredients are needed to turn the machinery of paraproducts into an efficient tool. To understand how nonlinear functions act on Hölder functions $\mathcal{C}^\alpha$, with $0<\alpha<1$, and to understand how one can compose two paraproducts. The first point is the object of point 1 in the following analogue of Bony's classical result on paralinearization [**7**], while the second point is dealt with in point 2.

Theorem 40 (**[P20]**)**.** (1) Let fix an integer $b \geqslant 2, \alpha \in (0,1)$, and consider a nonlinearity $\mathrm{F} \in C_b^3$.Then for every $f \in \mathcal{C}^\alpha$, we have $\mathrm{F}(f) \in \mathcal{C}^\alpha$ and

$$R_F(f) := \mathrm{F}(f) - \Pi^{(b)}_{\mathrm{F}'(f)}(f) \in \mathcal{C}^{2\alpha}.$$

More precisely

$$\Big\|\mathrm{F}(f) - \Pi^{(b)}_{\mathrm{F}'(f)}(f)\Big\|_{\mathcal{C}^{2\alpha}} \lesssim \|\mathrm{F}\|_{C_b^3}\Big(1 + \|f\|^2_{\mathcal{C}^\alpha}\Big).$$

If $F \in C_b^4$ then the remainder term $R_F(f)$ is Lipschitz with respect to f, in so far as we have

$$\big\|R_{\mathrm{F}}(f) - R_{\mathrm{F}}(g)\big\|_{\mathcal{C}^{2\alpha}} \lesssim \|\mathrm{F}\|_{C_b^4}\big(1 + \|f\|_{\mathcal{C}^\alpha} + \|g\|_{\mathcal{C}^\alpha}\big)^2\,\|f-g\|_{\mathcal{C}^\alpha}.$$

(2) Fix an integer $b \geqslant 2$, $\alpha \in (0,1)$, $\beta \in (0,\alpha]$ and consider $u \in \mathcal{C}^\alpha$ and $v \in \mathcal{C}^\beta$. Then for every $f \in \mathcal{C}^\alpha$, we have

$$\Pi_u^{(b)}\Big(\Pi_v^{(b)}(f)\Big) - \Pi_{uv}^{(b)}(f) \in \mathcal{C}^{\alpha+\beta}$$

with

$$\Big\|\Pi_u^{(b)}\Big(\Pi_v^{(b)}(f)\Big) - \Pi_{uv}^{(b)}(f)\Big\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha}\,\|u\|_{\mathcal{C}^\alpha}\|v\|_{\mathcal{C}^\beta}.$$

With these tools in hands, we were able to adapt the machinery of paracontrolled distributions of [**3**] to our setting.

Intermezzo : Paracontrolled distributions

The ideas of paracontrolled calculus, as introduced in [**3**], have their roots in Gubinelli's notion of controlled path. As recalled in section 3.5 on the regularity of the Itô-Lyons solution map, the setting of controlled paths provides an alternative formulation of Lyons' rough paths theory that offers a simple approach to the core of the theory, while rephrasing it in a very useful Banach setting.

Assume we are given an $\mathbb{R}^\ell$-valued (weak geometric) α-Hölder rough path

$$\mathbf{X} = \big((X_{ts}, \mathbb{X}_{ts})\big)_{0\leqslant s\leqslant t\leqslant T},$$

with $X_{ts} \in \mathbb{R}^\ell$ and $\mathbb{X}_{ts} \in \mathbb{R}^\ell \otimes \mathbb{R}^\ell$, and a map $\sigma \in C\Big(\mathbb{R}^d, \mathrm{L}\big(\mathbb{R}^\ell, \mathbb{R}^d\big)\Big)$. Following Lyons, an $\mathbb{R}^d$-valued path $x_\bullet$ is said to solve the rough differential equation

$$dx_t = \sigma(x_t)\,\mathbf{X}(dt) \tag{5.2.8}$$

if one has

$$x_t - x_s = \sigma(x_s)\,X_{ts} + \sigma'(x_s)\sigma(x_s)\,\mathbb{X}_{ts} + O\big(|t-s|^a\big) \tag{5.2.9}$$

for all $0 \leqslant s \leqslant t \leqslant T$, for some constant $a > 1$. Gubinelli's crucial remark was to notice that for a path $x_\bullet$ to satisfy equation (5.2.9), it needs to be controlled by X in the sense that one has

$$x_t - x_s = x'_s\,X_{ts} + O\big(|t-s|^{2\alpha}\big),$$

for some $\mathrm{L}(\mathbb{R}^\ell, \mathbb{R}^d)$-valued α-Hölder path $x'_\bullet$, here $x'_s = \sigma(x_s)$. The point of this remark is that, somewhat conversely, if we are given an $\mathrm{L}(\mathbb{R}^\ell, \mathbb{R}^d)$-valued α-Hölder path $z_\bullet$ controlled by $\mathbf{X}$, then there exists a unique $\mathbb{R}^d$-valued path $y_\bullet$ whose increments satisfy

$$y_t - y_s = z_s\,X_{ts} + z'_s\,\mathbb{X}_{ts} + O\big(|t-s|^a\big),$$

for some exponent $a > 1$. With a little bit of abuse, we write $\int_0^\bullet z_s\,\mathbf{X}(ds)$ for that path $y_\bullet$ – this path depends not only on z but rather on (z, z'). This path depends continuously on (z, z') and $\mathbf{X}$ in the right topologies. Given an $\mathbb{R}^d$-valued path $x_\bullet$ controlled by $\mathbf{X}$, and σ sufficiently regular, *the* $\mathrm{L}(\mathbb{R}^\ell, \mathbb{R}^d)$*-valued path* $z_s := \sigma(x_s)$ *is controlled by* $\mathbf{X}$*, with a control of the size of* (z, z') *given in terms of the size of* (x, x'). So, for a path $x_\bullet$ to solve the rough differential equation (5.2.8), it is necessary and sufficient that it satisfies

$$x_t - x_s = \int_s^t \sigma(x_r)\,\mathbf{X}(dr),$$

for all $0 \leqslant s \leqslant t \leqslant T$, that is, $x_\bullet$ is a fixed point of the continuous map

$$x_\bullet \mapsto \int_0^\bullet \sigma(x_r)\,\mathbf{X}(dr),$$

from the space of paths controlled by $\mathbf{X}$ to itself. The well-posed character of equation (5.2.8) is then shown by proving that this map is a contraction if one works on a sufficiently small time interval.

The setting of paracontrolled distribution does not differ much from the above description. We aim in the sequel at solving equations of the form

$$\big(\partial_t + \Delta\big)u = \mathrm{F}(u)\,\zeta,$$

for some distribution ζ. Comparing this equation with (5.2.8), the role of the rough path will be played in that setting by a pair $\mathbf{X} = (\zeta, Z)$ of distributions, with ζ in the role of dX_t, with $\Pi(Z, \zeta)$, well-defined, somehow in the role of $d\mathbb{X}_t$, and $\big(\partial_t + \Delta\big)$ in the role of $\frac{d}{dt}$. The elementary insight that the/a solution u should behave at small space scales as ζ is turned into the definition of a distribution controlled by $\mathbf{X}$, using the paraproduct as a means of comparison, for writing a first order Taylor expansion of u similar to identity (5.2.2). The crucial point of this definition is that one can then make sense of the product $\mathrm{F}(u)\,\zeta$, in that controlled setting, by providing an analogue of the right hand side of identity (5.2.9) defining there $\sigma(x_s)\,\mathbf{X}(ds)$. To run formally in our present setting the above argument sketched for rough differential equations, we shall need to see how controlled distributions are transformed by a nonlinear map.

5.2.3. Generalized parabolic Anderson model As an illustration, we shall study the generalized parabolic Anderson model equation (gPAM)

$$\partial_t u + Lu = \mathrm{F}(u)\,\xi, \qquad u(0) = u_0,$$

on some possibly unbounded 2-dimensional Riemannian manifold M satisfying some mild geometric conditions. One can take as operator the Laplace-Beltrami operator or some sub-elliptic diffusion operator. The nonlinearity F is $\mathcal{C}_b^3$, and ξ stands above for a weighted Gaussian noise with weight in $L^2 \cap L^\infty$. For a generic distribution $\zeta \in \mathcal{C}^{\alpha-2}$, denote by X the solution to the equation

$$\partial_t X + LX = \zeta,$$

given by the formula $X(t) := \int_0^t P_{t-s}(\zeta)\,ds$.

Theorem 41 ([P20]). Let $\alpha \in \left(\frac{2}{3}, 1\right)$, an initial data $u_0 \in \mathcal{C}^{2\alpha}$, a nonlinearity $\mathrm{F} \in C_b^3$, and a time horizon $T > 0$ be fixed. Assume that $\zeta \in \mathcal{C}^{\alpha-2}$.

(1) **Local well-posedness for (gPAM).** If the resonant term $\Pi(X, \zeta)$ is well-defined as a continuous function from $[0, T]$ to $\mathcal{C}^{\alpha-2}$, then for a small enough time horizon T, the generalized PAM equation

$$\partial_t u + Lu = \mathrm{F}(u)\,\zeta, \qquad u(0) = u_0$$

has a unique solution in some function space.

(2) **Global well-posedness for (PAM).** If the resonant term $\Pi(X, \zeta)$ is well-defined as a continuous function from $\mathbb{R}_+$ to a weighted version of $\mathcal{C}^{\alpha-2}$, then the PAM equation

$$\partial_t u + Lu = u\,\zeta, \qquad u(0) = u_0$$

has a unique global in time solution in some function space.

The implementation of this result in the case where $\zeta = \xi$ is a random Gaussian noise takes the following form.

Theorem 42 ([P20]). Let ξ stand for a time-independent weighted noise in space, and set $\xi^\epsilon := P_\epsilon \xi$, and $X^\epsilon(t) = \int_0^t P_{t-s}\big(\xi^\epsilon\big)\,ds$.

(1) The pair $\big(\xi^\epsilon, X^\epsilon\big)$ converges in probability in some space to some extended noise (ζ, X), with $\zeta = \xi$, and $\Pi(X, \zeta)$ well-defined in the above sense.

(2) Furthermore, if u^ϵ stands for the solution of the renormalized equation

$$\partial_t u^\epsilon + Lu^\epsilon = \mathrm{F}\big(u^\epsilon\big)\,\xi^\epsilon - c^\epsilon\,\mathrm{F}'\big(u^\epsilon\big)\,\mathrm{F}(u^\epsilon), \qquad u^\epsilon(0) = u_0$$

where $c^\epsilon(\cdot) := \mathbb{E}\Big[\Pi\big(L^{-1}\xi^\epsilon, \xi^\epsilon\big)(\cdot)\Big]$ is a deterministic real-valued function on M, then u^ϵ converges in probability to the solution u of equation (1) associated with (ζ, X), in some space whose definition depends on whether or not F is linear.

What next? The methods of paracontrolled calculus has only been introduced recently and their scope has not been delimitated well so far. Bringing together the tools of rough drivers of section 5.1 with the semigroup approach to paracontrolled calculus, I intend to investigate the integration theory of quasi-linear symmetric rough systems, in which the rough driver in the equation

$$df_s = \mathbf{A}(f_s\,; ds)\, f_s$$

is allowed to depend on the current state of the unknown. In another direction, I am currently investigating, with F. Bernicot, D. Frey and N. Perkowski, the use of the above semigroup approach to singular PDEs to the problem of their systematic discretization and the kind of convergence results one can expect.

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