A FLOW-BASED APPROACH TO ROUGH DIFFERENTIAL EQUATIONS

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1. Introduction

This course is dedicated to the study of some class of dynamics in a Banach space, index by time $\mathbb{R}_+$. Although there exists many recipes to cook up such dynamics, those generated by differential equations or vector fields on some configuration space are the most important from a historical point of view. Classical mechanics reached for example its top with the description by Hamilton of the evolution of any classical system as the solution of a first order differential equation with a universal form. The outcome, in the second half of the twentieth centary, of the study of random phenomena did not really change that state of affair, with the introduction by Itô of stochastic integration and stochastic differential equations.

Classically, one understands a differential equation as the description of a point motion, the set of all these motions being gathered into a single object called a *flow*. It is a familly $\varphi = \big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ of maps from the state space to itself, such that $\varphi_{tt} = \mathrm{Id}$, for all $0 \leqslant t \leqslant T$, and $\varphi_{ts} = \varphi_{tu} \circ \varphi_{us}$, for all $0 \leqslant s \leqslant u \leqslant t \leqslant T$. The first aim of the approach to some class of dynamics that is proposed is this course is the construction of flows, as opposed to the construction of trajectories started from some given point.

I will explain in the first par of the course a simple method for constructing a flow φ from a family $\mu = \big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ of maps that almost forms a flow. The two essential points of this construction are that

- **i)** φ_{ts} is loosely speaking the composition of infinitely many $\mu_{t_{i+1}t_i}$ along an infinite partition $s < t_1 < \cdots < t$ of the interval $[s, t]$, with infinitesimal mesh,
- **ii)** φ depends continuously on μ in some sense.

We will find back along the way the classical Cauchy-Lipschitz theory. It will appear that a good way of understanding what a solution to the ordinary differential equation on $\mathbb{R}^n$

$$\dot{x}_t = \sum_{i=1}^{\ell} V_i(x_t)\dot{h}^i_t =: V_i(x_t)\dot{h}^i_t$$

is, for some Lipschitz continuous vector fields V_i and some real-valued controls h^i of class $\mathcal{C}^1$, is to say that the path x satisfies at any time s the Taylor-type expansion formula

$$x_t = x_s + \big(h^i_t - h^i_s\big)V_i(x_s) + o(t-s),$$

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and even

$$f\big(x_t\big) = f\big(x_s\big) + \big(h^i_t - h^i_s\big)\big(V_i f\big)(x_s) + o(t-s),$$

for any function f of class $\mathcal{C}^2_b$, with $V_i f$ standing for the derivative of f in the direction of V_i.

What insight does it provide to understand what a solution to the Stratonovich stochastic differential equation

$$\circ dx_t = V_i(x_t)\circ dw_t \tag{1.1}$$

driven by the Brownian motion w is? The use of this notion of differential enables to write the following kind of Taylor-type expansion of order 2 for any function f of class $\mathcal{C}^3$.

(1.2)

$$\begin{aligned} f\big(x_t\big) &= f\big(x_s\big) + \int_s^t \big(V_i f\big)(x_r)\circ dw_r \\ &= f\big(x_s\big) + \big(w^i_t - w^i_s\big)\big(V_i f\big)(x_s) + \int_s^t\int_s^r \big(V_j(V_i f)\big)(x_u)\circ dw_u \circ dw_r \\ &= f\big(x_s\big) + \big(w^i_t - w^i_s\big)\big(V_i f\big)(x_s) + \left(\int_s^t\int_s^r \circ dw_u \circ dw_r\right)\big(V_j(V_i f)\big)(x_s) + \int_s^t\int_s^r\int_s^u (\cdots) \end{aligned}$$

For any choice of $2<p<3$, the Brownian increments $w^i_{ts} := w^i_t - w^i_s$ have almost-surely a size of order $(t-s)^{\frac{1}{p}}$, the iterated integrals $\int_s^t\int_s^r \circ dw_u\circ dw_r$ have size $(t-s)^{\frac{2}{p}}$, and the triple integral size $(t-s)^{\frac{3}{p}} = o(t-s)$. What will come later out of this formula is that a solution to equation (1.1) is precisely a path x for which one can write for any function f of class $\mathcal{C}^3$ a Taylor-type expansion of order 2 of the form

$$f\big(x_t\big) = f\big(x_s\big) + \big(w^i_t - w^i_s\big)\big(V_i f\big)(x_s) + \left(\int_s^t\int_s^r \circ dw_u \circ dw_r\right)\big(V_j(V_i f)\big)(x_s) + o(t-s)$$

at any time s. This conclusion puts forward the fact that what the dynamics really see of the Brownian control w is not only its increments w_{ts} but also its iterated integrals $\int_s^t\int_s^r \circ dw_u\circ dw_r$. The notion of a p-rough path $\mathbf{X} = \big(X_{ts}, \mathbb{X}_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ is an abstraction of this family of pairs of quantities, for $2<p<3$ here. This *multi-level object* satisfies some constraints of analytic type (size of its increments) and algebraic type, coming from the higher level parts of the object. As they play the role of some iterated integrals, they need to satisfy some identities consequences of the Chasles relation for elementary integrals: $\int_s^t = \int_s^u + \int_u^t$. These constraints are all what these rough paths $\mathbf{X} = (X, \mathbb{X})$ need to satisfy to give a sense to the equation

$$dx_t = \mathrm{F}^{\otimes}(x_t)\mathbf{X}(dt) \tag{1.3}$$

for a collection $\mathrm{F} = \big(V_1,\dots,V_\ell\big)$ of vector fields on $\mathbb{R}^n$, by defining a solution as a path x for which on can write the Taylor-type expansion of order 2

$$f\big(x_t\big) = f\big(x_s\big) + X^i_{ts}\big(V_i f\big)(x_s) + \mathbb{X}^{jk}_{ts}\big(V_j(V_k f)\big)(x_s) + o(t-s), \tag{1.4}$$

for any function f of class $\mathcal{C}^3_b$. The notation $\mathrm{F}^{\otimes}$ is used here to insist on the fact that it is not only the collection F of vector fields that is used in this definition, but also the differential operators $V_j V_k$ constructed from F. The introduction and the study of p-rough paths and their collection is done in the second part of the course.

Guided by the results on flows of the first part, we shall reinterpret equation (1.3) to construct directly a flow φ solution to the equation

$$d\varphi = \mathrm{F}^{\otimes}\mathbf{X}(dt),$$

in a sense to be made precise in the third part of the course. The recipe of construction of φ will consist in associating to F and $\mathbf{X}$ a $\mathcal{C}^1$-approximate flow $\mu = \big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ having everywhere a behaviour similar to that described by equation (1.4), and then to apply the theory described in the first part of the course. The maps μ_{ts} will be constructed as the time 1 maps associated with some ordinary differential equation constructed from F and $\mathbf{X}_{ts}$ in a simple way. As they will depend continuously on $\mathbf{X}$, the continuous dependence of φ on $\mathbf{X}$ will come as a consequence of point **ii)** above.

All that will be done in a deterministic setting. We shall see in the fourth part of the course how this approach to dynamics is useful in giving a fresh viewpoint on stochastic differential equations and their associated dynamics. The key point will be the fundamental fact that Brownian motion has a natural lift to a Brownian p-rough path, for any $2 < p < 3$. Once this object will be constructed by probabilistic means, the deterministic machinery for solving rough differential equations, described in the third part of the course, will enable us to associate to any realization of the Brownian rough path a solution to the rough differential equation (1.3). This solution coincides almost-surely with the solution to the Stratonovich differential equation (1.1)! One shows in that way that this solution is a continuous function of the Brownian rough path, in striking contrast with the fact that it is only a measurable function of the Brownian path itself, with no hope for a more regular dependence in a generic setting. This fact will provide a natural and easy road to the deep results of Wong-Zakai, Stroock & Varadhan or Freidlin & Wentzell.

Several other approaches to rough differential equations are available, each with its own pros and cons. We refer the reader to the books [1] and [2] for an account of Lyons' original approach; she/he is refered to the book [3] for a thourough account of the Friz-Victoir approach, and to the lecture note [4] by Baudoin for an easier account of their main ideas and results, and to the forthcoming excellent lecture notes [5] by Friz and Hairer on Gubinelli's point of view. The present approach does not overlap with the above ones.[1]

[1]Comments on these lecture notes are welcome. Please send them to the email address *ismael.bailleul@univ-rennes1.fr*

2. Flows and approximate flows

Guide for this section

This first part of the course will present the backbone of our approach to rough dynamics under the form of a simple recipe for constructing flows of maps on some Banach space. Although naive, it happens to be robust enough to provide a unified treatment of ordinary, rough and stochastic differential equations. We fix throughout a Banach space V.

The main technical difficulty is to deal with the non-commutative character of the space of maps from V to itself, endowed with the composition operation. To understand the part of the problem that does not come from non-commutativity, let us consider the following model problem. Suppose we are given a family $\mu = \big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ of elements of some Banach space depending continuously on s and t, and such that $\big|\mu_{ts}\big| = o_{t-s}(1)$. Is it possible to construct from μ a family $\varphi = \big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ of elements of that Banach space, depending continuously on s and t, and such that we have

$$\varphi_{tu} + \varphi_{us} = \varphi_{ts} \tag{2.1}$$

for all $0 \leqslant s \leqslant u \leqslant t \leqslant 1$? This additivity property plays the role of the flow property. Would the time interval $[0, 1]$ be a finite discrete set $t_1 < \cdots < t_n$, the additivity property (2.1) would mean that φ_{ts} is the sum of the $\varphi_{t_{i+1}t_i}$, whose definition should be $\mu_{t_{i+1}t_i}$, as these are the only quantities we are given if no arbitrary choice is to be done. Of course, this will not turn φ into an additive map, in the sense that property (2.1) holds true, in this discrete setting, but it suggest the following attempt in the continuous setting of the time interval $[0, 1]$.

Given a partition $\pi = \big\{0 < t_1 < \cdots < 1\big\}$ of $[0, 1]$ and $0 \leqslant s \leqslant t \leqslant 1$, set

$$\varphi^{\pi}_{ts} = \sum_{s\leqslant t_i<t_{i+1}\leqslant t} \mu_{t_{i+1}t_i}.$$

This map almost satisfies relation (2.1) as we have

$$\varphi^{\pi}_{tu} + \varphi^{\pi}_{us} = \varphi^{\pi}_{ts} - \mu_{u^+u^-} = \varphi^{\pi}_{ts} + o_{|\pi|}(1),$$

for all $0 \leqslant s \leqslant u \leqslant t \leqslant 1$, where u^-, u^+ are the elements of π such that $u^- \leqslant u < u^+$, and $|\pi| = \max\{t_{i+1} - t_i\}$ stands for the mesh of the partition. So we expect to find a solution φ to our problem under the form φ^{π}, for a partition of $[0, 1]$ of infinitesimal mesh, that is as a limit of φ^{π}'s, say along a sequence of refined partitions π_n where π_{n+1} has only one more point than π_n, say u_n. However, the sequence φ^{π_n} has no reason to converge without assuming further conditions on μ. To fix further the setting, let us consider partitions π_n of $[0, 1]$ by dyadic times, where we exhaust first all the dyadic times multiples of 2^{-k}, in any order, before taking in the partition points multiples of $2^{-(k+1)}$. Two dyadic times s and t being given, both multiples of 2^{-k_0}, take n big enough for them to be points of π_n. Then, denoting by u_n^-, u_n^+ the two points of π_n such that $u_n^- < u_n < u_n^+$, the quantity $\varphi^{\pi_{n+1}}_{ts} - \varphi^{\pi_n}_{ts}$ will either be null if $u_n \notin [s,t]$, or

$$\varphi_{ts}^{\pi_{n+1}} - \varphi_{ts}^{\pi_n} = \big(\mu_{u_n^+ u_n} + \mu_{u_n u_n^-}\big) - \mu_{u_n^+ u_n^-}, \tag{2.2}$$

otherwise. A way to control this quantity is to assume that the map μ is *approximately additive*, in the sense that we have some positive constants c_0 and $a > 1$ such that the inequality

$$\big|\big(\mu_{tu} + \mu_{us}\big) - \mu_{ts}\big| \leqslant c_0\, |t-s|^a \tag{2.3}$$

holds for all $0 \leqslant s \leqslant u \leqslant t \leqslant 1$. Under this condition, we have

$$\Big|\varphi_{ts}^{\pi_{n+1}} - \varphi_{ts}^{\pi_n}\Big| \leqslant c_0 2^{-am},$$

where $\big|\pi_{n+1}\big| = 2^{-m}$. There will be 2^m such terms in the formal series $\sum_{n\geqslant 0}\big(\varphi_{ts}^{\pi_{n+1}} - \varphi_{ts}^{\pi_n}\big)$, giving a total contribution for these terms of size $2^{-(a-1)m}$, summable in m. So this sum converges to some quantity φ_{ts} which satisfies (2.1) by construction (on dyadic times only, as defined as above). Note that commutativity of the addition operation was used implicitly to write down equation (2.2).

Somewhat surprisingly, the above approach also works in the non-commutative setting of maps from V to itself under a condition which essentially amounts to replace the addition operation and the norm $|\cdot|$ in condition (2.3) by the composition operation and the $\mathcal{C}^1$ norm. This will be the essential content of theorem 2 below, taken from the work [6].

2.1. C^1**-approximate flows and their associated flows.** We start by defining what will play the role of an approximate flow, in the same way as μ above was understood as an approximately additive map under condition (2.3).

Definition **1.** *A* $\mathcal{C}^1$***-approximate flow on* V** *is a family* $\mu = \big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ *of* $\mathcal{C}^2$ *maps from* V *to itself, depending continuously on* s,t *in the topology of uniform convergence, such that*

$$\big\|\mu_{ts} - \mathrm{Id}\big\|_{\mathcal{C}^2} = o_{t-s}(1) \tag{2.4}$$

and there exists some positive constants c_1 *and* $a > 1$*, such that the inequality*

$$\big\|\mu_{tu}\circ\mu_{us} - \mu_{ts}\big\|_{\mathcal{C}^1} \leqslant c_1 |t-s|^a \tag{2.5}$$

holds for all $0 \leqslant s \leqslant u \leqslant t \leqslant T$.

Note that μ_{ts} is required to be $\mathcal{C}^2$ close to the identity while we ask it to be an approximate flow in a $\mathcal{C}^1$ sense. Given a partition $\pi_{ts} = \{s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t\}$ of an interval $[s,t] \subset [0,T]$, set

$$\mu_{\pi_{ts}} = \mu_{t_n t_{n-1}} \circ \cdots \circ \mu_{t_1 t_0}.$$

Theorem **2** (Constructing flows on a Banach space)**.** *A* $\mathcal{C}^1$*-approximate flow defines a unique flow* $\varphi = \big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ *on* V *such that the inequality*

$$\big\|\varphi_{ts} - \mu_{ts}\big\|_\infty \leqslant c|t-s|^a \tag{2.6}$$

holds for some positive constant c, for all $0 \leqslant s \leqslant t \leqslant T$ *sufficiently close, say* $t - s \leqslant \delta$. *This flow satisfies the inequality*

$$\big\|\varphi_{ts} - \mu_{\pi_{ts}}\big\|_\infty \leqslant \frac{2}{1-2^{1-a}}\, c_1^2 T \big|\pi_{ts}\big|^{a-1} \tag{2.7}$$

for any partition π_{ts} *of any interval* (s,t) *of mesh* $\big|\pi_{ts}\big| \leqslant \delta$.

Note that the conclusion of theorem 2 holds in $\mathcal{C}^0$-norm. This loss of regularity with respect to the controls on μ given by equations (2.4) and (2.5) roughly comes from the use of uniform $\mathcal{C}^1$-estimates on some functions f_{ts} to control some increments of the form $f_{ts} \circ g_{ts} - f_{ts} \circ g'_{ts}$, for some $\mathcal{C}^0$-close maps g_{ts}, g'_{ts}. Note that if μ depends continuously on some parameter, then φ also depends continuously on that parameter, as a uniform limit of continuous functions, equation (2.11). One proves in exercice 5 that φ is actually locally Lipschitz as a function of μ, in the sense that if

$$\Big\|\big(\mu_{tu} \circ \mu_{us} - \mu_{ts}\big) - \big(\mu'_{tu} \circ \mu'_{us} - \mu'_{ts}\big)\Big\|_{\mathcal{C}^1} \leqslant \epsilon\, |t-s|^a$$

for all $0 \leqslant s \leqslant u \leqslant t \leqslant T$, then

$$\Big\|\big(\varphi_{ts} - \mu_{ts}\big) - \big(\varphi'_{ts} - \mu'_{ts}\big)\Big\|_\infty \leqslant c\,\epsilon\, |t-s|^a$$

for some explicit constant c. The remainder of this section will be dedicated to the proof of theorem 2. We shall proceed in two steps, by proving first that one can construct φ as the *uniform limit* of the μ_π's provided one can control uniformly their Lipschitz norm. This controll will be proved in a second step.

2.1.1. *First step.* Let us introduce the following definition to prepare the first step.

Definition **3.** *Let* $\epsilon \in (0,1)$ *be given. A partition* $\pi = \{s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t\}$ *of* (s,t) *is said to be* **of special type** ϵ *if we have* $\epsilon \leqslant \frac{s_i - s_{i-1}}{s_{i+1} - s_{i-1}} \leqslant 1 - \epsilon$, *for all* $i = 1 \dots n-1$. *The trivial partition of any interval into the interval itself is also said to be of special type* ϵ.

A partition of any interval into sub-intervals of equal length has special type $\frac{1}{2}$. Given a partition $\pi = \{s = s_0 < s_1 < \cdots < s_{n-1} < s_n = t\}$ of (s,t) of special type ϵ and $u \in \{s_1, \dots, s_{n-1}\}$, the induced partitions of the intervals (s,u) and (u,t) are also of special type ϵ. Set $m_\epsilon = \sup_{\epsilon \leqslant \beta \leqslant 1-\epsilon} \beta^a + (1-\beta)^a < 1$, and a constant

$$L > \frac{2c_1}{1 - m_\epsilon},$$

where c_1 is the constant in equation (2.5).

Lemma **4.** *Let* $\mu = \big(\mu_{ts}\big)_{0 \leqslant s \leqslant t \leqslant T}$ *be a* $\mathcal{C}^1$*-approximate flow on* V. *Given* $\epsilon > 0$, *there exists a positive constant* δ *such that for any* $0 \leqslant s \leqslant t \leqslant T$ *with* $t - s \leqslant \delta$, *and any special partition of type* ϵ *of an interval* $(s,t) \subset [0,T]$, *we have*

$$\big\|\mu_{\pi_{ts}} - \mu_{ts}\big\|_\infty \leqslant L|t-s|^a. \tag{2.8}$$

Proof – We proceed by induction on the number n of sub-intervals of the partition. The case $n = 2$ is the $\mathcal{C}^0$ version of identity (2.5). Suppose the statement has been proved for $n \geqslant 2$. Fix $0 \leqslant s < t \leqslant T$ with $t - s \leqslant \delta$, and let $\pi_{ts} = \{s_0 = s < s_1 < \cdots < s_n < s_{n+1} = t\}$ be a partition of (s,t) of special type ϵ, splitting the interval (s,t) into $(n+1)$ sub-intervals. Set $u = s_{[n/2]}$, so the two partitions π_{tu} and π_{us} are both of special type ϵ, with respective cardinals no greater than n, and $\epsilon \leqslant \frac{t-u}{t-s} \leqslant 1-\epsilon$. Then

$$\begin{aligned}\big\|\mu_{\pi_{ts}} - \mu_{ts}\big\|_\infty &\leqslant \big\|\mu_{\pi_{tu}} \circ \mu_{\pi_{us}} - \mu_{tu} \circ \mu_{\pi_{us}}\big\|_\infty + \big\|\mu_{tu} \circ \mu_{\pi_{us}} - \mu_{ts}\big\|_\infty \\ &\leqslant \big\|\mu_{\pi_{tu}} - \mu_{tu}\big\|_\infty + \big\|\mu_{tu} \circ \mu_{\pi_{us}} - \mu_{tu} \circ \mu_{us}\big\|_\infty + \big\|\mu_{tu} \circ \mu_{us} - \mu_{ts}\big\|_\infty \\ &\leqslant L|t-u|^a + \big(1 + o_\delta(1)\big) L\,|u-s|^a + c_1|t-s|^a,\end{aligned}$$

by the induction hypothesis and (2.4) and (2.5). Set $u - s = \beta(t-s)$, with $\epsilon \leqslant \beta \leqslant 1-\epsilon$. The above inequality rewrites

$$\big\|\mu_{\pi_{ts}} - \mu_{ts}\big\|_\infty \leqslant \Big\{\big(1 + o_\delta(1)\big)\big((1-\beta)^a + \beta^a\big)L + c_1\Big\}\,|t-s|^a.$$

In order to close the induction, we need to choose δ small enough for the condition

$$c_1 + \big(1 + o_\delta(1)\big)\, m_\epsilon L \leqslant L \tag{2.9}$$

to hold; this can be done since $m_\epsilon < 1$. ▷

As a shorthand, we shall write μ^n_{ts} for $\bigcirc_{i=0}^{n-1}\mu_{t_{i+1}t_i}$, where $s_i = s + \frac{i}{n}(t-s)$.

Proposition **5** (Step 1). *Let $\mu = \big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ be a $\mathcal{C}^1$-approximate flow on* V. *Assume the existence of a positive constant δ such that the maps μ^n_{ts}, for $n \geqslant 2$ and $t - s \leqslant \delta$, are all Lipschitz continuous, with a Lipschitz constant uniformly bounded above by some constant c_2, then there exists a unique flow $\varphi = \big(\varphi_{ts}\big)_{0\leqslant s\leqslant t\leqslant T}$ on* V *such that the inequality*

$$\big\|\varphi_{ts} - \mu_{ts}\big\|_\infty \leqslant c|t-s|^a \tag{2.10}$$

holds for some positive constant c, for all $0 \leqslant s \leqslant t \leqslant T$ with $t - s \leqslant \delta$. This flow satisfies the inequality

$$\big\|\varphi_{ts} - \mu_{\pi_{ts}}\big\|_\infty \leqslant c_1 c_2 T \big|\pi_{ts}\big|^{a-1} \tag{2.11}$$

for any *partition π_{ts} of (s,t), of mesh $\big|\pi_{ts}\big| \leqslant \delta$.*

Proof – The existence and uniqueness proofs both rely on the elementary identity (2.12)

$$f_N\circ\cdots\circ f_1 - g_N\circ\cdots\circ g_1 = \sum_{i=1}^N \Big(g_N\circ\cdots\circ g_{N-i+1}\circ f_{N-i} - g_N\circ\cdots\circ g_{N-i+1}\circ g_{N-i}\Big)\circ f_{N-i-1}\circ\cdots\circ f_1,$$

where the g_i and f_i are maps from V to itself, and where we use the obvious convention concerning the summand for the first and last term of the sum. In particular, if all the maps $g_N \circ \cdots \circ g_k$ are Lipschitz continuous, with a common upper bound c' for their Lipschitz constants, then

$$\left\| f_N \circ \cdots \circ f_1 - g_N \circ \cdots \circ g_1 \right\|_\infty \leqslant c' \sum_{i=1}^N \|f_i - g_i\|_\infty. \tag{2.13}$$

a) Existence. Set $\mathrm{D}_\delta := \big\{0 \leqslant s \leqslant t \leqslant T\,;\, t-s \leqslant \delta\big\}$ and write $\mathbb{D}_\delta$ for the intersection of D_δ with the set of dyadic real numbers. Given $s = a2^{-k_0}$ and $t = b2^{-k_0}$ in $\mathbb{D}_\delta$, define for $n \geqslant k_0$

$$\mu^{(n)}_{ts} := \mu^{2^n}_{ts} = \mu_{s_{N(n)}s_{N(n)-1}} \circ \cdots \circ \mu_{s_1 s_0},$$

where $s_i = s + i2^{-n}$ and $s_{N(n)} = t$. Given $n \geqslant k_0$, write

$$\mu^{(n+1)}_{ts} = \bigcirc_{i=0}^{N(n)-1} \big(\mu_{s_{i+1}s_i+2^{-n-1}} \circ \mu_{s_i+2^{-n-1}s_i}\big)$$

and use (2.12) with $f_i = \mu_{s_{i+1}s_i+2^{-n-1}} \circ \mu_{s_i+2^{-n-1}s_i}$ and $g_i = \mu_{s_{i+1}s_i}$ and the fact that all the maps $\mu_{s_{N(n)}s_{N(n)-1}} \circ \cdots \circ \mu_{s_{N(n)-i+1}s_{N(n)-i}} = \mu^i_{s_{N(n)}s_{N(n)-i}}$ are Lipschitz continuous with a common Lipschitz constant c_2, by assumption, to get by (2.13) and (2.5)

$$\left\|\mu^{(n+1)}_{ts} - \mu^{(n)}_{ts}\right\|_\infty \leqslant c_2 \sum_{i=0}^{N(n)-1} \left\|\mu_{s_{i+1}s_i+2^{-n-1}} \circ \mu_{s_i+2^{-n-1}s_i} - \mu_{s_{i+1}s_i}\right\|_\infty \leqslant c_1 c_2 T\, 2^{-(a-1)n};$$

so $\mu^{(n)}$ converges uniformly on $\mathbb{D}_\delta$ to some continuous function φ. We see that φ satisfies inequality (2.6) on $\mathbb{D}_\delta$ as a consequence of (2.8). As φ is a uniformly continuous function of $(s,t) \in \mathbb{D}_\delta$, by (2.6), it has a unique continuous extension to D_δ, still denoted by φ. To see that it defines a flow on D_δ, notice that for dyadic times $s \leqslant u \leqslant t$, we have $\varphi^{(n)}_{ts} = \varphi^{(n)}_{tu} \circ \varphi^{(n)}_{us}$, for n big enough; so $\varphi_{ts} = \varphi_{tu} \circ \varphi_{us}$ for such triples of times in $\mathbb{D}_\delta$, hence for all times since φ is continuous. The map φ is easily extended as a flow to the whole of $\{0 \leqslant s \leqslant t \leqslant T\}$. Note that φ inherits from the μ^n's their Lipschitz character, for a Lipschitz constant bounded above by c_2.

b) Uniqueness. Let ψ be any flow satisfying condition (2.6). With formulas (2.12) and (2.13) in mind, rewrite (2.6) under the form $\psi_{ts} = \mu_{ts} + O_c\big(|t-s|^1\big)$, with obvious notations. Then

$$\begin{aligned}\psi_{ts} = \psi_{s_{2^n}s_{2^n-1}} \circ \cdots \circ \psi_{s_1s_0} &= \Big(\mu_{s_{2^n}s_{2^n-1}} + O_c\big(2^{-an}\big)\Big) \circ \cdots \circ \Big(\mu_{s_1s_0} + O_c\big(2^{-an}\big)\Big)\\ &= \mu_{s_{2^n}s_{2^n-1}} \circ \cdots \circ \mu_{s_1s_0} + \Delta_n = \mu^{(n)}_{ts} + \Delta_n,\end{aligned}$$

where Δ_n is of the form of the right hand side of (2.12), so is bounded above by a constant multiple of $2^{-(a-1)n}$, since all the maps $\mu_{s_{2^n}s_{2^n-1}} \circ \cdots \circ \mu_{s_{2^n-\ell+1}s_{2^n-\ell}}$ are Lipschitz continuous with a common upper bound for their Lipschitz constants, by assumption. Sending n to infinity shows that $\psi_{ts} = \varphi_{ts}$.

c) Speed of convergence. Given any partition $\pi = \{s_0 = s < \cdots < s_n = t\}$ of (s,t), writing $\varphi_{ts} = \bigcirc_{i=0}^{n-1} \varphi_{s_{i+1}s_i}$, and using their uniformly Lipschitz character, we see as a consequence of (2.13) that we have for $\left|\pi_{ts}\right| \leqslant \delta$

$$\left\|\varphi_{ts} - \mu_{\pi_{ts}}\right\|_\infty \leqslant c_2 \sum_{i=0}^{n-1} \left\|\varphi_{s_{i+1}s_i} - \mu_{s_{i+1}s_i}\right\|_\infty \leqslant c_1c_2 \sum_{i=0}^{n-1} |s_{i+1} - s_i|^a \leqslant c_1c_2T \left|\pi_{ts}\right|^{a-1}.$$

$\triangleright$

Compare what is done in the above proof with what was done in the introduction to this part of the course in a commutative setting.

2.1.2. *Second step.* The uniform Lipschitz control assumed in proposition 5 actually holds under the assumption that μ is a $\mathcal{C}^1$-approximate flow. Recall L stands for a constant strictly greater than $\frac{2c_1}{1-m_\epsilon}$.

Proposition **6** (Uniform Lipschitz controls). *Let $\mu = \left(\mu_{ts}\right)_{0\leqslant s\leqslant t\leqslant T}$ be a $\mathcal{C}^1$-approximate flow on* V*. Then, given $\epsilon > 0$, there exists a positive constant δ such that the inequality*

$$\left\|\mu_{\pi_{ts}} - \mu_{ts}\right\|_{\mathcal{C}^1} \leqslant L|t-s|^a$$

holds for any partition π_{ts} of $[s,t]$ of special type ϵ, whenever $t - s \leqslant \delta$.

Proof – We proceed by induction on the number n of sub-intervals of the partition as in the proof of lemma 4. The case $n = 2$ is identity (2.5). Suppose the statement has been proved for $n \geqslant 2$. Fix $0 \leqslant s < t \leqslant T$ with $t - s \leqslant \delta$, and let $\pi_{ts} = \{s_0 = s < s_1 < \cdots < s_n < s_{n+1} = t\}$ be a partition of (s,t) of special type ϵ, splitting the interval (s,t) into $(n+1)$ sub-intervals. Set $m = \left[\frac{n+1}{2}\right]$ and $u := s_m$, so the two partitions π_{tu} and π_{us} are both of special type ϵ, with respective cardinals no greater than n. Then we have for any $x \in$ V

$$\begin{aligned} D_x\mu_{\pi_{ts}} - D_x\mu_{ts} &= D_x\big(\mu_{\pi_{tu}} \circ \mu_{\pi_{us}}\big) - D_x\mu_{ts} \\ &= \Big(D_{\mu_{\pi_{us}}(x)}\mu_{\pi_{tu}} - D_{\mu_{\pi_{us}}(x)}\mu_{tu}\Big)\big(D_x\mu_{\pi_{us}}\big) + \Big(\big(D_{\mu_{\pi_{us}}(x)}\mu_{tu} - D_{\mu_{us}(x)}\mu_{tu}\big)\big(D_x\mu_{\pi_{us}}\big)\Big) \\ &\quad + \big(D_{\mu_{us}(x)}\mu_{tu}\big)\Big(D_x\mu_{\pi_{us}} - D_x\mu_{us}\Big) + \Big(\big(D_{\mu_{us}(x)}\mu_{tu}\big)\big(D_x\mu_{us}\big) - D_x\mu_{ts}\Big) \\ &=: (1) + (2) + (3) + (4). \end{aligned}$$

We treat each term separately using repeatedly the induction hypothesis, continuity assumption (2.4) for μ_{ts} in $\mathcal{C}^2$ topology, and lemma 4 when needed. We first have

$$\big|(1)\big| \leqslant L|t-u|^a \left(1 + o_\delta(1)\right).$$

Also,

$$\Big|D_{\mu_{\pi_{us}}(x)}\mu_{tu} - D_{\mu_{us}(x)}\mu_{tu}\Big| \leqslant o_{t-u}(1)\left|\mu_{\pi_{us}}(x) - \mu_{us}(x)\right| \leqslant o_{t-u}(1)\, L|u-s|^a,$$

As the term $D_x\mu_{\pi_{us}}$ has size no greater than $\big(1 + o_\delta(1)\big) + L|u-s|^a$, we have

$$\big|(2)\big| \leqslant o_\delta(1)\,|u-s|^a.$$

Last, we have the upper bound

$$\big|(3)\big| \leqslant \big(1 + o_\delta(1)\big)L|u-s|^a,$$

while $\big|(4)\big| \leqslant \big\|\mu_{tu}\circ\mu_{us} - \mu_{ts}\big\|_{\mathcal{C}^1} \leqslant c_1|t-s|^a$ by (2.5). All together, and writing $t-u = \beta(t-s)$, for some $\beta\in[\epsilon, 1-\epsilon]$, this gives

$$\begin{aligned}\big|D_x\mu_{\pi_{ts}} - D_x\mu_{ts}\big| &\leqslant \Big(\big(1+o_\delta(1)\big)\big(\beta^a + (1-\beta)^a\big)L + c_1 + o_\delta(1)\Big)|t-s|^a\\ &\leqslant L\,|t-s|^a\end{aligned}$$

for δ small enough, as $m_\epsilon < 1$. $\rhd$

Propositions 5 and 6 together prove theorem 2. Note that an explicit choice of δ is possible as soon as one has a quantitative version of the estimate $\big\|\mu_{ts} - \mathrm{Id}\big\|_{\mathcal{C}^2} = o_{t-s}(1)$.

2.2. **Exercices on flows.** To get a hand on the machinery of $\mathcal{C}^1$-approximate flows, we shall first see how theorem 2 gives back the classical Cauchy-Lipschitz theory of ordinary differential equations for *bounded* Lipschitz vector fields on $\mathbb{R}^d$. Working with *unbounded* Lipschitz vector fields requires a slightly different notion of *local* $\mathcal{C}^1$-approximate flow to be described in appendix.

Theorem 2 can be understood as a non-commutative analogue of Feyel-de la Pradelle's sewing lemma [7], first introduced by Gubinelli [9] as an abstraction of a fundamental mechanism invented by Lyons [10]. Exercices 2-4 are variations on this commutative version of theorem 2, as already sketched in the introduction to this part.

• **Ordinary differential equations.** Let $V_1,\dots,V_\ell$ be vector fields on $\mathbb{R}^d$ (or a Banach space), and $h_1,\dots,h_\ell$ be real-valued $\mathcal{C}^1$ controls. Let φ stand for the flow associated with the ordinary differential equation

$$dx_t = V_i(x_t)dh^i_t.$$

1. a) Show that one defines a $\mathcal{C}^1$-approximate flow setting for all $x\in\mathbb{R}^d$

$$\mu_{ts}(x) = x + \big(h_t - h_s\big)^i V_i(x).$$

b) Prove that φ is equal to the flow associated to μ by theorem (2). In that sense, a path x is a solution to the above ordinary differential equation if and only if it satisfies at any time s the Taylor-type expansion formula

$$f\big(x_t\big) = f\big(x_s\big) + \big(h^i_t - h^i_s\big)\big(V_i f\big)(x_s) + o(t-s),$$

for any function f of class $\mathcal{C}^2_b$. Show that the above reasoning holds true if we only assume that the $\mathbb{R}^\ell$-valued control h is globally Lipschitz continuous. (It is actually sufficient to suppose h is α-Hölder, for some $\alpha > \frac{1}{2}$.)

c) What goes wrong with the above reasoning if the Lipschitz continuous vector fields V_i qre not bounded?

d) Show that φ depends continuously on h in the uniform topology for φ and the Lipschitz topology for h, defined by the distance

$$d(h,h') = \big|h_0 - h'_0\big| + \mathrm{Lip}(h-h'),$$

where $\mathrm{Lip}(h-h')$ stands for the Lipschitz norm of $h-h'$. (A similar result holds if h is α-Hölder, for some $\alpha > \frac{1}{2}$, with the Lipschitz norm replaced by the Hölder norm.)

• **Commutative sewing lemma.** Let V be a Banach space and $\mu = \big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ be a V-valued continuous function.

2. Feyel-de la Pradelle' sewing lemma. The following commutative version of theorem 2 was first proved under this form by Feyel and de la Pradelle in [7]; see also [8]. Suppose there exists some positive constants c_0 and $a > 1$ such that we have

$$\big|(\mu_{tu} + \mu_{us}) - \mu_{ts}\big| \leqslant c_0|t-s|^a \tag{2.14}$$

for all $0 \leqslant s \leqslant u \leqslant t \leqslant 1$. Simplify the proof of theorem 2 to show that there exists a unique map $\varphi = \big(\varphi_t\big)_{0\leqslant t\leqslant 1}$, whose increments $\varphi_{ts} := \varphi_t - \varphi_s$, satisfy

$$\big|\varphi_{ts} - \mu_{ts}\big| \leqslant c|t-s|^a$$

for some positive constant c and all $0 \leqslant s \leqslant t \leqslant 1$.

3. Controls and finite-variation paths. A *control* is a non-negative map $\omega = \big(\omega_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$, null on the diagonal, and such that we have

$$\omega_{tu} + \omega_{us} \leqslant \omega_{ts}$$

for all $0 \leqslant s \leqslant u \leqslant t \leqslant 1$.

a) Show that Feyel and de la Pradelle' sewing lemma holds true if we replace $t-s$ by ω_{ts}, if we suppose that ω^a is a control.

b) Recall that a V-valued path $x = \big(x_t\big)_{0\leqslant t\leqslant 1}$ is said to have finite p-variation, for some $p \geqslant 1$, if the following quantity is finite for all $0 \leqslant s \leqslant t \leqslant 1$:

$$|x|^p_{p-\text{var};[s,t]} := \sup \sum \Big|x_{t_{i+1}} - x_{t_i}\Big|^p,$$

with a sum over the partition points t_i of a given partition of the interval $[s,t]$, and a supremum over the set of all partitions of $[s,t]$. Such a definition is invariant by any reparametrization of the time interval $[s,t]$. Given such a path, show that setting $\omega_{ts} = |x|^p_{p-\text{var};[s,t]}$, defines a control ω.

c) Show that a path with finite p-variation can be reparametrized into a $\frac{1}{p}$-Hölder path, with 1-Hölder paths being understood as Lipschitz continuous paths. Given an $\mathbb{R}^\ell$-valued path h with finite 1-variation, set

$$\zeta_s = \inf\{t \geqslant 0\,;\, |h|_{1-\text{var};[0,t]} \geqslant s\}.$$

We define a solution $x_\bullet$ to the ordinary differential equation

$$dx_t = V_i(x_t)dh^i_t$$

driven by h as a path $x_\bullet$ such that the reparametrized path $y := x \circ \zeta$ is a solution to the ordinary differential equation

$$dy_s = V_i(y_s)d(h\circ\zeta)^i_t,$$

driven by the globally Lipschitz path $h \circ \zeta$.

d) Prove that the flow φ constructed in this case from theorem 2 depends continuously on h in the uniform norm for φ and the 1-variation topology associated with the norm $|\cdot|_{1-\text{var}}$ for h. (Following the remarks of exercice 1, one can actually prove the results of questions c) and d) for paths with finite p-variation, for $1 \leqslant p < 2$.)

4. Young integral. Given another Banach space E, denote by $\mathrm{L}_c(\mathrm{V,E})$ the space of continuous linear maps from V to E equipped with the operator norm. Let α and β be positive real numbers such that $\alpha + \beta > 1$. Given any $0 < \alpha < 1$, we denote by $\mathrm{Lip}_\alpha(\mathrm{E})$ the set of α-Hölder maps. This unusual notation will be justified in the third path of the course.

a) Given an $\mathrm{L}_c(\mathrm{V},\mathrm{E})$-valued α-Lipschitz map $x = \big(x_s\big)_{0\leqslant s\leqslant 1}$ and a V-valued β-Lipschitz map $y = \big(y_s\big)_{0\leqslant s\leqslant 1}$, show that setting

$$\mu_{ts} = x_s\big(y_t - y_s\big)$$

for all $0 \leqslant s \leqslant t \leqslant 1$, defines an E-valued function μ that satisfies equation (2.14), with a constant c_0 to be made explicit.

b) The associated function φ is denoted by $\varphi_t = \int_0^t x_s dy_s$, for all $0 \leqslant t \leqslant 1$. Show that it is a continuous function of $x \in \mathrm{Lip}_\alpha\big(\mathrm{L}_c(\mathrm{V},\mathrm{E})\big)$ and $y \in \mathrm{Lip}_\beta(\mathrm{V})$.

• **Lipschitz dependence of φ on μ.** As emphasized in the remark following theorem 2, inequality (2.11) implies that φ, understood as a function of μ, is continuous in the $\mathcal{C}^0$-norm on the sets of μ's of the form $\big\{\mu\,;\,(2.5) \text{ holds uniformly}\big\}$, equipped with the $\mathcal{C}^0$-norm. One can actually prove that it depends Lipschitz continuously on μ in the following sense.

5. Let $\mu = \big(\mu_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ and $\mu' = \big(\mu'_{ts}\big)_{0\leqslant s\leqslant t\leqslant 1}$ be $\mathcal{C}^1$-approximate flows on V, with associated flows φ and φ'. Suppose that we have

$$\big\|\big(\mu_{tu}\circ\mu_{us} - \mu_{ts}\big) - \big(\mu'_{tu}\circ\mu'_{us} - \mu'_{ts}\big)\big\|_{\mathcal{C}^1} \leqslant \epsilon\,|t-s|^a$$

for a positive constant ϵ, with $a > 1$ as in the definition of the $\mathcal{C}^1$-approximate flows μ, μ', for all $0 \leqslant s \leqslant u \leqslant t \leqslant 1$. Prove that one has

$$\big\|\big(\varphi_{ts} - \mu_{ts}\big) - \big(\varphi'_{ts} - \mu'_{ts}\big)\big\|_\infty \leqslant c\,\epsilon\,|t-s|^a,$$

for all for $0 \leqslant s \leqslant t \leqslant 1$, some explicit positive constant c.

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