Uniqueness of the Φ_3^4 measures on closed Riemannian 3-manifolds

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Abstract. We constructed in a previous work the Φ_3^4 measures on compact boundaryless 3-dimensional Riemannian manifolds as some invariant probability measures of some Markovian dynamics. We prove in the present work that these dynamics have unique invariant probability measures. This is done by using an explicit coupling by change of measure that does not require any a priori information on the support of the law of the solution to the dynamics. The coupling can be used to see that the semigroup generated by the dynamics satisfies a Harnack-type inequality, which entails that the semigroup has the strong Feller property.

1 – Introduction

Let M stand for an arbitrary compact boundaryless 3-dimensional Riemannian manifold. Following the flow of research that grew out of the recent development of the domain of singular stochastic partial differential equations (PDEs), we constructed in our previous work [4] the Φ_3^4 measure on M as an invariant probability measure of some Markovian dynamics on the space $C^{-1/2-\varepsilon}(M)$ of $(-\frac{1}{2}-\varepsilon)$-Hölder/Besov distributions on M, for an arbitrary $\varepsilon > 0$. The dynamics is given by Parisi & Wu's paradigm of stochastic quantization and takes the form

$$\partial_t u = (\Delta - 1)u - u^3 + \xi, \tag{1.1}$$

where ξ stands for a spacetime white noise. When set on a discrete 3-dimensional torus this PDE rewrites as a coupled system of stochastic differential equations whose invariant measure is unique and has a density with respect to the massive discrete Gaussian free field measure proportional to $\exp\big(-\frac{1}{4}\sum_i \phi_i^4\big)$. Its continuous counterpart is 'the' Φ_3^4 measure; it has density $\exp\big(-\frac{1}{4}\int_M \phi^4\big)$ with respect to the massive Gaussian free field measure on M. However this reference measure has support in the spaces $C^{-1/2-\varepsilon}(M)$, for all $\varepsilon > 0$, and essentially no better. The fourth power of ϕ is thus almost surely ill-defined and a renormalization procedure is needed to construct such a measure from its density. In particular this makes the unique characterization of the Φ_3^4 measure a non-trivial question. The stochastic quantization approach to the construction of the Φ_3^4 measure postulates that Equation (1.1) is well-posed for all times and that it defines a Markovian dynamics which has a unique invariant probability measure, defined as the Φ_3^4 measure. This approach to the construction of the Φ_3^4 measure does not avoid the need of a renormalization process. Indeed spacetime white noise has Hölder parabolic regularity $-5/2-\varepsilon$, for all $\varepsilon > 0$, and no better, so a solution to Equation (1.1) has at best parabolic Hölder regularity $-1/2-\varepsilon$, and the quantity u^3 is ill-defined. This problem is what makes Equation (1.1) a *singular* stochastic PDE. Its proper formulation requires a priori the use of an ad hoc setting such as regularity structures [14, 7, 6, 8, 17], paracontrolled calculus [13, 1, 2, 3] or Duch's renormalization group setting [9, 10]. (So far only paracontrolled calculus has been developed in a manifold setting. A forthcoming work of Hairer & Singh will extend the analytic core of regularity structures to that setting.) Either way one gets (in a Euclidean setting) from the use of any of these tools a proper definition of a solution to Equation (1.1) and a local in time well-posedness result that needs to be supplemented by some ad hoc arguments to prove the long time existence of its solution. The Markovian character of the dynamics on $C^{-1/2-\varepsilon}(M)$ generated Equation (1.1) is inherited from its discrete counterpart. A compactness argument related to the property of '*coming down from infinity*' satisfied by the solutions of Equation (1.1) then gives the long-time existence of the local solution and the existence of an invariant measure for the semigroup on $C^{-1/2-\varepsilon}(M)$ generated by this equation. This was first proved in the setting of the torus by Mourrat & Weber in [20]. The uniqueness of such an invariant measure was proved in the 3-dimensional torus using a robust argument from dynamical systems: If the semigroup generated by the dynamics (1.1) has the strong Feller property and there is in the state space an accessible point then the semigroup has at most one invariant probability measure. Hairer & Mattingly proved in [15] a general result that shows in particular that the Φ_3^4 dynamics on the 3-dimensional torus has the strong Feller property. Hairer & Schönbauer proved in [16] a very general and deep result on the support of the law of a certain class of random models that gives as a by-product the existence of an accessible point for the Φ^4_3 dynamics on the 3-dimensional torus. None of these results are available in a manifold setting, and proving them in a manifold setting in the same generality as in [15] and [16] appears to us as a considerable task.

We did not use regularity structure, paracontrolled calculus or renormalization group methods in our construction of 'a' Φ^4_3 measure on an arbitrary boundaryless 3-dimensional Riemannian manifold M in [4]; rather we followed Jagannath & Perkowski who noticed in [18] that a clever change of variable allows to rewrite the proper formulation of Equation (1.1) as a PDE with random coefficients

$$(\partial_t - \Delta + 1)v = B \cdot \nabla v - Av^3 + Z_2 v^2 + Z_1 v + Z_0, \tag{1.2}$$

where $B \in C\big([0,T], C^{-\eta}(M, TM)\big), A \in C\big([0,T], C^{1-\eta}(M)\big)$ and

$$Z_i \in C\big([0,T], C^{-\frac{1}{2}-\eta}(M)\big) \qquad (0 \leq i \leq 2)$$

are random variables built from the noise, for any $0 < T < \infty$ and $\eta > 0$. The dynamics (1.2) is completed with the datum of an initial condition in a space of the form $C^{-1/2-\varepsilon}(M)$, for $\varepsilon > 0$ small enough. Note that no singular product is involved in Equation (1.2); the renormalization problem in (1.1) is involved in the definition and construction of the random variables A, Z_2, Z_1, Z_0. We were able in [4] to construct these random fields and prove an L^p coming down from infinity result for the solutions to Equation (1.2) that entails the existence of an invariant probability measure for the Markovian dynamics (1.1). The question of uniqueness of such an invariant probability measure was left aside in [4]; this is the point that we address in the present work.

Theorem 1 – *The semigroup on $C^{-1/2-\varepsilon}(M)$ generated by the dynamics* (1.1) *has a unique invariant probability measure.*

The change of variable $(u \mapsto v)$ from (1.1) to (1.2) is explicit: Adding for instance a (possibly random adapted) drift h in the dynamics of u adds an explicit h-dependent drift in the dynamics of v. We use Equation (1.2) as a convenient description of the Markovian dynamics of u to construct a coupling by change of measure between two solutions of Equation (1.1) started from two arbitrary initial conditions in the state space. We obtain some explicit control on the probability of a successful coupling that is independent of the pair of initial conditions. This allows us to infer the uniqueness of an invariant probability measure for the dynamics generated by (1.1). To run this approach we need to strengthen the L^p coming down from infinity result proved in [4] into an L^∞ coming down result for the solution v to Equation (1.2). This is what Section 2 is about. We adapt there to our setting Moinat & Weber' seminal approach [19] to the coming down phenomenon. This kind of control is actually needed not only for v but also for the solution v_ℓ of an equation similar to Equation (1.1), with an additional drift that depends on a real parameter ℓ. Section 4 deals with that perturbed equation. As a matter of fact it turns out to be necessary to also have a quantitative control on the sizes of $v(t)$ and $v_\ell(t)$ in stronger norms, not just L^∞; such controls are provided in Section 3 and Section 4. Equipped with the quantitative estimates proved in these sections we construct in Section 5 a coupling by change of measure that leads to a proof of uniqueness of an invariant measure for the semigroup generated by (1.1). As a by-product of our analysis we prove in Section 6 a Harnack-type inequality for the semigroup that provides a short proof that this semigroup has the strong Feller property. A reader interested only in the uniqueness result can skip Section 2 and Section 3, look at Theorem 7 in Section 4 and read Section 5.

Our uniqueness result gives a characterization of our Φ^4_3 measure as the unique invariant probability measure of a Markovian dynamics on some distribution space over M. As this dynamics depends only on the Riemannian structure of M, the Φ^4_3 measure appears as depending only on the isometry class of the Riemannian manifold M.

Notation – *For an initial condition ϕ of* (1.1) *we will denote by ϕ' the corresponding initial condition of* (1.2) *given by the Jagannath & Perkowski transform*

$$\phi = \RSI(0) - \RSW(0) + e^{-3\RSV(0)}\big(\phi' + v_{ref}(0)\big), \tag{1.3}$$

with the notations of [4]. *The precise definition of the different terms above plays no role here, so we refer the interested reader to* [4].

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2 – An L^∞ coming down from infinity result

The well-posed character of Equation (1.2) on the whole time interval $[0,\infty)$ was proved in Section 2 of [4]. We also obtained therein an explicit control on the L^p norm of the solution v to Equation (1.2) that is independent of its initial condition. This phenomenon is called '*coming down from infinity*'. It was first proved for a transform of the solutions of Equation (1.1) by Mourrat & Weber in their seminal work [20] on the Φ^4 equation on the 3-dimensional torus. It was later extended to the Euclidean setting of $\mathbb{R}^3$ by Moinat & Weber [19] and Gubinelli & Hofmanová [12] using different methods. We obtain in this section a corresponding uniform L^∞ control on v; this is the content of Theorem 2 below. The L^p coming down from infinity result proved in [4], for $1 \leq p < \infty$, is not sufficient for our needs here.

Throughout this section we will use the shorthand notation $\|\cdot\|$ for $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_D$ for $\|\cdot\|_{L^\infty(D)}$, for a parabolic domain D. Set

$$\mathcal{T} := \big\{A, B, Z_2, Z_1, Z_0\big\}$$

and

$$\begin{aligned} m_A^{-1} &= \varepsilon\Big(\frac{1}{2} - \varepsilon\Big), \quad m_B^{-1} = 1 - \varepsilon\Big(\frac{1}{2} - 2\varepsilon\Big) - 3\varepsilon, \\ m_{Z_2}^{-1} &= \frac{1}{2} - \varepsilon'', \quad m_{Z_1}^{-1} = \frac{3}{2} - \varepsilon'', \quad m_{Z_0}^{-1} = \frac{5}{2} - \varepsilon'', \end{aligned} \tag{2.1}$$

with

$$\frac{1}{2} + \varepsilon'' := (1+\varepsilon)\Big(\frac{1}{2} + \varepsilon\Big).$$

Fix $T \geq 2$; its precise value does not matter here. For $\lambda > 0$ we define the parabolic domain

$$\mathcal{D}_s := (s^2, T) \times M \subset \mathbb{R} \times M.$$

For $\tau \in \mathcal{T}$ we define $[\tau]_{|\tau|}$ as the norm of $\tau \in C_T C^{|\tau|}(M)$, where $|A| = 1 - \varepsilon, |B| = -\varepsilon$ and $|Z_i| = -1/2 - \varepsilon$. Set

$$A_+ := \sup_{\mathcal{D}} A$$

and

$$A_- := \min_{\mathcal{D}} A$$

and

$$c_A := \big(1 + \max(A_+, A_-^{-1})\big)^2.$$

Recall that we proved in Theorem 5 of [4] that Equation (1.2) is well-posed globally in time. The following statement provides an L^∞ coming down from infinity result; its proof follows the seminal work [19] of Moinat & Weber.

Theorem 2 – *There exists a positive constant C such that any solution of Equation* (1.2) *satisfies for all $0 < s \leq 1$ the estimate*

$$\|v\|_{\mathcal{D}_s} \leq C \max\left\{\frac{1 + (\min_{\mathcal{D}} A)^{-1/2}}{s}, \big((c_A[\tau]_{|\tau|})^{m_\tau}\big)_{\tau\in\mathcal{T}}\right\}. \tag{2.2}$$

2.1 Tools for the proof

We collect in this section two ingredients that will play a key role in the proof of Theorem 2: A Schauder type estimate and a corollary of the maximum principle. Given $\alpha \in \mathbb{R}$ denote by $\mathcal{C}^\alpha(\mathbb{R}\times M)$ the parabolic Besov space of regularity exponent α and integrability exponents (∞,∞). The statement of Schauder's estimate involves a regularization procedure

$$(\cdot)_\delta : h \in \mathcal{C}^\alpha(\mathbb{R}\times M) \mapsto h_\delta \in \mathcal{C}^2(\mathbb{R}\times M), \qquad (\alpha \in \mathbb{R})$$

indexed by $0 < \delta \leq 1$ adapted to the parabolic setting of $\mathbb{R} \times M$. The particular choice of regularization is not particularly important. To fix the ideas we can proceed as follows. Denote by p the kernel of the semigroup generated by the non-positive elliptic operator $\partial_t^2 - \Delta^2$ on the parabolic space $\mathbb{R}\times\mathbb{R}^3$. Let $(f(z,\cdot))_{z\in\mathbb{R}\times M}$ be a smooth family of diffeomorphisms between a z-dependent neighbourhood of $z\in\mathbb{R}\times M$ and $\mathbb{R}\times\mathbb{R}^3$. (We can use the compactness of M, hence the fact that it has a positive injectivity radius, to construct such a map.) We choose it in such a way that $f\big((t,x),\cdot\big)$ has support in $[t-1,t+1]\times M$ and $f\big((t_1,x_1),(t_2,x_2)\big) = f\big((0,x_1),(s-t_2-t-1,x_2)\big)$. Set

$$\varphi_\delta(z,z') := p_\delta\big(0, f(z,z')\big).$$

There is a positive constant c such that $\varphi_\delta(z,\cdot)$ has support in a $(c\delta)$-neighbourhood of z, uniformly in $z\in\mathbb{R}\times M$. This regularization map has the property that

$$\|h\|_{\mathcal{C}^\alpha} \simeq \sup_{0<\delta\leq 1} \delta^{-\alpha}\|h_\delta\|$$

for any $\alpha<0$. For $0<\alpha<1$ and a domain $D\subset\mathbb{R}\times M$ we define the Hölder seminorm

$$[h]_{\alpha,D} := \sup_{z\neq z'\in D} \frac{|h(z)-h(z')|}{|z-z'|^\alpha}$$

and for $1<\alpha<2$ we set

$$[h]_{\alpha,D} := \sup_{z'\neq z\in D}\ \sup_{\theta\in T_xM} \frac{\big|h(z') - h(z) - d(z',z)\,dh(z)(\theta)\big|}{d(z',z)^\alpha}.$$

We write $[h]_\alpha$ for $[h]_{\alpha,\mathbb{R}\times M}$. For a function $h\in\mathcal{C}^\beta(\mathbb{R}\times M)$ with $0<\beta<1$ we have

$$\|h_\delta - h\|_{L^\infty} \lesssim \delta^\beta [h]_\beta,$$

for an implicit multiplicative constant independent of h. The following notation is used in the statement and proof of Theorem 3. For $0<\gamma<1$ and for any continuous function W on $\mathcal{D}_s^2$ and subset $\mathcal{E}$ of $\mathcal{D}_s$ we set for $\rho>0$

$$\|W\|_{(\gamma\,;\,\rho),\mathcal{E}} := \sup_{z'\neq z\in\mathcal{E}, |z'-z|\leq\rho} \frac{|W(z',z)|}{|z'-z|^\gamma};$$

this is a kind of ρ-local γ-Hölder norm of W on the set $\mathcal{E}$. Theorem 3 below provides a strong control on a function w in terms of a control on $\mathcal{L}w$ and a 'weak' control of w itself. It is a simplified version of a subtler and finer estimate proved by Moinat & Weber in [19], Lemma 2.11 therein. (The latter was itself a generalization of Proposition 2 of Otto, Sauer, Smith & Weber's work [21].)

Theorem 3 – *Let a regularity exponent $\kappa\in(1,2)$ and a constant $\delta_0>0$ be given. There is a constant $c_1>0$ with the following property. If for all $0<4\delta\leq\lambda\leq\lambda_0$ one has*

$$\delta^{2-\kappa}\|\,(\mathcal{L}w)_\delta\,\|_{\mathcal{D}_{s+\lambda-\delta}} \leq C_\lambda \tag{2.3}$$

for some function w on $\mathcal{D}_s$ then one has

$$\sup_{0<\lambda\leq\lambda_0} \lambda^\kappa [w]_{\kappa,\mathcal{D}_{s+\lambda}} \leq c_1\Big(\sup_{\lambda\leq\lambda_0}\lambda^\kappa C_\lambda + \|w\|_{\mathcal{D}_s}\Big). \tag{2.4}$$

Moreover one can associate to any $0<\delta<\delta_0$ a constant $\rho_\delta>0$ such that for $0<\rho<\rho_\delta$ one has

$$\|dw\|_{\mathcal{D}_{s+\delta}} \lesssim \rho^{\kappa-1}[w]_{\kappa,\mathcal{D}_{s+\delta}} + \frac{1}{\rho}\,\|w\|_{(0\,;\,\rho),\mathcal{D}_{s+\delta}} \tag{2.5}$$

and

$$[dw]_{\kappa-1,\mathcal{D}_{s+\delta}} \lesssim [w]_{\kappa,\mathcal{D}_{s+\delta}} + \frac{1}{\rho^\kappa}\,\|w\|_{(0\,;\,\rho),\mathcal{D}_{s+\delta}}. \tag{2.6}$$

Sketch of proof – We only give a sketch of proof of this statement; the details will be given elsewhere. The assumption on $(\mathcal{L}w)_\delta$ gives an estimate of the size of $\mathcal{L}w$ seen as an element of $\mathcal{C}^{\kappa-2}(\mathcal{D}_{s+\lambda})$. Duhamel's formula gives for $0<s\leq t$

$$w(t) = e^{(t-s)(\Delta-1)}(w(s)) + \int_s^t e^{(t-r)(\Delta-1)}(\mathcal{L}w)(r)\,dr.$$

Write $(\mathcal{F}f)(t) := e^{t(\Delta-1)}f$ for the free propagation operator. It is classic that

$$\big\|\mathcal{F}w(s)\big\|_{\mathcal{C}^\kappa(\mathcal{D}_{s+\lambda})} \lesssim \lambda^{-\kappa}\|w(s)\|_{L^\infty}.$$

Now let $\bigcup_{i\in I} U_i$ be a finite cover of M by chart domains and let $(\chi_i)_{i\in I}$ be an associated partition of unity. Let $\chi_i^+ \in C_c^\infty(U_i)$ be equal to 1 on $\mathrm{supp}(\chi_i)$ for all $i \in I$. Writing the operator

$$\begin{aligned} e^{(t-r)(\Delta-1)} f &= \sum_{i\in I} \chi_i^+ e^{(t-r)(\Delta-1)}(\chi_i f) + \sum_{i\in I} (1-\chi_i^+) e^{(t-r)(\Delta-1)}(\chi_i f) \\ &=: \sum_{i\in I} \mathcal{A}_i^{t-r}(f) + \sum_{i\in I} \mathcal{B}_i^{t-r}(f), \end{aligned}$$

the second sum involves operators that are supported off-diagonal and are smoothing, uniformly in $t - r \geq 0$. They satisfy for each $i \in I$ an estimate of the form

$$\Big\| \int_s^\bullet \mathcal{B}_i^{\bullet - r}(\mathcal{L}v)(r)\, dr \Big\|_{\mathcal{C}^2(\mathcal{D}_s)} \lesssim \|v\|_{L^\infty(\mathcal{D}_s)}.$$

The kernels $\mathcal{K}_i(t-r)$ of the operators $\mathcal{A}_i^{t-r}$ have support near the the diagonal of $M \times M$, and in a chart where a generic point x has coordinates $\overline{x}$ near $\mathbf{0} \in \mathbb{R}^3$ one has

$$\mathcal{K}_i(t-r,x,y) = \chi_i^+(x)\,(t-r)^{-3/2} K_i\Big(t-r, \frac{\overline{x}-\overline{y}}{\sqrt{t-r}}, \overline{y}\Big)$$

for some function $\mathcal{K}_i$ in the heat calculus, as described e.g. in Grieser's lecture notes [11] – Definition 2.1 therein; the function K_i is, in particular, a smooth function of the square root of its first argument on the semiclosed interval $[0,\infty)$. We now decompose the functions K_i into their 'restrictions' to parabolic annuli using the dyadic decomposition

$$a_{-1}(\overline{s},\overline{z}) + \sum_{j\geq 0} a\big(2^{2j}\overline{s}, 2^j\overline{z}\big)$$

of a function in $C_c^\infty([0,\infty)\times\mathbb{R}^3)$ equal to 1 in a neighbourhood of $\mathbf{0} \in [0,\infty)\times\mathbb{R}^3$, with the support of $a(s_1,z_1)$ included in a parabolic annulus

$$0 < c_1 \leq |s_1| + |z_1|^2 \leq c_2 < \infty.$$

We have an associated decomposition for

$$\begin{aligned} \mathcal{K}_i(t-r,x,y) &= \chi_i^+(x)\chi_i(y)a_{-1}(t-r,\overline{x}-\overline{y})\mathcal{K}_i(t-r,x,y) \\ &\quad + \sum_{j\geq 0} \chi_i^+(x)\chi_i(y) a\big(2^{2j}(t-r), 2^j(\overline{x}-\overline{y})\big)\, \frac{1}{(t-r)^{3/2}}\, K_i\Big(t-r, \frac{\overline{x}-\overline{y}}{\sqrt{t-r}}, \overline{y}\Big) \\ &=: \chi_i^+(x)\chi_i(y)a_{-1}(t-r,\overline{x}-\overline{y})\mathcal{K}_i(t-r,x,y) + \sum_{j\geq 0} K_i^j\Big(t-r, \overline{x}-\overline{y}, \overline{y}\,;\,\overline{x}\Big) \end{aligned}$$

with

$$\sum_{j\geq 0} K_i^j = \Bigg(\sum_{0\leq j\leq n} + \sum_{j>n}\Bigg) K_i^j =: K_i^{\leq n} + K_i^{>n}$$

for any $n \geq 0$. Denote by $\mathrm{Op}(K_i^{\leq n}), \mathrm{Op}(K_i^{>n})$ the integral operators on spacetime associated with the kernels $K_i^{\leq n}, K_i^{>n}$ of the variables $\big((t,\overline{x}),(s,\overline{y})\big)$. In those terms, one has for each $n \geq 0$

$$\mathcal{L}^{-1} \simeq \sum_{i\in I} \mathrm{Op}(K_i^{\leq n})(\mathcal{L}v) + \mathrm{Op}(K_i^{>n})(\mathcal{L}v),$$

up to the regularizing operators associated with a_{-1}. For an integer n_λ such that $2^{n_\lambda} \simeq \lambda/2$ one can decompose

$$\begin{aligned} \big(\mathcal{L}^{-1}(\mathcal{L}w)\big)_{|\mathcal{D}_{s+\lambda}} &= \big(\mathrm{Op}(K_i^{\leq n_\lambda})(\mathcal{L}w)\big)_{|\mathcal{D}_{s+\lambda}} + \big(\mathrm{Op}(K_i^{>n_\lambda})(\mathcal{L}w)\big)_{|\mathcal{D}_{s+\lambda}} \\ &= \big(\mathrm{Op}(K_i^{\leq n_\lambda})(\mathcal{L}w)\big)_{|\mathcal{D}_{s+\lambda}} + \mathrm{Op}(K_i^{>n_\lambda})\big((\mathcal{L}w)_{|\mathcal{D}_{s+\lambda/2}}\big)_{|\mathcal{D}_{s+\lambda}} \end{aligned} \tag{2.7}$$

using the fact that the kernel $K_i^{>n_\lambda}$ has support in a parabolic ball of radius approximately equal to $\lambda/2$. The corresponding operator from $\mathcal{C}^{\kappa-2}(\mathcal{D}_{s+\lambda/2})$ into $\mathcal{C}^\kappa(\mathcal{D}_{s+\lambda})$ has norm $O(1)$ uniformly in λ, so the corresponding term in (2.7) has size in $\mathcal{C}^\kappa(\mathcal{D}_{s+\lambda})$ of order $C_{\lambda/2}$ from the assumption (2.3). The operator $\mathrm{Op}(K_i^{\leq n_\lambda})$ is regularizing and its norm as an operator from $L^\infty(\mathcal{D}_s)$ into $\mathcal{C}^\kappa(\mathcal{D}_{s+\lambda})$ is of order $\lambda^{-\kappa}$. Collecting the above four contributions to the estimate on the size of $w \in \mathcal{C}^\kappa(\mathcal{D}_{s+\lambda})$

gives for $0 < \lambda \leq 1$

$$\begin{aligned}\|w\|_{\mathcal{C}^\kappa(\mathcal{D}_{s+\lambda})} &\lesssim \lambda^{-\kappa}\|w(s)\|_{L^\infty} + \|w\|_{L^\infty(\mathcal{D}_s)} + \lambda^{-\kappa}\|w\|_{L^\infty(\mathcal{D}_s)} + C_{\lambda/2} \\ &\lesssim \lambda^{-\kappa}\|w\|_{L^\infty(\mathcal{D}_s)} + C_{\lambda/2}.\end{aligned}$$

The estimate (2.4) follows as a consequence. The proofs of the estimates (2.5) and (2.6) on the uniform and $(\kappa-1)$-Hölder norms of dv are left to the reader. ▷

In addition to Theorem 3 we will also use in the proof of Theorem 2 the following statement which provides the form of the bound (2.2).

Lemma 4 – *Let f, g, h be continuous functions on $[0,1]\times M$ with $\min g =: g_- > 0$. Any continuous function w on $[0,1]\times M$ such that*

$$(\partial_t - \Delta + f\cdot\nabla)w = -gw^3 + h \tag{2.8}$$

on $(0,1)\times M$ satisfies for all $0 < t \leq 1$ the estimate

$$\|w\|_{[t,1]\times M} \leq \max\Big((g_-\,t)^{-\frac{1}{2}}, (\|h\|/g_-)_\infty^{\frac{1}{3}}\Big).$$

Indeed one can check that the functions

$$\pm\Big((2g_- t)^{-\frac{1}{2}} + (\|h\|/g_-)_\infty^{\frac{1}{3}}\Big)$$

are supersolution (+) and subsolution (-) of Equation (2.8), so the conclusion comes from the comparison/maximum principle. The strong damping effect of the superlinear term $-Av^3$ in Equation (1.2) will only be used in the proof of Theorem 2 by appealing to Lemma 4 on a regularized version of Equation (1.2).

2.2 Proof of Theorem 2

The main step of the proof of Theorem 2 consists in showing that if one has

$$\|v\|_{\mathcal{D}_s} \geq 32$$

and

$$[Z_1 - 1]_{-1/2-\varepsilon} \leq \frac{c}{c_A}\,\|v\|_{\mathcal{D}_s}^{1/m_{Z_1}}$$

and

$$[\tau]_{|\tau|} \leq \frac{c}{c_A}\,\|v\|_{\mathcal{D}_s}^{1/m_\tau} \tag{2.9}$$

for all $\tau \in \big\{A, B, Z_2, Z_0\big\}$, for a well-chosen fixed positive constant c independent of v, then

$$\|v\|_{\mathcal{D}_{s+s_1}} \leq \max\left\{\frac{2\,(\inf_{\mathcal{D}} A)^{-\frac{1}{2}}}{s_1}, \frac{\|v\|_{\mathcal{D}_s}}{2}\right\} \tag{2.10}$$

for all s_1 with $s + s_1 \leq 1/2$ and $s_1 \geq 1/\|v\|_{\mathcal{D}_s}$. The proof of this inequality is the content of item *(a)* below. We explain in item *(b)* how the statement of Theorem 2 follows from that fact.

(a) The main step: Proof of (2.10). The proof of (2.10) proceeds in three steps.

1. We prove that $\|v\|_{\mathcal{D}_s}$ controls the $(3/2-\varepsilon)$ and $(1/2\pm\varepsilon)$ seminorms of v on the smaller parabolic domains $\mathcal{D}_{s+\lambda}$. (The Schauder estimate from Theorem 3 is used for that purpose.)
2. We apply Lemma 4 to a regularized version of Equation (1.2) to get with the result of Step 1 a uniform bound on v on domains of the form $\mathcal{D}_{s+s_1}$. Both s_1 and the regularization parameter are free in that step.
3. We tune the regularization parameter to optimize the bound from Step 2.

We use below the shorthand notation $\mathcal{L}$ for the differential operator $\partial_t - \Delta + 1$. The constant c in (2.9) will be chosen later, just before (2.23), and we write

$$c'_A := \frac{c}{c_A}.$$

Step 1. *We first derive from Equation* (1.2) *and the Schauder estimate from Theorem 3 some λ-dependent bound on the $(3/2-\varepsilon)$-Hölder norm of v on $\mathcal{D}_{s+\lambda}$ in terms of its uniform norm on* *the larger domain* $\mathcal{D}_s$. (The bound explodes as λ goes to 0.) We start from the regularized version

$$(\mathcal{L}v)_\delta = -(Av^3)_\delta + (B\cdot\nabla v)_\delta + (Z_2v^2)_\delta + (Z_1v)_\delta + (Z_0)_\delta. \tag{2.11}$$

of Equation 1.2. Note that the conclusion of Theorem 3 makes it possible to use the $(3/2-\varepsilon)$-Hölder seminorm of v in our estimates of the right-hand side of (2.11) provided it comes with a small factor that can eventually be absorbed in the left-hand side of (2.4). For the Z_0 and A terms in (2.11) we simply bound

$$\begin{aligned}\|(Z_0)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} &\leq c'_A\,\delta^{-\frac{1}{2}-\varepsilon}\|v\|_{\mathcal{D}_s}^{1/m_{Z_0}}\\ \|(Av^3)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} &\leq \sqrt{c_A}\,\|v\|_{\mathcal{D}_s}^3, \qquad \text{since } \|A\|_{\mathcal{D}}\leq\sqrt{c_A}.\end{aligned}$$

For the Z_2 and Z_1 terms in (2.11) one gets from the assumption (2.9) for $\delta\leq\lambda/4$

$$\begin{aligned}\|(Z_2v^2)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} &\leq c'_A\delta^{-\frac{1}{2}-\varepsilon}\|v\|_{\mathcal{D}_s}^{2+1/m_{Z_2}} + 2c'_A\delta^{-2\varepsilon}[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}\|v\|_{\mathcal{D}_s}^{1/m_{Z_2}+1},\\ \|(Z_1v)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} &\leq c'_A\delta^{-\frac{1}{2}-\varepsilon|}\|v\|_{\mathcal{D}_s}^{1+1/m_{Z_1}} + c'_A\delta^{-2\varepsilon}[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}\|v\|_{\mathcal{D}_s}^{1/m_{Z_1}}.\end{aligned}$$

It turns out to be useful to introduce the commutator $B_\delta\cdot\nabla v-(B\cdot\nabla v)_\delta$ to estimate $(B\cdot\nabla v)_\delta$ itself as we get from Lemma 5 the bound

$$\begin{aligned}\|(B\cdot\nabla v)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} &\leq \left\|B_\delta\cdot\nabla v-(B\cdot\nabla v)_\delta\right\|_{\mathcal{D}_{s+\lambda-\delta}} + \|B_\delta\cdot\nabla v\|_{\mathcal{D}_{s+\lambda-\delta}}\\ &\lesssim \delta^{1/2-2\varepsilon}[B]_{-\varepsilon}[\nabla v]_{\frac{1}{2}-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \delta^{-\varepsilon}[B]_{-\varepsilon}\|\nabla v\|_{\mathcal{D}_{s+\lambda/2}}.\end{aligned}$$

Write $V(z',z)$ for $v(z')-v(z)$ and note that for all λ sufficiently small

$$\begin{aligned}\|\nabla v\|_{\mathcal{D}_{s+\lambda/2}} &\lesssim \lambda^{\frac{1}{2}-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{-1}\|V\|_{(0;\lambda),\mathcal{D}_{s+\lambda/2}},\\ [\nabla v]_{1/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} &\lesssim [v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{-\frac{3}{2}+\varepsilon}\|V\|_{(0;\lambda),\mathcal{D}_{s+\lambda/2}}.\end{aligned} \tag{2.12}$$

Combining these estimates all together yields the bound

$$\delta^{1/2+\varepsilon}\|(\mathcal{L}v)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} \lesssim C_\lambda$$

where

$$\begin{aligned}C_\lambda = \lambda^{\frac{1}{2}+\varepsilon}\sqrt{c_A}\,\|v\|_{\mathcal{D}_s}^3 &+ c'_A\lambda\|v\|_{\mathcal{D}_s}^{1/m_B}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + c'_A\lambda^{-\frac{1}{2}}\|v\|_{\mathcal{D}_s}^{1+1/m_B}\\ &+ c'_A\|v\|_{\mathcal{D}_s}^{2+1/m_{Z_2}} + c'_A\lambda^{\frac{1}{2}-\varepsilon}[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}\|v\|_{\mathcal{D}_s}^{1/m_{Z_2}+1}\\ &+ c'_A\|v\|_{\mathcal{D}_s}^{1+1/m_{Z_1}} + c'_A\lambda^{\frac{1}{2}-\varepsilon}[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}\|v\|_{\mathcal{D}_s}^{1/m_{Z_1}} + c'_A\|v\|_{\mathcal{D}_s}^{1/m_{Z_0}}.\end{aligned}$$

Note that $\lambda/2$ is involved in C_λ rather than λ itself. The Schauder estimate from Proposition 3 gives us

$$\sup_{\lambda\leq\frac{\lambda_0}{2}}\lambda^{\frac{3}{2}-\varepsilon}[u]_{\frac{3}{2}-\varepsilon,\mathcal{D}_{s+\lambda}} \leq \sup_{\lambda\leq\lambda_0}\lambda^{\frac{3}{2}-\varepsilon}[u]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}} \lesssim \sup_{\lambda\leq\lambda_0}\lambda^{\frac{3}{2}-\varepsilon}C_\lambda + \|v\|_{\mathcal{D}_s}, \tag{2.13}$$

where the same domains are involved in the supremum on both sides. Even though $\lambda^{3/2-\varepsilon}C_\lambda$ depends on the $3/2-\varepsilon$ seminorm of v on $\mathcal{D}_{s+\lambda/2}$ it comes with a factor that will eventually be small for the choice of λ_0 made below. The constant $\lambda^{3/2-\varepsilon}C_\lambda$ still depends on some $1/2-\varepsilon$ seminorm of v on $\mathcal{D}_{s+\lambda/2}$. To eventually have a bound on the supremum in the right hand side of (2.13) that only involves $\|v\|_{\mathcal{D}_s}$ we use the estimates (2.5) and (2.6)

$$\begin{aligned}\|\nabla v\|_{\mathcal{D}_{s+\lambda}} &\lesssim \lambda^{\frac{1}{2}-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}} + \lambda^{-1}\|V\|_{(0;\lambda),\mathcal{D}_{s+\lambda}}\\ [v]_{1/2-\varepsilon,\mathcal{D}_{s+\lambda}} &\leq \lambda^{1-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}} + \lambda^{\frac{1}{2}+\varepsilon}\|\nabla v\|_{\mathcal{D}_{s+\lambda}}\end{aligned} \tag{2.14}$$

to see that

$$[v]_{1/2-\varepsilon,\mathcal{D}_{s+\lambda}} \lesssim \lambda^{1-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}} + \lambda^{-\frac{1}{2}+\varepsilon}\|v\|_{\mathcal{D}_{s+\lambda}}.$$

Overall we have a $[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}}$ term in an upper bound for $\lambda^{3/2-\varepsilon}C_\lambda$ that comes with a factor

$$\lambda^{3-3\varepsilon}\|v\|_{\mathcal{D}_s}^{1+1/m_{Z_2}}.$$

The choice

$$\lambda_0 \leq \|v\|_{\mathcal{D}_s}^{-1}$$

ensures that this term can be absorbed in the left hand side of (2.13) and we get

$$\sup_{\lambda\leq\lambda_0/2}\ \lambda^{\frac{3}{2}-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}}\lesssim\|v\|_{\mathcal{D}_s}. \tag{2.15}$$

(We used here the inequality $\varepsilon''>\varepsilon$.) Then it follows from (2.14) that

$$\sup_{\lambda\leq\lambda_0/2}\ \lambda^{\frac{1}{2}-\varepsilon}[v]_{\frac{1}{2}-\varepsilon,\mathcal{D}_{s+\lambda}}\lesssim\|v\|_{\mathcal{D}_s}. \tag{2.16}$$

A similar estimate holds for $\sup_{\lambda\leq\lambda_0/2}\lambda^{\frac{1}{2}+\varepsilon}[v]_{\frac{1}{2}+\varepsilon,\mathcal{D}_{s+\lambda}}$; we will use it below in Step 2.

We now state an elementary result that will be useful in the next step. For $z\in\mathbb{R}\times M$ we denote by $B(z,\delta)\subset\mathbb{R}\times M$ the parabolic ball of center z and radius δ.

Lemma 5 – *Pick $\alpha<0$ and $\beta\in(0,1)$ such that $\alpha+\beta>0$. Let $f\in C^\alpha(B(z,\delta))$ and $g\in C^\beta(B(z,\delta))$. Then we have*

$$\big|\big((fg)_\delta-f_\delta g\big)(z)\big|\lesssim\delta^{\alpha+\beta}[f]_{\alpha,B(z,\delta)}\,[g]_{\beta,B(z,\delta)}. \tag{2.17}$$

Moreover if $f\in L^\infty(B(z,\delta))$ we have

$$\big|(fg)_\delta(z)-(f_\delta g)(z)\big|\lesssim\delta^{\beta}\|f\|_\infty\,[g]_{\beta,B(z,\delta)}. \tag{2.18}$$

Step 2. *We regularize Equation* (1.2) *and apply the maximum principle of Lemma 4 to its solution.* The regularized version of Equation (1.2) takes the form

$$\begin{aligned}(\partial_t-\Delta+B_\delta\cdot\nabla)v_\delta&=-Av_\delta^3+[\mathcal{L},(\cdot)_\delta](v)+\big(B_\delta\cdot\nabla v_\delta-(B\cdot\nabla v)_\delta\big)+\big(Av_\delta^3-(Av^3)_\delta\big)\\&\qquad+(Z_2v^2)_\delta+\big((Z_1-1)v\big)_\delta+(Z_0)_\delta.\\&=:-Av_\delta^3+h_\delta.\end{aligned}$$

(The -1 in the linear term in v come from the fact that $\mathcal{L}=\partial_t-\Delta+1$ while Lemma 4 involves the operator $\partial_t-\Delta$.) For all $s_1>0$ such that $s+s_1<\frac{1}{2}$, the pointwise estimate from Lemma 4 gives here for $\|v_\delta\|_{\mathcal{D}_{s+s_1}}$ the upper bound

$$\begin{aligned}\max\Big\{&\big(\inf_{\mathcal{D}_0}A\big)^{-\frac{1}{2}}\frac{2}{s_1},\big(\inf_{\mathcal{D}_0}A\big)^{-\frac{1}{3}}\big\|[\mathcal{L},(\cdot)_\delta](v)\big\|^{\frac{1}{3}}_{\mathcal{D}_{s+s_1/2}},\big(\inf_{\mathcal{D}_0}A\big)^{-\frac{1}{3}}\|Av_\delta^3-(Av^3)_\delta\|^{\frac{1}{3}}_{\mathcal{D}_{s+s_1/2}},\\&\big(\inf_{\mathcal{D}_0}A\big)^{-\frac{1}{3}}\|B_\delta\cdot\nabla v_\delta-(B\cdot\nabla v)_\delta\|^{\frac{1}{3}}_{\mathcal{D}_{s+s_1/2}},\big(\inf_{\mathcal{D}_0}A\big)^{-\frac{1}{3}}\|(Z_2v^2)_\delta\|^{\frac{1}{3}}_{\mathcal{D}_{s+s_1/2}},\\&\big(\inf_{\mathcal{D}_0}A\big)^{-\frac{1}{3}}\|(Z_1v)_\delta\|^{\frac{1}{3}}_{\mathcal{D}_{s+s_1/2}},\big(\inf_{\mathcal{D}_0}A\big)^{-\frac{1}{3}}\|(Z_0)_\delta\|^{\frac{1}{3}}_{\mathcal{D}_{s+s_1/2}}\Big\},\end{aligned} \tag{2.19}$$

and one gets a bound on v writing for $0<\delta\leq s_1$

$$\|v\|_{\mathcal{D}_{s+s_1}}\leq\|v_\delta\|_{\mathcal{D}_{s+s_1}}+\delta^{\frac{1}{2}-\varepsilon}[v]_{(\frac{1}{2}-\varepsilon\,;\,2\delta),\mathcal{D}_{s+s_1-\delta}}\lesssim\|v_\delta\|_{\mathcal{D}_{s+s_1}}+\Big(\frac{\delta}{s_1}\Big)^{\frac{1}{2}-\varepsilon}\|v\|_{\mathcal{D}_s}, \tag{2.20}$$

as a consequence of (2.18) for the first inequality and (2.16) for the second inequality. Let us introduce a positive parameter $k\geq4$ that will be chosen below. Leaving aside the exponent 1/3, the different terms in the above maximum can be bounded as follows for $0<\delta\leq s_1/k$

$$\begin{aligned}&\big\|[\mathcal{L},(\cdot)_\delta](v)\big\|_{\mathcal{D}_{s+s_1/2}}\lesssim\delta^{-1}\|v\|_{\mathcal{D}_s},\\&\big\|Av_\delta^3-(Av^3)_\delta\big\|_{\mathcal{D}_{s+s_1/2}}\lesssim\Big(\delta^{\frac{1}{2}}\|A\|_{1/2,\mathcal{D}_s}+k^{-\frac{1}{2}+\varepsilon}\|A\|_{\mathcal{D}_s}\Big)\|v\|^3_{\mathcal{D}_s},\\&\big\|B_\delta\cdot\nabla v_\delta-(B\cdot\nabla v)_\delta\big\|_{\mathcal{D}_{s+s_1/2}}\lesssim[B]_{-\varepsilon}\|v\|_{\mathcal{D}_s}k^{-\frac{1}{2}+2\varepsilon}s_1^{-1+3\varepsilon},\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\|(Z_2v^2)_\delta\|_{\mathcal{D}_{s+s_1/2}}\lesssim[Z_2]_{-1/2-\varepsilon}\|v\|^2_{\mathcal{D}_s}\delta^{-\frac{1}{2}-\varepsilon},\\&\|(Z_1v)_\delta\|_{\mathcal{D}_{s+s_1/2}}\lesssim\delta^{-\frac{1}{2}-\varepsilon}[Z_1]_{-1/2-\varepsilon}\|v\|_{\mathcal{D}_s},\\&\|(Z_0)_\delta\|_{\mathcal{D}_{s+s_1/2}}\lesssim\delta^{-\frac{1}{2}-\varepsilon}[Z_0]_{-1/2-\varepsilon}.\end{aligned} \tag{2.22}$$

Indeed for the A term one has

$$(Av_\delta^3)(z)-(Av^3)_\delta(z)=\int\varphi_\delta(z,z')v(z')\Big\{\big(A(z)-A(z')\big)v_\delta^2(z)+A(z')\big(v_\delta^2(z)-v^2(z')\big)\Big\}dz'.$$

Denote by $w_A(\cdot, D)$ the modulus of continuity of A on a domain D. Above, the term with the increment of A gives a contribution bounded above by $w_A(\delta, \mathcal{D}_{s+s_1/4})\|v\|^3_{\mathcal{D}_s}$. As $0<\delta\leq s_1/k$ one has for $z\in\mathcal{D}_{s+s_1/2}$ and $z'\in\mathcal{D}_{s+s_1/4}$

$$\big|v_\delta^2(z)-v^2(z')\big|\lesssim \delta^{1/2-\varepsilon}[v]_{1/2-\varepsilon,\mathcal{D}_{s+s_1/4}}\|v\|_{\mathcal{D}_s},$$

and we get from the fact that A is (better than) 1/2-Hölder and the bound (2.16) the estimate

$$\big|(Av_\delta^3)(z)-(Av^3)_\delta(z)\big|\lesssim \Big(\delta^{\frac12}\|A\|_{1/2,\mathcal{D}_s}+\Big(\frac{\delta}{s_1}\Big)^{\frac12-\varepsilon}\|A\|_{\mathcal{D}_s}\Big)\|v\|^3_{\mathcal{D}_s}.$$

The condition $0<\delta\leq s_1/k$ gives the A estimate from (2.21).

For the B term write

$$B_\delta\cdot\nabla v_\delta-(B\cdot\nabla v)_\delta=\big(B_\delta\cdot\nabla v-(B\cdot\nabla v)_\delta\big)+B_\delta\cdot\nabla(v-v_\delta).$$

We use Lemma 5 to estimate the term in the big parenthesis on the right hand side. This gives

$$\big\|B_\delta\cdot\nabla v_\delta-(B\cdot\nabla v)_\delta\big\|_{\mathcal{D}_s+s_1/2}\lesssim \delta^{\frac12-2\varepsilon}[B]_{-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+s_1/4}},$$

and using the estimate (2.15) we get

$$\big\|B_\delta\cdot\nabla v_\delta-(B\cdot\nabla v)_\delta\big\|_{\mathcal{D}_s+s_1/2}\lesssim [B]_{-\varepsilon}\|v\|_{\mathcal{D}_s}\Big(\frac{\delta}{s_1}\Big)^{\frac12-2\varepsilon}s_1^{-1+3\varepsilon}\lesssim [B]_{-\varepsilon}\|v\|_{\mathcal{D}_s}k^{-\frac12+2\varepsilon}s_1^{-1+3\varepsilon}.$$

We also use Lemma 5 to deal with the Z_2 term and write

$$\begin{aligned}(Z_2v^2)_\delta&=(Z_2)_\delta v^2+O\big(\delta^\varepsilon[Z_2][v^2]_{1/2+2\varepsilon,\mathcal{D}_{s+s_1/4}}\big)\\&=O\big(\delta^{-\frac12-\varepsilon}[Z_2]\|v\|^2_{\mathcal{D}_s}\big)+O\big(\delta^\varepsilon[Z_2]\|v\|_{\mathcal{D}_s}[v]_{1/2+2\varepsilon,\mathcal{D}_{s+s_1/4}}\big),\end{aligned}$$

and using the variation of (2.16) we obtain the estimate on the Z_2 term in (2.22) since $\delta\leq s_1$. Similarly one has

$$\begin{aligned}(Z_1v)_\delta&=(Z_1)_\delta v+O\big(\delta^\varepsilon[Z_1][v]_{2\varepsilon,\mathcal{D}_{s+s_1/4}}\big)\\&=O\big(\delta^{-\frac12-\varepsilon}[Z_1]\|v\|_{\mathcal{D}_s}\big)+O\big(\delta^\varepsilon[Z_1]\|v\|_{\mathcal{D}_s}[v]_{2\varepsilon,\mathcal{D}_{s+s_1/4}}\big)\\&=O\big(\delta^{-\frac12-\varepsilon}[Z_1]\|v\|_{\mathcal{D}_s}\big)+O\big(\delta^\varepsilon[Z_1]\|v\|_{\mathcal{D}_s}s_1^{-2\varepsilon}]\|v\|_{\mathcal{D}_s}\big)=O\big(\delta^{-\frac12-\varepsilon}[Z_1]\|v\|_{\mathcal{D}_s}\big).\end{aligned}$$

Step 3. Choice of scales λ_0 and δ. We choose

$$s_1\geq\lambda_0=\|v\|^{-1}_{\mathcal{D}_s},\qquad \delta=c_1\|v\|^{-1-\varepsilon}_{\mathcal{D}_s}$$

for a positive constant $c_1\geq 1$ to be chosen below, so

$$k=c_1^{-1}\|v\|^\varepsilon_{\mathcal{D}_s}$$

makes the job. One then has

$$\begin{aligned}\big\|[\mathcal{L},(\cdot)_\delta](v)\big\|_{\mathcal{D}_{s+s_1/2}}&\lesssim c_1^{-1}\|v\|^{2+\varepsilon}_{\mathcal{D}_s},\\ \big\|Av_\delta^3-(Av^3)_\delta\big\|_{\mathcal{D}_{s+s_1/2}}&\lesssim c_1^{-\frac12}\|A\|_{1/2,\mathcal{D}_s}\|v\|^{3-\varepsilon(\frac12-\varepsilon)}_{\mathcal{D}_s},\\ \big\|B_\delta\cdot\nabla v_\delta-(B\cdot\nabla v)_\delta\big\|_{\mathcal{D}_{s+s_1/2}}&\lesssim c_1^{\frac12}[B]_{-\varepsilon}\|v\|^{2-\varepsilon(\frac12-2\varepsilon)-3\varepsilon}_{\mathcal{D}_s},\end{aligned}$$

and

$$\begin{aligned}\|(Z_2v^2)_\delta\|_{\mathcal{D}_{s+s_1/2}}&\lesssim c_1^{-\frac12-\varepsilon}[Z_2]_{-1/2-\varepsilon}\|v\|^{2+(1+\varepsilon)(\frac12+\varepsilon)}_{\mathcal{D}_s},\\ \|(Z_1v)_\delta\|_{\mathcal{D}_{s+s_1/2}}&\lesssim c_1^{-\frac12-\varepsilon}[Z_1]_{-1/2-\varepsilon}\|v\|^{1+(1+\varepsilon)(\frac12+\varepsilon)}_{\mathcal{D}_s},\\ \|(Z_0)_\delta\|_{\mathcal{D}_{s+s_1/2}}&\lesssim c_1^{-\frac12-\varepsilon}[Z_0]_{-1/2-\varepsilon}\|v\|^{(1+\varepsilon)(\frac12+\varepsilon)}_{\mathcal{D}_s}.\end{aligned}$$

We then choose c_1 so that the estimate on $[\mathcal{L},(\cdot)_\delta](v)$ reads

$$\big\|[\mathcal{L},(\cdot)_\delta](v)\big\|_{\mathcal{D}_{s+s_1/2}}\leq 2^{-3}\,\|v\|^{2+\varepsilon}_{\mathcal{D}_s},$$

and then one chooses c in (2.9) so that the A,B,Z_i terms above are all smaller than $2^{-3}(\inf_\mathcal{D}A)\|v\|^3_{\mathcal{D}_s}$. The choice of exponents m_τ in (2.1) was done precisely for that purpose. The estimates (2.20) and

(2.19) together give

$$\|v\|_{\mathcal{D}_{s+s_1}} \leq \max\left\{ \frac{2\,(\inf_{\mathcal{D}} A)^{-1/2}}{s_1} \,,\, \frac{1}{2}\,\|v\|_{\mathcal{D}_s} \right\}. \tag{2.23}$$

(b) Proof of Theorem 2 from (2.23). We now use an argument due to Moinat & Weber [19] to derive the L^∞ bound (2.2) from the estimate (2.23). We proceed differently depending on whether $\min_{\mathcal{D}_s} A \leq 1$ or $\min_{\mathcal{D}_s} A > 1$. For $\min_{\mathcal{D}_s} A \leq 1$ we have for

$$\widetilde{v} := \big(\min_{\mathcal{D}_s} A\big)^{\frac{1}{2}}\, v.$$

the estimate

$$\|\widetilde{v}\|_{\mathcal{D}_{s+s'}} \leq \max\left\{ \frac{2}{s'}, \frac{1}{2}\,\|\widetilde{v}\|_{\mathcal{D}_s} \right\}, \tag{2.24}$$

We set $s_1 := 4\,\|\widetilde{v}\|_{\mathcal{D}_s}^{-1}$ and define the times $s = s_0 < s + s_1 < \ldots < s + s_N = \frac{1}{2}$ from the relation

$$s_{n+1} - s_n = 4\,\|\widetilde{v}\|_{\mathcal{D}_{s+s_n}}^{-1}. \tag{2.25}$$

The sequence terminates once $s + s_{n+1} \geq 1/2$, in which case we set $s_{n+1} = s_N = 1/2$, or once the assumption (2.9) fails for $\mathcal{D}_{s+s_{n+1}}$. Since $4\|\widetilde{v}\|_{\mathcal{D}_{s+s_n}}^{-1}$ is increasing in n the sequence terminates after finitely many steps and we have

$$\|\widetilde{v}\|_{\mathcal{D}_{s+s_n}} \leq \frac{1}{2}\|\widetilde{v}\|_{\mathcal{D}_{s+s_{n-1}}}. \tag{2.26}$$

It follows from (2.25) and (2.26) that for $1 \leq n \leq N-1$

$$s + s_n \lesssim \|\widetilde{v}\|_{\mathcal{D}_{s+s_n}}^{-1}. \tag{2.27}$$

Therefore the bound in Theorem 2 holds at the time $s + s_n$, for $n \leq N-1$. For the last domain $\mathcal{D}_{s_N}$, if the assumption (2.9) fails for $\mathcal{D}_{s_N}$, then we get the bound immediately. Otherwise we have either $s + s_{N-1} \geq 1/4$ or $s_N - s_{N-1} \geq 1/4$. In the first case, we use the estimate (2.27) for $s + s_{N-1}$

$$s + s_N = \frac{1}{2} \leq 2(s + s_{N-1}) \lesssim \|\widetilde{v}\|_{\mathcal{D}_{s+s_{N-1}}}^{-1}. \tag{2.28}$$

In the second case, by (2.25), we have

$$s + s_N = \frac{1}{2} \leq 2(s_N - s_{N-1}) = 4\,\|\widetilde{v}\|_{\mathcal{D}_{s+s_{N-1}}}^{-1}. \tag{2.29}$$

Finally, for any time $r \in (s + s_n, s + s_{n+1})$ with $0 \leq n \leq N-2$, using (2.25) and (2.27) we infer that

$$r \leq s + s_{n+1} = s + s_n + (s_{n+1} - s_n) \lesssim \|\widetilde{v}\|_{\mathcal{D}_{s+s_n}}^{-1} \lesssim \|\widetilde{v}\|_{\mathcal{D}_r}^{-1},$$

and for $t \in (s + s_{N-1}, s + s_N)$, (2.28) and (2.29) imply that

$$r \leq s + s_N \lesssim \|\widetilde{v}\|_{\mathcal{D}_{s+s_{N-1}}}^{-1} \leq \|\widetilde{v}\|_{\mathcal{D}_r}^{-1}.$$

This gives the desired estimate

$$\|v\|_{\mathcal{D}_s} \leq \frac{2\,(\min_{\mathcal{D}} A)^{-\frac{1}{2}}}{s}.$$

In the case where $\min_{\mathcal{D}_s} A > 1$ we infer from (2.23) that

$$\|v\|_{\mathcal{D}_{s+s_1}} \leq \max\left\{ \frac{2}{s_1}, \frac{1}{2}\|v\|_{\mathcal{D}_s} \right\}.$$

We get the estimate $\|v\|_{\mathcal{D}_s} \leq 2/s$ by repeating the preceding argument.

3 – Controlling stronger norms of v

We quantify part of Step 1 in the proof of Theorem 2 to get the following result. It requires that $0 < \varepsilon \leq 1/4$. The variant of this statement proved in Section 4 will play a key role in our proof of the uniqueness of the Φ^4_3 measure in Section 5.

Theorem 6 – *There are two functions $C_\emptyset$ and $C'_\emptyset$ of the natural sizes of A, B, Z_1, Z_2 that do not depend on Z_0 such that setting*

$$\lambda_0 = C_\emptyset \wedge \big(\|Z_2\|\|v\|_{\mathcal{D}_s}\big)^{-\frac{2}{3-4\varepsilon}}$$

one has

$$[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda_0}} \lesssim C'_\emptyset\big(1 \vee \|v\|_{\mathcal{D}_s}\big)^3 + \|Z_0\|.$$

The precise formulas for $C_\emptyset$ and $C'_\emptyset$ are given in the proof of Theorem 6 and have no importance in this work.

Proof – The proof of this statement is essentially a variation on the content of Step 1 in the proof of Theorem 2. We repeat it here for the reader's convenience. We start with the equation

$$(\mathcal{L}v)_\delta = -(Av^3)_\delta + (B\cdot\nabla v)_\delta + (Z_2 v^2)_\delta + (Z_1 v)_\delta + (Z_0)_\delta. \tag{3.1}$$

Note that the conclusion of Proposition 3 makes it possible to use the $(3/2-\varepsilon)$-Hölder seminorm of v in our estimates of the right-hand side of (3.1) provided it comes with a small factor that can eventually be absorbed in the left-hand side of (2.4). We work with $0<\delta\leq\lambda/4$. For the Z_0 and a terms in (3.1) we have

$$\|(Z_0)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} \leq \delta^{-\frac{1}{2}-\varepsilon}\|Z_0\|$$
$$\|(Av^3)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} \leq \|A\|\|v\|^3_{\mathcal{D}_s}.$$

For the Z_2 and Z_1 terms in (3.1) one gets

$$\|(Z_2v^2)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} \leq \delta^{-\frac{1}{2}-\varepsilon}\|Z_2\|\,\|v\|^2_{\mathcal{D}_s} + 2\delta^{-2\varepsilon}\|Z_2\|\,[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}\|v\|_{\mathcal{D}_s},$$
$$\|(Z_1v)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} \leq \delta^{-\frac{1}{2}-\varepsilon|}\|Z_1\|\,\|v\|_{\mathcal{D}_s} + \delta^{-2\varepsilon}\|Z_1\|\,[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}.$$

It turns out to be useful to introduce the commutator $B_\delta\cdot\nabla v - (B\cdot\nabla v)_\delta$ to estimate $(B\cdot\nabla v)_\delta$ itself as we get from Lemma 5 the bound

$$\begin{aligned}\|(B\cdot\nabla v)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} &\leq \big\|B_\delta\cdot\nabla v - (B\cdot\nabla v)_\delta\big\|_{\mathcal{D}_{s+\lambda-\delta}} + \|B_\delta\cdot\nabla v\|_{\mathcal{D}_{s+\lambda-\delta}}\\ &\lesssim \delta^{\frac{1}{2}-2\varepsilon}\|B\|\,[\nabla v]_{1/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \delta^{-\varepsilon}\|B\|\,\|\nabla v\|_{\mathcal{D}_{s+\lambda/2}}.\end{aligned}$$

Write $V(z,z')$ for $v(z)-v(z')$ and note that for all λ sufficiently small

$$\begin{aligned}\|\nabla v\|_{\mathcal{D}_{s+\lambda/2}} &\lesssim \lambda^{\frac{1}{2}-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{-1}\|V\|_{(0;\lambda),\mathcal{D}_{s+\lambda/2}},\\ [\nabla v]_{1/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} &\lesssim [v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{-\frac{3}{2}+\varepsilon}\|V\|_{(0;\lambda),\mathcal{D}_{s+\lambda/2}}.\end{aligned} \tag{3.2}$$

Combining these estimates all together and $\|V\|_{(0;\lambda),\mathcal{D}_{s+\lambda/2}} \leq 2\|v\|_{\mathcal{D}_s}$ yields the bound

$$\delta^{2-\frac{3}{2}+\varepsilon}\|(\mathcal{L}v)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}} \lesssim C_\lambda$$

with

$$\begin{aligned}C_\lambda := \lambda^{\frac{1}{2}+\varepsilon}\|A\|\|v\|^3_{\mathcal{D}_s} &+ \|B\|\,[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{-\frac{1}{2}}\|B\|\,\|v\|_{\mathcal{D}_s}\\ &+ \|Z_2\|_{-1/2-\varepsilon}\|v\|^2_{\mathcal{D}_s} + \lambda^{\frac{1}{2}-\varepsilon}\|Z_2\|_{-1/2-\varepsilon}[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}\|v\|_{\mathcal{D}_s}\\ &+ \|Z_1\|\,\|v\|_{\mathcal{D}_s} + \lambda^{\frac{1}{2}-\varepsilon}\|Z_1\|\,[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}}\|v\|_{\mathcal{D}_s} + \|Z_0\|.\end{aligned}$$

Note that $\lambda/2$ is involved in C_λ rather than λ itself. The Schauder estimate from Proposition 3 gives us

$$\sup_{\lambda\leq\frac{\lambda_0}{2}} \lambda^{\frac{3}{2}-\varepsilon}[u]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}} \leq \sup_{\lambda\leq\lambda_0} \lambda^{\frac{3}{2}-\varepsilon}[u]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda}} \lesssim \sup_{\lambda\leq\lambda_0/2} \lambda^{\frac{3}{2}-\varepsilon}C_\lambda + \|v\|_{\mathcal{D}_s} \tag{3.3}$$

Even though the C_λ depend on the $(3/2-\varepsilon)$ seminorm of v on $\mathcal{D}_{s+\lambda}$ it comes with a small multiplicative factor if one chooses

$$\lambda_0 = \Big(4\max\Big\{\|B\|^{\frac{1}{1-\varepsilon}}\,;\,\|Z_1\|^{\frac{1}{3/2-2\varepsilon}}\,;\,\big(\|v\|_{\mathcal{D}_s}\|Z_2\|\big)^{\frac{1}{3/2-2\varepsilon}}\Big\}\Big)^{-1}.$$

The term of C_λ that involves the $(3/2-\varepsilon)$ seminorm of v can then be absorbed in the left-hand side of (3.3). Still the constant C_λ depends on some $(1/2-\varepsilon)$ seminorm of v. To eventually have a bound on C_λ that only involves $\|v\|_{\mathcal{D}_s}$ we use the elementary estimate

$$[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}} \leq [v]_{1/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} \leq \lambda^{1-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{\frac{1}{2}+\varepsilon}\|\nabla v\|_{\mathcal{D}_{s+\lambda/2}}$$

and the gradient bound (3.2) on v in uniform norm to see that

$$[v]_{(1/2-\varepsilon;\delta),\mathcal{D}_{s+\lambda/2}} \lesssim \lambda^{1-\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{-\frac{1}{2}+\varepsilon}\|v\|_{\mathcal{D}_{s+\lambda/2}}.$$

One therefore has

$$\lambda^{2-\varepsilon_0-\varepsilon}\|Z_i\|\,[v]_{(1/2-\varepsilon\,;\,\delta),\mathcal{D}_{s+\lambda/2}} \lesssim \|Z_i\|\big(\lambda^{3-\varepsilon_0-2\varepsilon}[v]_{3/2-\varepsilon,\mathcal{D}_{s+\lambda/2}} + \lambda^{\frac{3}{2}-\varepsilon_0}\|v\|_{\mathcal{D}_s}\big)$$

for $1 \leq i \leq 2$, so our choice of λ_0 ensures that this term can also be absorbed in the left-hand side of (3.3). We get in the end the bound

$$\sup_{\lambda\leq\lambda_0/2} \lambda^{\frac{3}{2}-\varepsilon}[v]_{\frac{3}{2}-\varepsilon,\mathcal{D}_{s+\lambda}} \lesssim \sup_{\lambda\leq\lambda_0} C'_\lambda + \|v\|_{\mathcal{D}_s}$$

where

$$\begin{aligned} C'_\lambda = \lambda^2\|A\|\,\|v\|^3_{\mathcal{D}_s} + \lambda^{1-\varepsilon}\|B\|\,\|v\|_{\mathcal{D}_s} + \lambda^{\frac{3}{2}-\varepsilon}\|Z_2\|\,\|v\|^2_{\mathcal{D}_s} + \lambda^{\frac{3}{2}-\varepsilon_0}\|Z_1\|\,\|v\|_{\mathcal{D}_s} \\ + \lambda^{\frac{3}{2}-\varepsilon_0}\|Z_2\|\,\|v\|^2_{\mathcal{D}_s} + \lambda^{\frac{3}{2}-\varepsilon}\|Z_1\|\,\|v\|_{\mathcal{D}_s} + \lambda^{\frac{3}{2}-\varepsilon}\|Z_0\|. \end{aligned}$$

This is an increasing function of λ and writing

$$[v]_{\frac{3}{2}-\varepsilon,\mathcal{D}_{s+\lambda_0}} \lesssim \lambda_0^{-\frac{3}{2}+\varepsilon}\big(C'_{\lambda_0} + \|v\|_{\mathcal{D}_s}\big)$$

gives the conclusion. ▷

4 – Variation on a theme

We will use in our proof of the uniqueness of an invariant measure of (1.2) a perturbed version of this dynamics that contains an additional drift with a particular form. This section is dedicated to proving for the solution of this perturbed dynamics some explicit control on its L^∞ and stronger norm similar to Theorem 2 and Theorem 6. We use the convention that $\mathbf{1}_{1<\cdot<\tau} = 0$ if $\tau = 1$. Below, the space $(\!|\alpha_0, 1+2\varepsilon|\!)$ was introduced in [4] for a particular value of α_0 that does not matter here. The norm on this space quantifies the explosion of a function of time $t > 0$ as t goes to 0; its precise definition here does not really matter. For $0 < T < \tau$ the restriction to $[T,\tau)$ of an element of the space $C\big([0,\tau), C^{-1/2-\varepsilon}(M)\big) \cap (\!|\alpha_0, 1+2\varepsilon|\!)$ is an element of $C\big([T,\tau], C^{1+2\varepsilon}(M)\big)$.

Theorem 7 – *Pick a constant $\ell \in \mathbb{R}$ and an initial condition $\phi_2' \in C^{-1/2-\varepsilon}(M)$. The equation*

$$\begin{aligned} \partial_t v_\ell &= (\Delta-1)v_\ell - Av_\ell^3 + B\nabla v_\ell + Z_2 v_\ell^2 + Z_1 v_\ell + Z_0 + \ell\,\mathbf{1}_{1<t<\tau}\,\frac{v(t)-v_\ell(t)}{\|v(t)-v_\ell(t)\|_{L^2}}\,\exp\big(3\Upsilon(t)\big) \\ &=: F(v_\ell) + \ell\,\mathbf{1}_{1<t<\tau}\,\frac{v(t)-v_\ell(t)}{\|v(t)-v_\ell(t)\|_{L^2}}\,\exp\big(3\Upsilon(t)\big) =: F_\ell(t,v_\ell), \qquad (0\leq t<\tau), \end{aligned} \tag{4.1}$$

where

$$\tau = \tau(\ell,\phi_1,\phi_2) := \inf\big\{s\geq 1\,;\, v_\ell(s) = v(s)\big\} \wedge 2,$$

has a unique solution in $C\big([0,\tau), C^{-1/2-\varepsilon}(M)\big) \cap (\!|\alpha_0, 1+2\varepsilon|\!)$. Furthermore, it satisfies the estimates

$$\|v_\ell(s)\|_{L^\infty} \leq c_1(\widehat{\xi}\,)\,(1+\ell^{\frac{1}{3}}) \tag{4.2}$$

and

$$\|v_\ell(s)\|_{C^{1+2\varepsilon}} \leq c_2(\widehat{\xi}\,)\,(1+\ell) \tag{4.3}$$

for all $1\leq s<\tau$, for some explicit functions $c_1(\widehat{\xi}\,), c_2(\widehat{\xi}\,)$ of $\widehat{\xi}$ whose precise values play no role in what follows.

Proof – *Local in time well-posedness beyond time* 1. There is no loss of generality in assuming that $v(1,\phi_1') \neq v_\ell(1,\phi_2')$. Denote by $C_{v_\ell(1,\phi_2')}\big([1,T], C^{1+2\varepsilon}(M)\big)$ the set of continuous functions from the interval $[1,T]$ into $C^{1+2\varepsilon}(M)$ with value $v_\ell(1,\phi_2')$ at time 1. Pick a positive constant

$$m < \big\|v(1,\phi_1') - v_\ell(1,\phi_2')\big\|_{L^2} \wedge 1$$

(think of it as being small) and set

$$\mathcal{V}_{v_\ell(1,\phi_2')}(m,T) := \Big\{ w \in C_{v_\ell(1,\phi_2')}\big([1,T], C^{1+2\varepsilon}(M)\big) \,;\, \min_{1\le t\le T} \|v(t)-w(t)\|_{L^2} > m \Big\}.$$

This is an open subset of $C_{v_\ell(1,\phi_2')}\big([1,T], C^{1+2\varepsilon}(M)\big)$ with closure $\overline{\mathcal{V}}_{v_\ell(1,\phi_2')}(m,T)$ included in

$$\Big\{ w \in C_{v_\ell(1,\phi_2')}\big([1,T], C^{1+2\varepsilon}(M)\big) \,;\, \min_{1\le t\le T} \|v(t)-w(t)\|_{L^2} \ge m \Big\}.$$

Lemma 8 – *There exists a positive time $T(\ell,m)$ such that the map $\mathscr{F}$ defined as*

$$\mathscr{F}(w)(t) := e^{(t-1)(\Delta-1)}\big(v_\ell(1,\phi_2')\big) + \int_0^{t-1} e^{(t-1-s)(\Delta-1)}\big(F_\ell(1+s,w)\big)ds$$

is a contraction of $\overline{\mathcal{V}}_{v_\ell(1,\phi_2')}\big(m,T(\ell,m)\big)$ into itself. One can choose $T(\ell,m)$ as a decreasing function of m.

Proof – Indeed, for $w\in\mathcal{V}_{v_\ell(1,\phi_2')}(C,T)$ one has $F_\ell(w)\in C\big([1,T],C^{-1/2-\varepsilon}(M)\big)$ with

$$\begin{aligned} F_\ell(w_1)-F_\ell(w_2) = (\Delta-1)(w_1-w_2) - A\big(w_1^3-w_2^3\big) + B\nabla(w_1-w_2) + Z_2\big(w_1^2-w_2^2\big) + Z_1(w_1-w_2) \\ + \ell\Big(\frac{v-w_1}{\|v-w_1\|_{L^2}} - \frac{v-w_2}{\|v-w_2\|_{L^2}}\Big) \end{aligned}$$

and

$$\begin{aligned} \big\|F_\ell(w_1)-F_\ell(w_2)\big\|_{C_TC^{-1/2-\varepsilon}} \lesssim 1 + 2\big(\|w_1\|_{L^\infty}^2 + \|w_2\|_{L^\infty}^2\big)\|w_1-w_2\|_{L^\infty} + \|w_1-w_2\|_{C^{1+\varepsilon}} \\ + \big(\|w_1\|_{L^\infty} + \|w_2\|_{L^\infty}+1\big)\|w_1-w_2\|_{C^{1/2+\varepsilon}} \\ + \ell\, m^{-2}\big(\|v\|_{L^\infty} + \|w_1\|_{L^\infty} + \|w_2\|_{L^\infty}\big)\|w_1-w_2\|_{L^\infty}. \end{aligned}$$

(We estimated the ℓ-term in L^∞ by bounding the operator norm of the map $\|f\|_{L^2}f$ from L^∞ into itself. We left the constants on the right hand side to make the verification easier.) The small-time estimate

$$\big\|\big(\partial_t-(\Delta-1)\big)^{-1}(f)\big\|_{C_TC^{1+2\varepsilon}} \lesssim T^{\frac12-3\varepsilon}\|f\|_{C_TC^{-1/2-\varepsilon}},$$

then entails the statement of the lemma. ►

Two solutions corresponding to $m_2<m_1$ satisfy $T(\ell,m_2)\ge T(\ell,m_1)$ and coincide on the interval $[0,T(\ell,m_1)]$ from uniqueness. We prove the non-explosion of the solution to Equation (4.1) before the coupling time with the path $(u(t))_{t\ge0}$ by showing that any solution to (4.1) on a time interval $(0,T]$ satisfies the size estimates (4.2) and (4.3). By repeated use of the local in time result of Lemma 8 these estimates allow to define the maximal existence time $\tau(\ell,m)\ge T(\ell,m)$ before $\|v_\ell(t)-v(t)\|_{L^2}=m$, with $\tau(\ell,m_2)\ge\tau(\ell,m_1)$ if $m_2<m_1$. The coupling time τ is defined as

$$\tau=\tau(\ell)=\lim_{m\downarrow0}\tau(\ell,m).$$

We assume in the next paragraph that v_ℓ is a solution to (4.1) on a time interval $(0,T]$, so $\|v(t)-v_\ell(t)\|_{L^2}\ge m$ for some $m>0$ below.

Quantitative estimates (4.2) *and* (4.3)*.* The proof of these statements is similar to the proofs of Theorem 2 and Theorem 6 but one has to deal with the fact that the drift

$$G_\ell(v_\ell)(t) := \ell\,\frac{v(t)-v_\ell(t)}{\|v(t)-v_\ell(t)\|_{L^2}}\,\exp\big(3\Upsilon(t)\big)$$

does not have good controls in $C^{-1/2-\varepsilon}(M)$ for all times. Rather it has good controls when seen as an element of $L^2(M)$. We only point out the modifications of the proof of Theorem 2 needed to provide a proof of (4.2) and (4.3). We go linearly along the proof of the former.

In Step 1, looking at the drift as a continuous function of time with values in a fixed ball of $L^2(M)$ of radius proportional to ℓ, one gets an additional term

$$\big\|\big(G_\ell(v_\ell)\big)_\delta\big\|_{\mathcal{D}_{s+\lambda-\delta}} \lesssim_{\widehat{\xi}} \delta^{-\frac34}\,\ell.$$

The worst exponent in Step 1 in Section 2 was $-1/2-\varepsilon$. This difference between the worst exponents in the two situations explains why the reasoning of Step 1 cannot give control of the $3/2-\varepsilon$ seminorm of v_ℓ as the quantity $\delta^{1/2+\varepsilon}\|(\mathcal{L}v_\ell)_\delta\|_{\mathcal{D}_{s+\lambda-\delta}}$ is not bounded anymore. It gives

however a control of the $1+2\varepsilon$ seminorm of v_ℓ in terms of $\|v_\ell\|_{\mathcal{D}_s}+\ell$, following verbatim what was done in Section 2.2.
The contribution of the drift $G_\ell(v_\ell)$ in Step 2 is the same as in Step 1 and one needs in addition to replace here the use of the $(3/2-\varepsilon)$ seminorm of v made in Section 2.2 in the estimate of the B term by the use of the $(1+2\varepsilon)$ seminorm of v_ℓ. Step 3 works similarly as in Section 2.2 and provides the estimate

$$\|v_\ell\|_{\mathcal{D}_{s+s_1}} \leq \max\left\{\frac{2\,(\inf_{\mathcal{D}} A)^{-\frac{1}{2}}}{s_1}\,,\,\frac{1}{2}\,\|v_\ell\|_{\mathcal{D}_s}+\ell^{\frac{1}{3}}\right\},$$

so we have

$$\|v_\ell\|_{\mathcal{D}_{s+s_1}} \leq \max\left\{\frac{2\,(\inf_{\mathcal{D}} A)^{-\frac{1}{2}}}{s_1}\,,\,\frac{3}{4}\,\|v_\ell\|_{\mathcal{D}_s}\right\}$$

as long as $\|v_\ell\|_{\mathcal{D}_s}\geq 4\,\ell^{\frac{1}{3}}$. Proceeding as in part *(b)* of the proof of Theorem 2 with the contraction coefficient 1/2 replaced by 3/4 one gets

$$\|v\|_{\mathcal{D}_s} \lesssim \max\left\{\frac{1}{s},\,\big((c_A[\tau]_{|\tau|})^{m_\tau}\big)_{\tau\in\mathcal{T}}\,,\,4\,\ell^{\frac{1}{3}}\right\},$$

from which (4.2) follows. We repeat the proof of Theorem 6 to obtain (4.3), adding the contribution of the drift and replacing the $(3/2-\varepsilon)$ seminorm by the $(1+2\varepsilon)$ seminorm. ▷

5 – Uniqueness of the invariant measure

We proved in Proposition 21 of [4] that the dynamics on $C^{-1/2-\varepsilon}(M)$ generated by (1.1) is Markovian; we prove in this section that it has at most one invariant probability measure. We work for that purpose with Jagannath & Perkowski's formulation (1.2) of (1.1) and use a *coupling argument* to prove the uniqueness. Given two points $\phi_1',\phi_2'\in C^{-1/2-\varepsilon}(M)$, an elementary coupling of two solutions to (1.2) started from ϕ_1' and ϕ_2' would consist in constructing on some probability space a pair of spacetime white noises such that the solutions to Equation (1.2) built from each of these noises take the same value at some fixed positive finite time T outside of an event of arbitrarily small probability independent of ϕ_1',ϕ_2'. The corresponding solutions of Equation (1.1) would also coincide at that time. One could then take some random initial conditions with law two invariant probability measures μ_1,μ_2 for the dynamics generated by (1.1) and write for any continuous function f on $C^{-1/2-\varepsilon}(M)$

$$\mu_1(f)=\mathbb{E}\big[f(u(T;\mu_1))\big]=\mathbb{E}\big[f(u(T;\mu_2))\big]=\mu_2(f),$$

with $u(\cdot\,;\mu_i)$ denoting the solution to (1.1) with random initial condition with law μ_i. We are not able to produce such a strong coupling here; rather, given the trajectory $u(\cdot\,;\mu_1)$, we can add to the dynamics of $u(\cdot\,;\mu_2)$ a drift that forces the latter to meet the former by a fixed time T with high probability. With the dynamics of $u(\cdot\,;\mu_2)$ changed the measure μ_2 is not invariant anymore for this new dynamics and the above simple argument for uniqueness does not apply per se. However, for a particular drift there is an equivalent probability measure on our probability space for which the new dynamics has the same law as the original dynamics with random initial condition with law μ_2. A variation on the above pattern of proof then gives the equality of μ_1 and μ_2.

Denote by $(\Omega,\mathcal{F},\mathbb{P})$ the probability space on which all our random variables have been implicitly defined so far. We write $\mathscr{L}_{\mathbb{P}}(X)$ for the law under $\mathbb{P}$ of a random variable X and use a similar notation $\mathscr{L}_{\mathbb{Q}}(X)$ for any other probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$. We write $\mathbb{E}_{\mathbb{P}}$ and $\mathbb{E}_{\mathbb{Q}}$ for the corresponding expectation operators. We use a coupling by a change of measure argument to prove that the semigroup $(\mathcal{P}_t)_{t\geq 0}$ on $C^{-1/2-\varepsilon}(M)$ generated by (1.1) has at most one invariant probability measure.

Theorem 9 – *The semigroup $(\mathcal{P}_t)_{t\geq 0}$ has at most one invariant probability measure.*

Proof – We proceed in two steps and first construct a coupling by a change of measure between two trajectories of the Jagannath & Perkowski version of the Φ^4_3 dynamics started from different points. As a preliminary remark note that shifting the noise ξ by a (possibly random) element h

of its Cameron-Martin space with support in time in the interval $[1,2]$ is equivalent to adding a drift $h\exp\big(3\curlyvee\big)$ to the dynamics of v. Indeed, let ϕ and ϕ' be related by the relation

$$\phi' = \exp\big(3\curlyvee(0)\big)\Big(\phi - \mathsf{I}(0) + \Psi(0)\Big) - v_{\mathrm{ref}}(0).$$

One sees that if v_r^h solves an equation of the form

$$(\partial_t - \Delta)v_r^h = B_r\cdot\nabla v_r^h - A_r(v_r^h)^3 + Z_{2,r}(v_r^h)^2 + Z_{1,r}v_r^h + Z_{0,r} + (e^{r\Delta}h)\,e^{3\curlyvee_r}, \qquad v_r^h(0) = \phi',$$

then

$$u_r^h = \mathsf{I}_r - \Psi_r + e^{-3\curlyvee_r}(v_r^h + v_{\mathrm{ref},r}), \tag{5.1}$$

solves the equation

$$(\partial_t - \Delta)u_r^h = -(u_r^h)^3 + 3(a_r - b_r)u_r^h + e^{r\Delta}(\xi + h), \qquad u_r(0) = \phi.$$

The convergence of v_r^h to the solution v^h of the equation

$$(\partial_t - \Delta)v^h = B\cdot\nabla v^h - A(v^h)^3 + Z_2(v^h)^2 + Z_1 v^h + Z_0 + he^{3\curlyvee}, \qquad v^h(0) = \phi',$$

ensures the convergence of u_r^h to a limit.

Step 1 – The coupling. Pick $\phi_1,\phi_2\in C^{-1/2-\varepsilon}(M)$ with corresponding ϕ_1',ϕ_2'. We adopt as above the notation $v = v(\cdot,\phi_1')$ for the solution of the Jagannath-Perkowski equation with initial condition ϕ_1', with $u = u(\cdot,\phi_1)$ the corresponding function given by the inverse Jagannath-Perkowski transform. Recall Theorem 7 provides some quantitative estimates on the solutions of the equation

$$\begin{aligned}\partial_t v_\ell &= (\Delta-1)v_\ell - Av_\ell^3 + B\cdot\nabla v_\ell + Z_2 v_\ell^2 + Z_1 v_\ell + Z_0 + \ell\,\mathbf{1}_{1<t<\tau}\,\frac{v(t)-v_\ell(t)}{\|v(t)-v_\ell(t)\|_{L^2}}\,\exp\big(3\curlyvee(t)\big)\\ &=: F(v_\ell) + \ell\,\mathbf{1}_{1<t<\tau}\,\frac{v(t)-v_\ell(t)}{\|v(t)-v_\ell(t)\|_{L^2}}\,\exp\big(3\curlyvee(t)\big) =: F_\ell(t,v_\ell), \qquad (0\le t<\tau),\end{aligned}$$

where

$$\tau = \tau(\ell,\phi_1,\phi_2) := \inf\big\{s\ge 1\,;\, v_\ell(s) = v(s)\big\}\wedge 2.$$

The random time τ is called the *coupling time* – we take as a convention $\inf\emptyset = +\infty$. A *successful coupling* corresponds to the event $\{\tau<2\}$, in which case we let $v_\ell(t) = v(t)$ for $t\ge\tau$. We have

$$\mathbf{1}_{\tau<2}\,v_\ell(2,\phi_2') = \mathbf{1}_{\tau<2}\,v(2,\phi_1')$$

and

$$\mathbf{1}_{\tau<2}\,u_\ell(2,\phi_2) = \mathbf{1}_{\tau<2}\,u(2,\phi_1),$$

with u_ℓ corresponding to v_ℓ via the inverse Jagannath-Perkowski transform (5.1).

Lemma 10 – *Take $\ell\ge 1$. There is an absolute constant $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$ and for $1<t<\tau$ one has*

$$\big\langle F(v) - F_\ell(t,v'), v - v'\big\rangle_{L^2} \lesssim -\big(\ell - o_{\widehat{\xi}}(\ell)\big)\|v-v'\|_{L^2} \tag{5.2}$$

for all $v,v'\in C^{1+2\varepsilon}(M)$ with

$$\|v\|_{L^\infty}\vee\|v'\|_{L^\infty} \le c_1(\widehat{\xi}\,)\,\ell^{\frac13},$$

and

$$\|v\|_{C^{1+2\varepsilon}}\vee\|v'\|_{C^{1+2\varepsilon}} \le c_2(\widehat{\xi}\,)\,\ell, \tag{5.3}$$

for a $\widehat{\xi}$-dependent non-negative function $o_{\widehat{\xi}}(\ell)$ of ℓ such that $o_{\widehat{\xi}}(\ell)/\ell$ goes to 0 as ℓ goes to ∞.

Proof – Note that there are no absolute values in (5.2). We give upper bounds for each term in this expression. We make the common abuse of notation of writing $\langle Z_i f, g\rangle_{L^2}$ for $\langle Z_i fg, \mathbf{1}\rangle$, the result of testing a well-defined distribution $Z_i fg$ on the constant function $\mathbf{1}$. We use a similar convention for $\langle B\cdot\nabla f, g\rangle_{L^2}$.

– *The A term.* As A is positive one has

$$\big\langle -A\big(v^3 - v'^3\big), v - v'\big\rangle_{L^2} \le 0,$$

and this term does not contribute to the upper bound (5.2).

– *The B term.* We start from the identity

$$(v-v')^2 = 2(v-v') \prec (v-v') + (v-v') \odot (v-v'),$$

with the left $(v-v')$ seen as an element of $L^2(M)$ and the right $(v-v')$ seen as an element of $C^{1+2\varepsilon}(M)$ in the paraproduct and resonant terms, and estimate each term in $B^{1+\varepsilon}_{2,\infty}(M)$. Losing a little bit on the regularity exponent allows using the interpolation size estimate between different Besov spaces and estimate

$$\|v-v'\|_{B^{1+\varepsilon}_{\infty\infty}} \lesssim_{\widehat{\xi}} \ell^{\frac{1}{3}\frac{\varepsilon}{1+2\varepsilon}+\frac{1+\varepsilon}{1+2\varepsilon}},$$

with an exponent strictly smaller than 1. We write

$$\|v-v'\|_{B^{1+\varepsilon}_{\infty\infty}} \lesssim_{\widehat{\xi}} \ell^{<1}.$$

One then gets from the classical continuity estimates on the paraproduct and resonant operators that

$$\|(v-v')^2\|_{B^{1+\varepsilon}_{2\infty}} \lesssim_{\widehat{\xi}} \ell^{<1}\, \|v-v'\|_{L^2}.$$

– *The Z_1 term.* We take advantage of the fact that $Z_1 \in C_T B_{21}^{-\frac{1}{2}-\frac{\varepsilon}{2}}(M)$ almost surely. We use the previous estimate to see that

$$\big|\langle Z_1(v-v'), v-v'\rangle\big| \lesssim_{\widehat{\xi}, Z_1} \ell^{<1}\, \|v-v'\|_{L^2}.$$

– *The Z_2 term.* Here as well we consider Z_2 as an element of $C_T B_{21}^{-\frac{1}{2}-\frac{\varepsilon}{2}}(M)$. First we obtain by interpolation between the L^∞ and $C^{1+2\varepsilon}$ estimate on v and v' that

$$\|v \pm v'\|_{B^{\frac{1}{2}+2\varepsilon}_{\infty\infty}} \lesssim_{\widehat{\xi}} \ell^{\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon}},$$

with an exponent slightly bigger than 5/9. We thus get from the classical continuity estimates on the paraproduct and resonant operators that

$$\|(v-v')^2\|_{B^{\frac{1}{2}+2\varepsilon}_{2\infty}} \lesssim_{\widehat{\xi}} \ell^{\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon}}\, \|v-v'\|_{L^2}. \tag{5.4}$$

Now write

$$\big(v^2-(v')^2\big)(v-v') = (v-v')^2(v+v') = (v-v')^2 \prec (v+v') + \Big\{(v+v') \prec (v-v')^2 + (v+v')\odot(v-v')^2\Big\}.$$

To estimate the contribution of the first paraproduct in the Z_2 term we use the elementary refined continuity estimate from Lemma 13 in Appendix A to get the best of the L^∞ and $C^{1+2\varepsilon}$ estimates on $(v+v')$. We have for all integers N

$$\begin{aligned}\big\|(v-v')^2 \prec (v+v')\big\|_{B^{(\frac{1}{2}+2\varepsilon)-\varepsilon}_{2\infty}} &\lesssim_{\widehat{\xi}} \|(v-v')^2\|_{B^{\frac{1}{2}+2\varepsilon}_{2\infty}} \Big(2^{-N\varepsilon}\|v+v'\|_{B^{\frac{1}{2}+2\varepsilon}_{\infty\infty}} + N\|v+v'\|_{L^\infty}\Big) \\ &\lesssim_{\widehat{\xi}} \ell^{\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon}}\, \|v-v'\|_{L^2}\Big(2^{-N\varepsilon}\ell^{\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon}} + N\ell^{\frac{1}{3}}\Big).\end{aligned}$$

Choosing N such that $2^{-N\varepsilon}\ell^{\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon}} \simeq \ell^{\frac{1}{3}}$ gives

$$2^{-N\varepsilon/2}\ell^{\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon}} + N\ell^{\frac{1}{3}} \lesssim \ell^{\frac{1}{3}+\eta}$$

for every $\eta > 0$ and $\ell \geq \ell(\eta)$ large enough. One thus has

$$\big\|(v-v')^2 \prec (v+v')\big\|_{B^{\frac{1}{2}+\varepsilon}_{2\infty}} \lesssim_{\widehat{\xi}} \ell^{<1}\, \|v-v'\|_{L^2}$$

for an exponent $\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon} + \frac{1}{3} + \eta$ of ℓ strictly smaller than 1, for an appropriate choice of η. We can directly use (5.4) and the L^∞ estimate on v and v' to see that

$$\big\|(v+v') \prec (v-v')^2 + (v+v')\odot(v-v')^2\big\|_{B^{\frac{1}{2}+2\varepsilon}_{2\infty}} \lesssim_{\widehat{\xi}} \ell^{\frac{1}{3}\frac{5+6\varepsilon}{3-2\varepsilon}+\frac{1}{3}}\, \|v-v'\|_{L^2},$$

here again with an exponent of ℓ strictly smaller than 1.

– *The Z_0 term.* As we have a term $-\ell\|v-v'\|_{L^2}$ that comes from $F_\ell(t,v')$ and all the other contributions to $\big\langle F(v) - F_\ell(t,v'), v-v'\big\rangle_{L^2}$ add up to a quantity bounded above by a constant multiple of $\ell^{<1}\, \|v-v'\|_{L^2}$ we obtain (5.2) for an appropriate choice of $\varepsilon_0 > 0$. ▷

The proof makes it clear that one can take $o_{\widehat{\xi}}(\ell)$ of the form

$$o_{\widehat{\xi}}(\ell) = c\Big(\|B_{|[1,2]}\|_{C([1,2],C^{-\varepsilon}(M))} + \|Z_{2|[1,2]}\|_{C([1,2],C^{-1/2-\varepsilon})(M)} + \|Z_{1|[1,2]}\|_{C([1,2],C^{-1/2-\varepsilon})(M)}\Big)\ell^\gamma \tag{5.5}$$

for some positive constant c and some positive exponent $\gamma < 1$.

As in Lemma 4 of [4] one proves that the function of time $\|v(\cdot,\phi_1') - v_\ell(\cdot,\phi_2')\|_{L^2}$ is Young differentiable on the interval $(1,\tau)$ and one has for $1 < t < \tau$

$$\big\|v(t,\phi_1') - v_\ell(t,\phi_2')\big\|_{L^2} - \big\|v(1,\phi_1') - v_\ell(1,\phi_2')\big\|_{L^2} = \int_1^t \frac{\Big\langle F\big(v(s,\phi_1')\big) - F_\ell\big(t, v_\ell(s,\phi_2')\big), v(s,\phi_1') - v_\ell(s,\phi_2')\Big\rangle_{L^2}}{\big\|v(s,\phi_1') - v_\ell(s,\phi_2')\big\|_{L^2}}\,ds.$$

It follows from Lemma 10 that one has for $1 \leq t < \tau$ the inequality

$$\big\|v(t,\phi_1') - v_\ell(t,\phi_2')\big\|_{L^2} \leq \big\|v(1,\phi_1') - v_\ell(1,\phi_2')\big\|_{L^2} - \big(\ell - o_{\widehat{\xi}}(\ell)\big)(t-1), \tag{5.6}$$

for a positive quantity $o_{\widehat{\xi}}(\ell)$ that depends on $\widehat{\xi}_{|[1,2]}$ such that $o_{\widehat{\xi}}(\ell)/\ell$ goes to 0 as ℓ goes to ∞. So one has a successful coupling on the event

$$\left\{\big\|v(1,\phi_1') - v_\ell(1,\phi_2')\big\|_{L^2} \leq \frac{\ell - o_{\widehat{\xi}}(\ell)}{2}\right\} \subset \big\{\tau < 2\big\}.$$

As a consequence of this inclusion and the L^p or L^∞ coming down from infinity result of [4] or Theorem 2, one can choose ℓ big enough to have both $\mathbb{P}(\tau(\ell,\phi_1,\phi_2) = 2)$ and $\mathbb{Q}_\ell(\tau(\ell,\phi_1,\phi_2) = 2)$ strictly smaller than 1 independently of ϕ_1,ϕ_2, say

$$\max\big(\mathbb{P}(\tau(\ell,\phi_1',\phi_2') = 2), \mathbb{Q}_\ell(\tau(\ell,\phi_1',\phi_2') = 2)\big) \leq a < 1, \quad (\forall\, \phi_1',\phi_2' \in C^{-1/2-\varepsilon}(M)). \tag{5.7}$$

We fix such an ℓ and set

$$R_{\ell,\phi_1,\phi_2} := \exp\left(-\ell\,\xi\Big(\mathbf{1}_{1<\cdot<\tau}\,\frac{v(\cdot,\phi_1') - v_\ell(\cdot,\phi_2')}{\|v(\cdot,\phi_1') - v_\ell(\cdot,\phi_2')\|_{L^2}}\Big) - \frac{\ell^2(\tau-1)}{2}\right).$$

Since $\tau \leq 2$, Novikov's integrability criterion

$$\mathbb{E}\Big[\exp\Big(\frac{\ell^2(\tau-1)}{2}\Big)\Big] < \infty$$

is satisfied and it follows from Girsanov theorem that the process

$$\xi + \ell\,\mathbf{1}_{1<\cdot<\tau}\,\frac{v - v_\ell}{\|v - v_\ell\|_{L^2}}$$

is under the probability

$$d\mathbb{Q}_{\ell,\phi_1,\phi_2} := R_{\ell,\phi_1,\phi_2}\,d\mathbb{P}$$

a spacetime white noise. Pick $\alpha \in (0,1]$. We have

$$\mathscr{L}_{\mathbb{Q}_{\ell,\phi_1,\phi_2}}(u_\ell(\cdot,\phi_2)) = \mathscr{L}_{\mathbb{P}}(u(\cdot,\phi_2)) \tag{5.8}$$

and

$$u_\ell(2,\phi_2) = u(2,\phi_1) \text{ on the event } \big\{\tau < 2\big\}. \tag{5.9}$$

Step 2 – Uniqueness of an invariant probability measure. We can now prove that the semigroup $(\mathcal{P}_t)_{t\geq 0}$ has at most one invariant probability measure. Otherwise, there would be (at least) two extremal invariant, hence singular, probability measures μ,ν. We could take ϕ_1 random with law μ, and ϕ_2 random with law ν, and keep writing $\mathbb{E}$ for the expectation operator in this extended probability space. Simply write R_ℓ rather than R_{ℓ,ϕ_1,ϕ_2}. Write $u_\ell(\cdot,\nu)$ and $u(\cdot,\mu)$ to emphasize the law of the initial condition. For a measurable set $A \subset C^{-1/2-\varepsilon}(M)$ with $\mu(A) = 0$ we prove below that $\nu(A) = 0$. The measure ν would thus be absolutely continuous with respect to μ, a contradiction with the fact that ν is singular with respect to μ. We give two proofs.

1. Assuming $\mu(A) = 0$, one would have from the identity in law (5.8) and the fact that

$$\mathbb{P}\big(u(2,\mu) \in A\big) = \mu(\mathcal{P}_t\mathbf{1}_A) = \mu(A) = 0$$

the identity

$$\begin{aligned}\nu(A) = \nu(\mathcal{P}_2 \mathbf{1}_A) &= \mathbb{Q}_\ell\big(u_\ell(2,\nu) \in A\big) = \mathbb{E}\big[R_\ell \mathbf{1}_A(u_\ell(2,\nu))\big] \\ &\overset{(5.9)}{=} \mathbb{E}\big[R_\ell \mathbf{1}_A(u(2,\mu))\mathbf{1}_{\tau<2}\big] + \mathbb{E}\big[R_\ell \mathbf{1}_A(u_\ell(2,\nu))\mathbf{1}_{\tau=2}\big] \\ &= \mathbb{E}\big[R_\ell \mathbf{1}_A(u_\ell(2,\nu))\mathbf{1}_{\tau=2}\big] \leq \mathbb{Q}_\ell(\tau=2) \overset{(5.7)}{\leq} a < 1.\end{aligned}$$

This is not enough to conclude that $\nu(A) = 0$, but instead of coupling the two dynamics on a single time interval $[1,2]$ we can repeat if necessary our attempts to couple them a fixed finite number of times, during the time intervals $[2k-1, 2k]$ after a coupling-free evolution on the time interval $[2k-2, 2k-1]$, for $k \leq n$, say. Denote by $u_\ell^{(n)}(\cdot, \phi_2')$ the corresponding dynamics. Write $\tau_1 \in [1,2], \dots, \tau_n \in [n+1, n+2]$ for the successive coupling times and set

$$\ln R^{(n)}_{\ell,\phi_1,\phi_2} := -\sum_{k=1}^n \left(\ell\, \xi\Big(\mathbf{1}_{2k-1<\cdot<\tau_k} \frac{v(\cdot,\phi_1') - v_\ell(\cdot,\phi_2')}{\|v(\cdot,\phi_1') - v_\ell(\cdot,\phi_2')\|_{L^2}}\Big) + \frac{\ell^2\big(\tau_k - 2k+1\big)}{2} \right)$$

and

$$d\mathbb{Q}_\ell^{(n)} := R_\ell^{(n)} d\mathbb{P}.$$

The probability measure $\mathbb{Q}_\ell^{(n)}$ implicitly depends on ϕ_1 and ϕ_2 and we also set

$$\overline{\mathbb{Q}}_\ell^{(n)} := \int \mathbb{Q}_\ell^{(n)} \nu(d\phi_2)\mu(d\phi_1).$$

The process $u_\ell^{(n)}(\cdot,\phi_2)$ has under $\mathbb{Q}_\ell^{(n)}$ the same distribution as $u(\cdot,\phi_2)$ and the pair

$$\big(u(\cdot,\phi_1), u_\ell^{(n)}(\cdot,\phi_2)\big)$$

is Markovian under both $\mathbb{P}$ and $\mathbb{Q}_\ell^{(n)}$. Denote by $\theta_r : \Omega \to \Omega$ a family of measurable measure preserving maps on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\theta_r \circ \theta_{r'} = \theta_{r+r'}$ and $\theta_r \xi(\cdot) = \xi(\cdot + r)$. The shift acts on measurables functions of ξ such as u and $u_\ell^{(n)}$. One then has as above

$$\begin{aligned}\nu(A) &= \int \mathbb{E}_\ell^{(n)}\Big[\mathbf{1}_A\big(u_\ell^{(n)}(2n,\phi_2)\big)\mathbf{1}_{\tau_n(\phi_1,\phi_2)=2n+2}\Big]\nu(d\phi_2)\mu(d\phi_1) \\ &= \int \mathbb{E}_\ell^{(n)}\Big[\mathbf{1}_A\big(\theta_{2n-2}u_\ell(2, u_\ell^{(n-1)}(2n-2,\phi_2))\big)\mathbf{1}_{\tau_n(\phi_1,\phi_2)=2n+2}\Big]\nu(d\phi_2)\mu(d\phi_1) \\ &= \int \mathbb{E}_\ell^{(n-1)}\Big[\mathbb{E}_\ell^{(1)}\Big[\mathbf{1}_A\big(\theta_{2n-2}u_\ell^{(1)}(2, u_\ell^{(n-1)}(2n-2,\phi_2))\big)\mathbf{1}_{\tau_1=2}\Big| u_\ell^{(n-1)}(2n-2,\phi_2)\Big] \times \\ &\qquad\qquad \mathbf{1}_{\tau_{n-1}(\phi_1,\phi_2)=2n}\Big]\nu(d\phi_2)\mu(d\phi_1) \\ &\leq a\, \overline{\mathbb{Q}}_\ell^{(n-1)}(\tau_{n-1}=2) \leq a^n,\end{aligned}$$

by induction. The conclusion $\nu(A) = 0$ follows from the fact that n is arbitrary.

2. Alternatively, one can assume the two invariant probability measures μ and ν singular and proceed as follows to get a contradiction. Denote by $\mathbb{P}_1$ the law of $u(\cdot,\mu)$ and by $\mathbb{P}_2$ the law of $u(\cdot,\nu)$, with a time parameter running in the time interval $[0,2]$. Step 1 produces a coupling between $\mathbb{P}_1$ and a probability with positive density D with respect to $\mathbb{P}_2$. This coupling is a probability measure $\mathbb{Q}$ on $C\big([0,2], C^{-1/2-\varepsilon}(M)\big) \times C\big([0,2], C^{-1/2-\varepsilon}(M)\big)$ that gives a positive probability to the event $\big\{u_1(2) = u_2(2)\big\}$, denoting by u_1 and u_2 the canonical marginal processes on the product space. Denote by π_1 and π_2 the canonical projections and set

$$d\mathbb{Q}^- := (1 \wedge D^{-1}) d\mathbb{Q}$$

and

$$\mathbb{Q}^+ := \mathbb{Q}^- + (\mathbb{P}_1 - \pi_{1\star}\mathbb{Q}^-) \otimes (\mathbb{P}_2 - \pi_{2\star}\mathbb{Q}^-).$$

The measure $\mathbb{Q}^+$ on $C\big([0,2], C^{-1/2-\varepsilon}(M)\big) \times C\big([0,2], C^{-1/2-\varepsilon}(M)\big)$ is a probability measure with marginals $\mathbb{P}_1$ and $\mathbb{P}_2$ – that is, a coupling of these two probability measures. We further have that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{Q}^+$, so

$$\mathbb{Q}^+\big(u_1(2) = u_2(2)\big) > 0. \tag{5.10}$$

Since under $\mathbb{Q}^+$ the random variable $u_1(2)$ has law μ and the random variable $u_2(2)$ has law ν we cannot have at the same time (5.10) and the fact that μ and ν are singular. We thank M. Hairer for sharing his insight on this reasoning. ▷

Together with the existence result proved in [4] Theorem 9 allows us to define the Φ^4_3 measure on M as the unique invariant probability measure of the semigroup $(\mathcal{P}_t)_{t\geq 0}$. This shows that the Φ^4_3 measure is associated with the Riemannian manifold M. It follows in particular that any smooth isometry between two 3-dimensional boundaryless Riemannian manifolds sends the Φ^4_3 measure of the former on the Φ^4_3 measure of the latter.

6 – Strong Feller property

The coupling used in the proof Theorem 9 can be used to prove the strong Feller property of the semigroup $(\mathcal{P}_t)_{t\geq 0}$ by showing that it satisfies some Harnack-type inequality. As a preliminary remark to the next statement, note that since one has the inclusion

$$\big\{\tau(\ell,\phi_1,\phi_2)=2\big\}\subset\left\{\big\|v(1,\phi_1')-v_\ell(1,\phi_2')\big\|_{L^2}>\frac{\ell-o_{\widehat{\xi}}(\ell)}{2}\right\}$$

with $v_\ell(1,\phi_2')=v(1,\phi_2')$ it follows from the L^p or L^∞ coming down from infinity result that

$$\mathbb{P}\big(\tau(\ell,\phi_1,\phi_2)=2\big)=:a_\ell=o_\ell(1)$$

is a function of ℓ that goes to 0 as ℓ goes to infinity.

Theorem 11 – *Pick a finite exponent $p_1>1$ and a time $t>0$. For any $\ell>0$ there exists a function*

$$\Psi_\ell : C^{-1/2-\varepsilon}(M)\times C^{-1/2-\varepsilon}(M)\to\mathbb{R},$$

that is null on the diagonal and continuous, such that the inequality

$$(\mathcal{P}_tf)^{p_1}(\phi_2)\leq\mathcal{P}_t(f^{p_1})(\phi_1)\,e^{\Psi_\ell(\phi_1,\phi_2)}\Big(1+a_\ell^{\frac{1}{p_1}}\,\|f\|_\infty\,e^{\Psi_\ell(\phi_1,\phi_2)}\Big)^{p_1},\tag{6.1}$$

holds for any measurable bounded function $f\geq 1$ on $C^{-1/2-\varepsilon}(M)$, and any $\phi_1,\phi_2\in C^{-1/2-\varepsilon}(M)$.

Proof – We use the notations of the proof of Theorem 9. Since $u_\ell(2,\phi_2)=u(2,\phi_1)$ on the event $\{\tau(\ell,\phi_1,\phi_2)<2\}$ one has

$$\begin{aligned}(\mathcal{P}_tf)(\phi_2)&=\mathbb{E}_{\mathbb{Q}_{\ell,\phi_1,\phi_2}}\big[f(u_\ell(2,\phi_2))\big]=\mathbb{E}\big[R_{\ell,\phi_1,\phi_2}f(u_\ell(2,\phi_2))\big]\\&=\mathbb{E}\big[R_{\ell,\phi_1,\phi_2}f\big(u(2,\phi_1)\big)\mathbf{1}_{\tau<2}\big]+\mathbb{E}_{\mathbb{Q}_{\ell,\phi_1,\phi_2}}\big[f\big(u_\ell(2,\phi_2)\big)\mathbf{1}_{\tau=2}\big]\end{aligned}$$

and we obtain an inequality of the form (6.1) from Hölder inequality and the condition $f\geq 1$, that allows to factorize in the second inequality below,

$$\begin{aligned}(\mathcal{P}_tf)^{p_1}(\phi_2)&\leq\Big(\mathbb{E}\big[R_{\ell,\phi_1,\phi_2}f(u(2,\phi_1))\big]+\|f\|_\infty\,\mathbb{E}\big[R_{\ell(\alpha),\phi_1,\phi_2}^{\frac{p_1}{p_1-1}}\big]^{\frac{p_1-1}{p_1}}a_\ell^{\frac{1}{p_1}}\Big)^{p_1}\\&\leq\mathbb{E}\big[R_{\ell,\phi_1,\phi_2}f(u(2,\phi_1))\big]^{p_1}\Big(1+\|f\|_\infty\,\mathbb{E}\big[R_{\ell(\alpha),\phi_1,\phi_2}^{\frac{p_1}{p_1-1}}\big]^{\frac{p_1-1}{p_1}}a_\ell^{\frac{1}{p_1}}\Big)^{p_1}\\&\leq\mathbb{E}\big[f^{p_1}(u(2,\phi_1))\big]\,\mathbb{E}\big[R_{\ell,\phi_1,\phi_2}^{\frac{p_1}{p_1-1}}\big]^{p_1-1}\Big(1+\|f\|_\infty\,\mathbb{E}\big[R_{\ell(\alpha),\phi_1,\phi_2}^{\frac{p_1}{p_1-1}}\big]^{\frac{p_1-1}{p_1}}a_\ell^{\frac{1}{p_1}}\Big)^{p_1}.\end{aligned}$$

This is (6.1) with

$$e^{\Psi_\ell(\phi_1,\phi_2)}=\mathbb{E}\big[R_{\ell,\phi_1,\phi_2}^{\frac{p_1}{p_1-1}}\big]^{p_1-1}$$

(The function Ψ_ℓ also depends on p_1 but we do not emphasize that dependence in the notation as it is irrelevant for us here.) We check from classical arguments that $\Psi_\ell(\phi_1,\phi_2)$ is a continuous function of ϕ_1 and ϕ_2. ▷

Corollary 12 – *The semigroup $(\mathcal{P}_t)_{t\geq 0}$ has the strong Feller property.*

Proof – We follow F.Y. Wang's classical proof – see e.g. Theorem 1.4.1 in [22]. Fix $t>0$. Applying (6.1) to $f=1+rg$, with a measurable function $0\leq g\leq 1$ and $0<r\leq 1$, one gets

$$\big(1+r(\mathcal{P}_tg)(\phi_2)\big)^{p_1}\leq\mathcal{P}_t\big((1+rg)^{p_1}\big)(\phi_1)\,e^{\Psi_\ell(\phi_1,\phi_2)}\Big(1+a_\ell^{\frac{1}{p_1}}\,(1+\|g\|_\infty)\,e^{\Psi_\ell(\phi_1,\phi_2)}\Big)^{p_1}\tag{6.2}$$

so

$$1 + p_1 r(\mathcal{P}_t g)(\phi_2) + o(r) \le \Big(1 + p_1 r(\mathcal{P}_t g)(\phi_1) + o(r)\Big) e^{\Psi_\ell(\phi_1,\phi_2)} \Big(1 + a_\ell^{\frac{1}{p_1}} (1 + \|g\|_\infty)\, e^{\Psi_\ell(\phi_1,\phi_2)}\Big)^{p_1}.$$

Sending ϕ_2 to ϕ_1 one gets from the continuity of Ψ_ℓ

$$1 + p_1 r \limsup_{\phi_2 \to \phi_1} (\mathcal{P}_t g)(\phi_2) + o(r) \le \Big(1 + p_1 r(\mathcal{P}_t g)(\phi_1) + o(r)\Big) \Big(1 + a_\ell^{\frac{1}{p_1}} (1 + \|g\|_\infty)\Big)^{p_1}$$

and since $\ell > 0$ and $r > 0$ are arbitrary and a_ℓ goes to 0 as ℓ goes to infinity

$$\limsup_{\phi_2 \to \phi_1} (\mathcal{P}_t g)(\phi_2) \le (\mathcal{P}_t g)(\phi_1).$$

Exchanging ϕ_1 and ϕ_2 in (6.2), the same reasoning gives

$$(\mathcal{P}_t g)(\phi_1) \le \liminf_{\phi_2 \to \phi_1} (\mathcal{P}_t g)(\phi_2),$$

from which the continuity of $\mathcal{P}_t g$ at ϕ_1 follows. The conclusion follows since ϕ_1 is arbitrary. ▷

A – An elementary continuity result

The following statement is a simple variation on the classical proof of continuity of the paraproduct and resonant operators; we learned it from V.N. Dang although it is likely to be known already.

Lemma 13 – *Let $(p_1, q_1), (p_2, q_2), (p, q)$ in $[1, +\infty]$ be such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p},$$
$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}.$$

For any $\gamma > 0$ there is a constant C_γ such that for all integers N and real numbers $a_1 \ge 0$ one has

$$\|f \prec g\|_{B^{a_2-\gamma}_{pq}} \le C_\gamma \, \|f\|_{B^{a_1}_{p_1 q_1}} \Big(2^{-N\gamma} \|g\|_{B^{a_2}_{p_2 q_2}} + N \|g\|_{L^{p_2}}\Big). \tag{A.1}$$

We give here the details for the reader's convenience, when things are set in a Euclidean space. An elementary adaptation of the pattern of proof is needed to make it work in the setting of a 3-dimensional Riemannian manifold with the paraproduct and resonant operators defined in Appendix A of [4] or in [5]. We use the usual Δ_k notation for the Littlewood-Paley projectors.

Proof – Write

$$f \prec g = \sum_{\ell < N, k \le \ell - 2} (\Delta_k f)(\Delta_\ell f) + \sum_{\ell \ge N, k \le \ell - 2} (\Delta_k f)(\Delta_\ell g) \tag{A.2}$$

for any integer N. On the one hand, one has for all $m \in \mathbb{N}$

$$\begin{aligned}
\Big\| \Delta_m \Big(\sum_{\ell < N, k \le \ell - 2} (\Delta_k f)(\Delta_\ell g) \Big) \Big\|_{L^p} &\lesssim \mathbf{1}_{m-1 \le N} \sum_{|\ell - m| \le 1, k \le \ell - 2} \big\| (\Delta_k f)(\Delta_\ell g) \big\|_{L^p} \\
&\lesssim \mathbf{1}_{m-1 \le N} \sum_{|\ell - m| \le 1, k \le \ell - 2} 2^{-\ell a_1}\, 2^{k a_1} \|\Delta_k f\|_{L^{p_1}} \|\Delta_\ell g\|_{L^{p_2}} \\
&\lesssim \mathbf{1}_{m-1 \le N}\, m\, \|f\|_{B^{a_1}_{p_1, q_1}} \|g\|_{L^{p_2}} \lesssim N \|f\|_{B^{a_1}_{p_1, q_1}} \|g\|_{L^{p_2}},
\end{aligned}$$

using in the penultimate inequality Hölder inequality and the fact that $a_1 \ge 0$, and Young convolution inequality to see that $\|\Delta_\ell g\|_{L^{p_2}} \le \|g\|_{L^{p_2}}$. On the other hand, one has for $m \ge N - 1$

$$\begin{aligned}
\Big\| \Delta_m \Big(\sum_{\ell \ge N, k \le \ell - 2} (\Delta_k f)(\Delta_\ell g) \Big) \Big\|_{L^p} &\lesssim \sum_{|\ell - m| \le 1, k \le \ell - 2} \big\| (\Delta_k f)(\Delta_\ell g) \big\|_{L^p} \\
&\lesssim \sum_{|\ell - m| \le 1, k \le \ell - 2} 2^{-k a_1}\, 2^{k a_1} \|\Delta_k f\|_{L^{p_1}} \|\Delta_\ell g\|_{L^{p_2}}
\end{aligned}$$

$$\lesssim m^{\frac{q_1-1}{q_1}} \|f\|_{B^{a_1}_{p_1,q_1}} \sum_{|\ell-m|\leq 1} \|\Delta_\ell g\|_{L^{p_2}}$$

from Hölder inequality in the sum over k. Estimate (A.1) follows as a consequence. ▷

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