Regularity structures for quasilinear singular SPDEs

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Abstract. We prove the well-posed character of a regularity structure formulation of the quasilinear generalized (KPZ) equation and give an explicit form for a renormalized equation in the full subcritical regime. Convergence results for the solution of the regularized renormalized equation are obtained in regimes that cover the spacetime white noise case.

Contents

1 – Introduction

Denote by $\mathbf{T}$ the one dimensional torus. We consider the one dimensional space-periodic quasilinear generalized (KPZ) equation

$$\big(\partial_t - a(u)\partial_x^2\big)u = f(u)\xi + g(u)(\partial_x u)^2, \tag{1.1}$$

for regular enough functions a, f, g, where a takes values in a compact interval of $(0,\infty)$ and ξ is a random spacetime distribution – with main example spacetime white noise. The initial condition $u_0 \in C^{0+}(\mathbf{T}) := \bigcup_{\alpha>0} C^\alpha(\mathbf{T})$ is given. This equation falls within the class of subcritical singular stochastic partial differential equations (SPDEs) of parabolic type. All equations of this class share the same defect: The low regularity of some terms in a singular SPDE prevents the expected regularizing effect of the dynamics to give sense to a number of products in the equations. In the case at hand, equation (1.1), one expects a parabolic type dynamics to have a resolvent that improves regularity by 2. The 'subcritical' nature of the dynamics is here encoded in the fact that the spacetime distribution ξ is (almost surely) assumed to have regularity α_0-2, for $0<\alpha_0<2$. It is then formally consistent to expect a solution u of equation (1.1) to have parabolic regularity α_0, as $(\partial_x u)^2$ will then have regularity $2(\alpha_0-1)$, bigger than α_0-2, the expected regularity of the term $f(u)\xi$. With a right hand side of regularity α_0-2 a Schauder type continuity estimate satisfied by the resolvent of the evolution gives indeed u a regularity α_0. The problem with that regularity analysis is that for u of regularity α_0 none of the products $f(u)\xi$ and $|\partial_x u|^2$ make sense, even less $g(u)|\partial_x u|^2$, when $0<\alpha_0<1$, the case of interest.

The development of the study of semilinear subcritial singular SPDEs was launched by the two groundbreaking works [23] of M. Hairer, on regularity structures, and [22] of M. Gubinelli, P. Imkeller & N. Perkowski, on paracontrolled calculus. Both of them introduced new settings and

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new tools to make sense of such equations and solve them uniquely under some small parameter condition. Despite the difference of languages and tools used in regularity structures and paracontrolled calculus both settings provide a similar understanding of a subcritical singular parabolic SPDE. The mantra of their common approach to the product problem is that if one can make sense of a number of analytically ill-defined 'reference products' that only involve the noise ξ, not in an ω-wise sense but as random variables, then one can make sense of the ill-defined products in the equation for all functions u that locally look like linear combinations of the reference random variables. Regularity structures and paracontrolled calculus differ in the tools used to make sense of that comparison with reference random variables. In both settings, working with a random noise turns out to be crucial to construct these reference random variables by probabilistic means.

We refer the reader to the overviews [13, 14] of Chandra & Weber and Corwin & Shen for non-technical introductions to the domain of semilinear singular SPDEs, to the books [17, 7] of Friz & Hairer and Berglund for a mildly technical introduction to regularity structures, and to Bailleul & Hoshino's Tourist's Guide [4] for a thorough tour of the analytic and algebraic sides of the theory. Readers interested in paracontrolled calculus will find a nice account of the fundamentals in Gubinelli's panorama [21].

The first works on quasilinear singular SPDEs by Otto & Weber [30], Furlan & Gubinelli [18] and Bailleul, Debussche, & Hofmanová [3] all three investigated the generalized (PAM) equation in the regime where the noise is $(\alpha_0 - 2)$ regular and $\alpha_0 > 2/3$. Interestingly each of these works used a different method: A variant of regularity structures in [30], a variant of paracontrolled calculus based on the use of the paracomposition operator for [18], and the initial form of paracontrolled calculus in [3]. On the paracontrolled side Bailleul & Mouzard [6] extended the high order paracontrolled calculus toolbox to deal with the paracontrolled equivalent of equation (1.4) in the spacetime white noise regime $\alpha_0 > 2/5$. On the regularity structures side Otto & Weber deepened their framework in their works [29] with Sauer & Smith, dedicated to the study of the equation with linear additive forcing

$$\partial_t u - a(u)\partial_x^2 u = \xi. \tag{1.2}$$

They obtained in particular in [29] an explicit form of a renormalized equation for (1.2) backed up by the general convergence result proved by Linares, Otto, Tempelmayr & Tsatsoulis in [27] that holds for a large class of random noises in the full subcritical regime. Our general formula for the counterterm in the renormalized equation generalizes theirs. The algebraic machinery behind their approach was further analysed by Linares, Otto & Tempelmayr in [26]. Meanwhile Gerencsér & Hairer provided in [20] an analysis of a regularity structure counterpart of equation (1.1), in the full subcritical regime. Their method allowed for an analysis of the renormalized equation only in the regime $\alpha_0 > 1/2$. By implementing some tricky integration by parts-type formulas Gerencsér was able in [19] to obtain the renormalized equation for the special case of equation (1.2) from the analysis of [20] in the spacetime white noise regime $\alpha_0 > 2/5$. We prove in the present work the well-posed character of a regularity structure formulation of the quasilinear generalized (KPZ) equation and give an explicit form for a renormalized equation in the full subcritical regime, with a simple expression in a number of cases. Convergence results for the solution of the regularized renormalized equation are obtained in regimes that cover the spacetime white noise case.

Following [3, 6] we set

$$L^{a(v)} := a(v)\partial_x^2$$

for a sufficiently regular function v on $[0,\infty) \times \mathbf{T}$ and rewrite equation (1.1) under the form

$$\big(\partial_t - L^{a(v)} + c\big)u = f(u)\xi + g(u)(\partial_x u)^2 + cu + \big(a(u) - a(v)\big)\partial_x^2 u \tag{1.3}$$

for a large positive constant c. We consider (1.3) as a 'perturbation' of the non-translation invariant generalized (KPZ) equation

$$\big(\partial_t - L^{a(v)} + c\big)u = f(u)\xi + g(u)(\partial_x u)^2 + cu.$$

Below we will set the scene to reformulate equation (1.3) in a regularity structure where it takes the form

$$\begin{cases} \boldsymbol{u} = \mathsf{P}_{<2}\big(Q^{a(v)}u_0\big) + \mathsf{K}_{\gamma}^{a(v),\mathsf{M}}(\boldsymbol{v}+\boldsymbol{w}), \\ \boldsymbol{v} = \mathsf{Q}_{\leq 0}\Big\{F(\boldsymbol{u})\zeta + \big\{G(\boldsymbol{u})(\boldsymbol{D}\boldsymbol{u})^2 + c\boldsymbol{u}\big\}\Big\}, \\ \boldsymbol{w} = \mathsf{Q}_{\leq 0}\Big\{\big\{A(\boldsymbol{u}) - A(\mathsf{P}_{<2}(v))\big\}\Big(\boldsymbol{D}^2\mathsf{P}_{\leq 2}Q^{a(v)}u_0 + \boldsymbol{D}^2\mathsf{K}_{\gamma+\alpha_0}^{a(v),\mathsf{M}}(\boldsymbol{v}+\boldsymbol{w})\Big)\Big\} \end{cases} \tag{1.4}$$

The operator $\mathsf{P}_{<2}$ (resp. $\mathsf{P}_{\leq 2}$) stands for the canonical lift operator of a spacetime/spatial function to the part of the polynomial regularity structure spanned by monomials of homogeneity less than (resp. less than or equal to) 2, and $Q^{a(v),c}u_0$ is the free propagation of the initial condition u_0 under the non-translation invariant operator $(\partial_t - L^{a(v)} + c)$. The operator $\mathsf{K}_{\gamma'}^{a(v),c,\mathsf{M}}$ is the model dependent integration operator on modelled distributions (up to order γ') intertwined to $(\partial_t - L^{a(v)} + c)$ via the reconstruction operator. The operator $\mathsf{Q}_{\leq 0}$ projects on elements of nonpositive homogeneity, and the operator $\boldsymbol{D}$ is a natural derivative operator on a space of modelled functions.

We will see in Theorem *16* that given any admissible model M on our regularity structure, equation (1.4) has a unique solution over a model-dependent time interval $(0, t_0(\mathsf{M}))$, in an appropriate class of modelled distributions. This analytical statement holds in the full subcritical range provided the model is part of the data. Such a statement was already proved by Gerencsér & Hairer in [20] in a different setting. However their choice of formulation for (1.1) did not allow them to write down in the full subcritical range the renormalized equation satisfied by the reconstruction of the model dependent solution $\boldsymbol{u}$ of (1.4) when the noise is smooth and one uses an appropriate admissible model. The spacetime white noise regime is in particular out of range of their result. The regularity structure in which we formulate equation (1.4) is different from the regularity structure used in [20]. Working with an appropriate choice of model M that is the natural analogue in our setting of the Bruned-Hairer-Zambotti (BHZ) renormalized model from [10] we are able to give in Theorem *1* below a renormalized equation in the full subcritical regime. Denote by ε a positive regularization parameter and by $\xi^\varepsilon \in C^\infty(\mathbf{R}\times\mathbf{T})$ an ε-regularized noise ξ. Denote by M^ε the BHZ renormalized model associated with ξ^ε and the operator $(\partial_t - L^{a(v)} + c)$, and denote by u^ε the M^ε-reconstruction of the solution $\boldsymbol{u}^\varepsilon$ of equation (1.4) with M^ε in place of M. (The model M^ε is described precisely in Section 4.3.2.) The function u^ε is defined on a time interval $[0, t_0(\mathsf{M}^\varepsilon))$. Our main results take a conditional form involving two 'assumptions'. ***Assumption 1*** is stated in Section 4.3.2 and assumes the convergence of the natural BHZ model associated with the non-translation invariant operator $(\partial_t - L^{a(v)} + c)$. There is no doubt that it holds true but we refrain from describing here the modifications of Chandra & Hairer's work [12], which need to be extended to our non translation-invariant setting.

1 – *Theorem. Choose any function $v(t,x)$ on $\mathbf{R}_+\times\mathbf{T}$ sufficiently close to the initial condition $u_0 \in C^{0+}(\mathbf{T})$ – see condition* (2.5) *for the precise meaning. Under* ***Assumption 1*** *there exists some continuous functions $\mathfrak{F}^a((\tau^{\boldsymbol{p}})^*) \in C(\mathbf{R}^3)$ and*

$$\ell^\varepsilon_{a(v)}(\cdot, \tau^{\boldsymbol{p}}) \in C(\mathbf{R}_+\times\mathbf{T})$$

indexed by an infinite set of symbols $\big\{\tau^{\boldsymbol{p}} \in \mathbb{B}^-_\circ\big\}$, such that the solution u^ε to

$$\big(\partial_t - a(u^\varepsilon)\partial_x^2\big)u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon + g(u^\varepsilon)(\partial_x u^\varepsilon)^2 + \sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}^-_\circ} \frac{\ell^\varepsilon_{a(v)}(\cdot,\tau^{\boldsymbol{p}})}{S(\tau^{\boldsymbol{p}})}\,\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon, \partial_x u^\varepsilon, v\big) \tag{1.5}$$

starting from u_0 converges in $C\big([0,t_0)\times\mathbf{T}\big)$ for a random time $t_0 > 0$ in probability as $\varepsilon > 0$ goes to 0.

Condition (2.5) only involves u_0. Let us emphasize that this convergence result holds in the whole subcritical regime $\alpha_0 > 0$ where $\alpha_0 - 2$ is the regularity of the noise ξ. The sum over $\tau^{\boldsymbol{p}}$

in (1.5) is called the 'counterterm'. The functions $\mathfrak{F}^a((\tau^{\boldsymbol{p}})^*)$ depend pointwisely on $u^\varepsilon, \partial_x u^\varepsilon, v$ in the sense that

$$\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon, \partial_x u^\varepsilon, v\big)(z) = \mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon(z), \partial_x u^\varepsilon(z), v(z)\big).$$

The functions $\ell^\varepsilon_{a(v)}(\cdot, \tau^{\boldsymbol{p}})$ are non-local functionals of the function $a(v(\cdot))$. Theorem *1* extends the results of [30, 18, 3, 6, 29, 20] and deals with the quasilinear generalized (KPZ) in the full subcritical regime. The reader familiar with regularity structures will see that our arguments extend immediately to coupled systems of generalized (KPZ) equations. Such a generalization is left to the reader and we concentrate here on the renormalized equation for (1.1).

The reader can feel uncomfortable about the fact that the functions

$$\ell^\varepsilon_{a(v)}(\cdot, \tau^{\boldsymbol{p}}) \text{ and } \mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon, \partial_x u^\varepsilon, v\big)$$

depend on the somewhat arbitrary choice of function v satisfying condition (2.5). One can give a simpler representation of the counterterm when the functions $\ell^\varepsilon_{a(v)}(\cdot, \tau^{\boldsymbol{p}})$ can be traded off for a local functional of $a(v(\cdot))$ – meaning that $\ell^\varepsilon_{a(v)}(z, \tau^{\boldsymbol{p}})$ can be replaced by a function of $a(v(z))$. This is the content of ***Assumption 2*** stated in Section 4.4.

2 – *Theorem. Under **Assumptions 1** and **2** there exist continuous functions $\chi^a_\tau \in C(\mathbf{R})$, $\mathfrak{F}(\tau^*) \in C(\mathbf{R}^2)$ and $l^\varepsilon_{(\cdot)}(\tau) \in C(\mathbf{R})$, all three indexed by a finite set of symbols $\big\{\tau \in \mathbb{B}^{-0}_\circ\big\}$, such that the third term of the right hand side of* (1.5) *is of the form*

$$\sum_{\tau \in \mathbb{B}^{-0}_\circ} \frac{l^\varepsilon_{a(u^\varepsilon)}(\tau)}{S(\tau)}\, \chi^a_\tau(u^\varepsilon)\, \mathfrak{F}(\tau^*)(u^\varepsilon, \partial_x u^\varepsilon) + O(1), \tag{1.6}$$

for a term $O(1)$ uniform in ε.

Above, the functions $\chi^a_\tau(\cdot)$ are polynomial functions of a and its derivatives and the coefficient $S(\tau)$ stands for a positive τ-dependent integer. Note that apart from the $O(1)$ term in (1.6), which we can discard in the renormalized equation, the counterterm is independent of v. We show in Section 4.5 that ***Assumption 2*** holds in particular for the quasilinear generalized (KPZ) equation driven by a spacetime white noise.

Dealing with quasilinear singular SPDEs rather than semilinear equations requires a twist that appears in the form of an infinite dimensional ingredient. It is related in our formulation (1.4) to the fact that our structure needs to be stable by the operator $\mathcal{I}_{(0,2)}$. In the previous works using regularity structures [30, 20, 29] this infinite dimensional feature appeared under the form of a one parameter family of heat kernels or abstract integration operators. Our regularity structure is different from the ones used in these works. Its model space $T = \bigoplus_{\beta \in A} T_\beta$ has infinite dimensional homogeneous spaces T_β whose basis elements are the usual trees associated with the generalized (KPZ) equation with an additional integer decoration p on each edge accounting for how many times the operator $\mathcal{I}_{(0,2)}$ is applied to this edge. The same infinite dimensional ingredient appeared in Bailleul & Mouzard's work [6] in a paracontrolled setting.

Organization of the work – We set the scene in Section 2, where the function spaces we work with are introduced together with our regularity structure. We introduced in particular a non-classical spacetime elliptic operator to define our parabolic spaces. For reader's convenience some properties of its heat kernel are proved ni full detail in Appendix A. Section 3 is dedicated to proving that equation (1.4) is locally well-posed in the full subcritical regime. The analysis of the renormalized equation problem is done in Section 4, where we give in particular an explicit description of the functions χ^a_τ in Section 4.3.

Notations – *We denote by* $\mathbf{R}$ *the set of real numbers and by* $\mathbf{N}$ *the set of nonnegative integers. We represent by $z = (t, x) \in \mathbf{R}^2$ a generic spacetime variable, for which we set*

$$\|z\|_{\mathfrak{s}} := |t|^{1/2} + |x|.$$

We also set for any multiindex $\mathbf{k} = (k_1, k_2) \in \mathbf{N}^2$

$$|\mathbf{k}|_{\mathfrak{s}} := 2k_1 + k_2,$$

and

$$\partial_z^{\mathbf{k}} := \partial_t^{k_1}\partial_x^{k_2}.$$

For $\alpha > 0$, *we define* $C^\alpha(\mathbf{T})$ *as the collection of functions* f *on* $\mathbf{T}$ *which is* $\lfloor\alpha\rfloor$*-th differentiable and such that* $\partial_x^{\lfloor\alpha\rfloor} f$ *is* $(\alpha - \lfloor\alpha\rfloor)$*-Hölder continuous. We also define* $\mathcal{C}_{\mathfrak{s}}^\alpha$ *as the set of functions* f *on* $\mathbf{R}\times\mathbf{T}$ *such that* $\partial_z^{\mathbf{k}} f$ *exists and is bounded for any* $\mathbf{k} \in \mathbf{N}^2$ *with* $|\mathbf{k}|_{\mathfrak{s}} < \alpha$, *and* $\partial_z^{\mathbf{k}} f$ *with* $|\mathbf{k}|_{\mathfrak{s}} = \lfloor\alpha\rfloor$ *is* $(\alpha - \lfloor\alpha\rfloor)$*-Hölder continuous with respect to the parabolic distance* $\|\cdot\|_{\mathfrak{s}}$.

An identity involving an element of the form $*^{(+)}$ *or* $*_{(+)}$, *whatever* $*$ *is, will be a shorthand notation for two identities: The identity with the element* $*$ *and the identity with the element* $*^+$ *or* $*_+$.

2 – The setting

We introduce in this section the functional setting and the regularity structure in which we set the study of equation (1.4).

2.1 – Function spaces. The following basic facts are proved in Appendix A – see Theorem *34*, Theorem *39*, and Corollary *43*. Pick $\alpha \in (0, 1]$ and an arbitrary function $v \in \mathcal{C}_{\mathfrak{s}}^\alpha$.

3 – Proposition. The fundamental solution $Q_{t,s}^{a(v),0}(x,y)$ *of the operator* $\partial_t - L^{a(v)}$ *satisfies the estimate*

$$\big|\partial_t^n \partial_x^k Q_{t,s}^{a(v),0}(x,y)\big| \leq \frac{c_0 e^{c_0(t-s)}}{(t-s)^{(1+k+2n)/2}} \exp\left(-c_1\frac{|x-y|^2}{t-s}\right) \tag{2.1}$$

for any $k + 2n \leq 2$, *for some positive constants* c_0, c_1 *depending only on* $\inf a > 0$, $\|a\|_{C^1}$, *and* $\|v\|_{\mathcal{C}_{\mathfrak{s}}^\alpha}$. *Moreover one has* $\int_{\mathbf{R}} Q_{t,s}^{a(v),0}(x,y)dy = 1$.

4 – Proposition. We define the spacetime elliptic operator

$$\mathcal{L}^{a(v)} := \big(\partial_t - L^{a(v)}\big)(\partial_t + \partial_x^2) = \partial_t^2 - a(v)\partial_x^4 - \big(a(v)-1\big)\partial_t\partial_x^2 \tag{2.2}$$

and introduce the parabolic operator with the additional variable $\theta > 0$

$$\partial_\theta - \mathcal{L}^{a(v)}.$$

The fundamental solution $\mathcal{Q}_\theta^{a(v),0}(\cdot,\cdot)$ *of* $\partial_\theta - \mathcal{L}^{a(v)}$ *satisfies the estimates*

$$\big|\partial_z^{\mathbf{k}} \mathcal{Q}_\theta^{a(v),0}\big((t,x),(s,y)\big)\big| \leq \frac{C_0 e^{C_0\theta}}{\theta^{|\mathbf{k}|_{\mathfrak{s}}/4}} \mathsf{G}_\theta(t-s, x-y) \tag{2.3}$$

and

$$\begin{aligned}\Big|\partial_z^{\mathbf{k}} \mathcal{Q}_\theta^{a(v),0}(z',w) - \sum_{|\mathbf{k}+\mathbf{l}|_{\mathfrak{s}}\leq 4} \frac{(z'-z)^{\mathbf{k}+\mathbf{l}}}{\mathbf{l}!}\partial_z^{\mathbf{k}+\mathbf{l}}\mathcal{Q}_\theta^{a(v),0}(z,w)\Big| \\ \leq \frac{C_0 e^{C_0\theta}\|z'-z\|_{\mathfrak{s}}^{4-|\mathbf{k}|_{\mathfrak{s}}+\delta}}{\theta^{(4+\delta)/4}}\big\{\mathsf{G}_\theta(z'-w) + \mathsf{G}_\theta(z-w)\big\},\end{aligned} \tag{2.4}$$

where

$$\mathsf{G}_\theta(t,x) := \frac{1}{\theta^{3/4}}\exp\left\{-C_1\left(\frac{t^2}{\theta} + \frac{|x|^{4/3}}{\theta^{1/3}}\right)\right\}$$

for any $|\mathbf{k}|_{\mathfrak{s}} \leq 4$ *and* $\delta \in (0,\alpha)$, *for some positive constants* C_0, C_1 *depending only on* $\inf a > 0$, $\|a\|_{C^1}$, *and* $\|v\|_{C^\alpha(\mathbf{T})}$. *Moreover one has* $\int_{\mathbf{R}^2} \mathcal{Q}_\theta^{a(v),0}(z,w)dw = 1$.

Next we describe a class of possible choices for v. Recall that $\alpha_0 - 2$ is the spacetime Hölder regularity of the noise ξ in equation (1.1). We consider some initial condition $u_0 \in C^\alpha(\mathbf{T})$ with $\alpha \in (0, \alpha_0)$.

5 – *Definition. For any $\alpha \in (0,1)$ and $T>0$, define $V^\alpha(0,T)$ as a collection of bounded continuous functions on $(0,T)\times\mathbf{T}$ such that the following quantity is finite.*

$$
\begin{aligned}
\|f\|_{V^\alpha(0,T)} := &\sup_{z,z'\in(0,T)\times\mathbf{R}^d} \frac{|f(z')-f(z)|}{\|z'-z\|_{\mathfrak{s}}^\alpha} \\
&+ \sup_{t\in(0,T)} t^{(1-\alpha)/2}\|\partial_x f(t,\cdot)\|_{L^\infty(\mathbf{T})} \\
&+ \sup_{t\in(0,T)} t^{(2-\alpha)/2}\big(\|\partial_x^2 f(t,\cdot)\|_{L^\infty(\mathbf{T})} + \|\partial_t f(t,\cdot)\|_{L^\infty(\mathbf{T})}\big) \\
&+ \sup_{0<t<t'<T} t^{(2-\alpha)/2}\frac{\|\partial_x f(t',\cdot)-\partial_x f(t,\cdot)\|_{L^\infty(\mathbf{T})}}{|t'-t|^{1/2}}.
\end{aligned}
$$

We will choose later a function $v\in V^\alpha(0,T)$ satisfying

$$
\begin{cases}
\|v\|_{V^\alpha(0,T)} \le C\|u_0\|_{C^\alpha(\mathbf{T})}, \\
\|e^{t\partial_x^2}u_0 - v\|_{L^\infty((0,T)\times\mathbf{T})} \le \delta\|u_0\|_{C^\alpha(\mathbf{T})}
\end{cases}
\tag{2.5}
$$

for some constant $C>0$ such that $\|e^{t\partial_x^2}u_0\|_{V^\alpha(0,T)} \le C\|u_0|_{C^\alpha(\mathbf{T})}$ holds (see Lemma *41* for the proof that $e^{t\partial_x^2}u_0\in V^\alpha(0,T)$) and a sufficiently small positive constant δ, which will be chosen later depending only on $\|u_0\|_{C^\alpha(\mathbf{T})}$. Other than the most natural choice $v(t,x)=e^{t\partial_x^2}u_0$ we can also choose a t-independent smooth function $v(x)=e^{\delta\partial_x^2}u_0$ for a sufficiently small $\delta>0$. The latter choice will be used only in Section 4.5.3. We then extend the domain of v to $\mathbf{R}^2$ setting

$$
v(t,x) = \begin{cases} v(0,x), & \text{for } (t\le 0), \\ v(T,x), & \text{for } (t\ge T). \end{cases}
$$

and consider the spacetime operator (2.2). Since $\|v\|_{\mathcal{C}^\alpha_{\mathfrak{s}}} \lesssim \|u_0\|_{C^\alpha(\mathbf{T})}$, the constants c_0,c_1,C_0,C_1 above can then be chosen to depend only on $\inf a>0$, $\|a\|_{C^1}$, and $\|u_0\|_{C^\alpha}$. Therefore, ***all the mutliplicative constants appearing sometime implicitly in some inequalities below are independent of the choice of v.***

For any bounded continuous functions f on $\mathbf{R}^2$, for $\mathbf{k}\in\mathbf{N}^2$ with $|\mathbf{k}|_{\mathfrak{s}}\le 4$, set

$$
\big(\partial^{\mathbf{k}}\mathcal{Q}_\theta^{a(v),0}f\big)(z) := \int_{\mathbf{R}^2} \partial_z^{\mathbf{k}}\mathcal{Q}_\theta^{a(v),0}(z,z')f(z')dz'.
$$

We use the operators $\mathcal{Q}_\theta^{a(v),0}$ to define the full scale of anisotropic parabolic Hölder spaces.

Definition – For $\beta<0$, define $\mathcal{C}^\beta_{\mathfrak{s}}(a(v))$ as the completion of the set of bounded continuous functions f on $\mathbf{R}^2$ under the norm

$$
\|f\|_{\mathcal{C}^\beta_{\mathfrak{s}}(a(v))} := \sup_{0<\theta\le 1} \theta^{-\beta/4}\big\|\mathcal{Q}_\theta^{a(v),0}f\big\|_{L^\infty(\mathbf{R}^2)}.
$$

Next we rewrite the resolvent of the operator $\partial_t - L^{a(v)}$ in the space $\mathcal{C}^\alpha_{\mathfrak{s}}(a(v))$ in terms of the operators $\mathcal{Q}_\theta^{a(v),0}$. With an eye on the heat kernel estimates (2.1) and (2.3) pick a positive constant $c>c_0\vee C_0$ and write

$$
Q_t^{a(v),c} := e^{-ct}Q_t^{a(v),0}
$$

and

$$
\mathcal{Q}_\theta^{a(v),c} := e^{-c\theta}\mathcal{Q}_\theta^{a(v),0}.
$$

Then the operators $c-\mathcal{L}^{a(v)}$ and $\partial_t - L^{a(v)}+c$ have inverses of the form

$$
\big(c-\mathcal{L}^{a(v)}\big)^{-1}f = \int_0^\infty \mathcal{Q}_\theta^{a(v),c}f\,d\theta = \int_0^1 \mathcal{Q}_\theta^{a(v),c}f\,d\theta + \mathcal{Q}_1^{a(v),c}\circ(c-\mathcal{L}^{a(v)})^{-1}f
$$

and

$$
\big((\partial_t - L^{a(v)}+c)^{-1}g\big)(t) = \int_{-\infty}^t Q_{t,s}^{a(v),c}g(s)\,ds.
$$

For any given bounded continuous function f on $\mathbf{R}^2$ one can write the resolvent operator of the parabolic operator $\partial_t - L^{a(v)} + c$ in terms of the spacetime elliptic operator $c - \mathcal{L}^{a(v)}$. Indeed setting $g = (c - \mathcal{L}^{a(v)})^{-1} f$ and $h = (\partial_t + \partial_x^2) g$ we have

$$(\partial_t - L^{a(v)} + c)h = \mathcal{L}^{a(v)} g + ch = -f + c(g+h),$$

thus

$$\begin{aligned}\big(\partial_t - L^{a(v)} + c\big)^{-1} f &= -h + c\big(\partial_t - L^{a(v)} + c\big)^{-1}(g+h) \\ &= -(\partial_t + \partial_x^2)(c - \mathcal{L}^{a(v)})^{-1} f \\ &\quad + c\big(\partial_t - L^{a(v)} + c\big)^{-1}(1 + \partial_t + \partial_x^2)(c - \mathcal{L}^{a(v)})^{-1} f.\end{aligned}$$

Thus setting

$$K^{a(v),c} f := -\int_0^1 (\partial_t + \partial_x^2)\mathcal{Q}_\theta^{a(v),c} f \, d\theta =: \int_0^1 K_\theta^{a(v),c} f \, d\theta$$

and

$$R^{a(v),c} f := K_1^{a(v),c}(c - \mathcal{L}^{a(v)})^{-1} f + c\big(\partial_t - L^{a(v)} + c\big)^{-1}(1 + \partial_t + \partial_x^2)(c - \mathcal{L}^{a(v)})^{-1} f,$$

one has the decomposition

$$\big(\partial_t - L^{a(v)} + c\big)^{-1} f = K^{a(v),c} f + R^{a(v),c} f. \tag{2.6}$$

The letter 'R' in $R^{a(v),c}$ is chosen for 'remainder'. This choise is justified by the regularizing properties of this operator stated in the next statement.

6 – *Theorem. Let $\beta \in [\alpha-2, 0)\backslash\{-1\}$. The map $K^{a(v),c}$ is a continuous operator from $\mathcal{C}_\mathfrak{s}^\beta(a(v))$ into $\mathcal{C}_\mathfrak{s}^{\beta+2}$ and the map $R^{a(v),c}$ is a continuous operator from $\mathcal{C}_\mathfrak{s}^\beta(a(v))$ into $\mathcal{C}_\mathfrak{s}^{\alpha+2-} := \bigcap_{\varepsilon>0} \mathcal{C}_\mathfrak{s}^{\alpha+2-\varepsilon}$.*

Proof – The former part is obtained from a similar argument to the proof of Theorem *40*. The latter part is obtained from a combination of Theorem *38* and Theorem *40*. Note that $(c - \mathcal{L}^{a(v)})^{-1}$ sends $\mathcal{C}_\mathfrak{s}^\beta(v)$ into $\mathcal{C}_\mathfrak{s}^{\beta+4}$ continuously by Theorem *40*, and the inverse operator $\big(\partial_t - L^{a(v)} + c\big)^{-1}$ sends $(1 + \partial_t + \partial_x^2)\big(c - \mathcal{L}^{a(v)}\big)^{-1} f \in \mathcal{C}_\mathfrak{s}^{\beta+2} \subset \mathcal{C}_\mathfrak{s}^\alpha$ into $\mathcal{C}_\mathfrak{s}^{\alpha+2-}$ by Theorem *38*. ▷

We fix from now on a constant $c > \max(c_0, C_0)$ and omit the letter 'c' in $Q^{a(v)}$, $\mathcal{Q}^{a(v)}$, $K^{a(v)}$, $R^{a(v)}$ unless it needs to be emphasized.

2.2 – The regularity structure.

We construct in this section the regularity structure associated with equation (1.4). It will be convenient, for notational purposes, to rewrite (1.4) under the form

$$\begin{cases} \boldsymbol{u} = \mathsf{P}_{<2}\big(Q^{a(v)} u_0\big) + \mathsf{K}_\gamma^{a(v),\mathsf{M}}(\boldsymbol{v} + \boldsymbol{w}), \\ \boldsymbol{v} = \mathsf{Q}_{\le 0}\Big\{F(\boldsymbol{u})\zeta_1 + \big\{G(\boldsymbol{u})(\boldsymbol{D}\boldsymbol{u})^2 + c\boldsymbol{u}\big\}\zeta_2\Big\}, \\ \boldsymbol{w} = \mathsf{Q}_{\le 0}\Big\{\big\{A(\boldsymbol{u}) - A(\mathsf{P}_{<2}(v))\big\}(\boldsymbol{D}^2 \mathsf{P}_{\le 2} Q^{a(v)} u_0 + \boldsymbol{D}^2 \mathsf{K}_{\gamma+\alpha_0}^{a(v),\mathsf{M}}(\boldsymbol{v} + \boldsymbol{w}))\zeta_3\Big\} \end{cases} \tag{2.7}$$

with three 'noise' symbols $\zeta_1, \zeta_2, \zeta_3$ in the regularity structure. This will help us distinguish three different types of terms.

We first define a 'preparatory' collection of rooted decorated trees

$$\overline{\mathbb{B}} = \overline{\mathbb{B}}_\bullet \cup \overline{\mathbb{B}}_\circ$$

with node decorations $\{X^\mathbf{k}\}_{\mathbf{k}\in\mathbf{N}^2}$ and $\{\zeta_l\}_{l\in\{1,2,3\}}$, and edge decorations $\{\mathcal{I}_\mathbf{n}\}_{\mathbf{n}\in\mathbf{N}^2}$. Write

$$\mathcal{I} := \mathcal{I}_\mathbf{0}$$

for simplicity and define $\overline{\mathbb{B}}_\bullet$ and $\overline{\mathbb{B}}_\circ$ by the smallest sets satisfying the following relations.

(a) $\overline{\mathbb{B}}_\bullet = \overline{\mathbb{B}}_\bullet^1 \cup \overline{\mathbb{B}}_\bullet^2 \cup \overline{\mathbb{B}}_\bullet^3$ with

$$\overline{\mathbb{B}}_\bullet^1 = \Big\{X^{\mathbf{k}} \prod_{i=1}^{n} \mathcal{I}(\tau_i) \; ; \; \mathbf{k} \in \mathbf{N}^2,\; n \in \mathbf{N},\; \tau_1, \dots, \tau_n \in \overline{\mathbb{B}}_\circ \setminus \{X^{\mathbf{k}} \zeta_l\}_{\mathbf{k}\in\mathbf{N}^2,\, l \in \{2,3\}}\Big\},$$

$$\overline{\mathbb{B}}_\bullet^2 = \Big\{X^{\mathbf{k}} \prod_{i=1}^{n} \mathcal{I}_{\mathbf{n}_i}(\tau_i) \; ; \; \mathbf{k} \in \mathbf{N}^2,\, n \in \mathbf{N},\; \tau_1, \dots, \tau_n \in \overline{\mathbb{B}}_\circ \setminus \{X^{\mathbf{k}} \zeta_l\}_{\mathbf{k}\in\mathbf{N}^2,\, l \in \{2,3\}},$$

$$\mathbf{n}_i = 0 \text{ except at most two } \mathbf{n}_i = (0,1)\Big\},$$

$$\overline{\mathbb{B}}_\bullet^3 = \Big\{X^{\mathbf{k}} \prod_{i=1}^{n} \mathcal{I}_{\mathbf{n}_i}(\tau_i) \; ; \; \mathbf{k} \in \mathbf{N}^2,\, n \in \mathbf{N},\; \tau_1, \dots, \tau_n \in \overline{\mathbb{B}}_\circ \setminus \{X^{\mathbf{k}} \zeta_l\}_{\mathbf{k}\in\mathbf{N}^2,\, l \in \{2,3\}},$$

$$\mathbf{n}_i = 0 \text{ except at most one } \mathbf{n}_i = (0,2)\Big\},$$

This definition ensures in particular that $X^{\mathbf{k}} \in \overline{\mathbb{B}}_\bullet$ by the convention that $\prod_{i=1}^{0} = 1$. We further assume that the product, called *tree product*, of the $\mathcal{I}_{\mathbf{n}_i}(\tau_i)$ is commutative. This means that we consider non-planar trees.

(b) $\overline{\mathbb{B}}_\circ = \overline{\mathbb{B}}_\circ^1 \cup \overline{\mathbb{B}}_\circ^2 \cup \overline{\mathbb{B}}_\circ^3$ with

$$\overline{\mathbb{B}}_\circ^l = \big\{\zeta_l\, \sigma \; ; \; \sigma \in \overline{\mathbb{B}}_\bullet^l\big\}, \qquad l \in \{1,2,3\}.$$

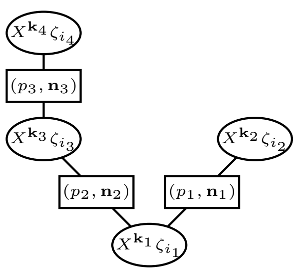
The set $\overline{\mathbb{B}}_\bullet$ contains all trees necessary to describe the right hand sides in (2.7). The set $\overline{\mathbb{B}}_\circ$ is a collection of trees in $\overline{\mathbb{B}}_\bullet$ multiplied by noise symbols ζ_l. As usual in a regularity structure setting we think of basis elements in $\overline{\mathbb{B}}$ as decorated trees. We define the homogeneity map $|\cdot| : \overline{\mathbb{B}} \to \mathbf{R}$ setting

$$|X^{\mathbf{k}}| := |\mathbf{k}|_{\mathfrak{s}}, \quad |\zeta_1| := \alpha_0 - 2, \quad |\zeta_2| = |\zeta_3| := 0,$$

$$|\mathcal{I}_{\mathbf{n}}(\tau)| := |\tau| + 2 - |\mathbf{n}|_{\mathfrak{s}}, \quad |\tau_1 \cdots \tau_n| := \sum_{i=1}^{n} |\tau_i|.$$

Since the operator $\mathcal{I}_{(0,2)}$ does not change the homogeneity, an infinite number of trees in $\overline{\mathbb{B}}$ have the same homogeneity. Modelled distributions which we will treat will then involve infinite linear combinations of trees. To deal with such infinite sums it will be convenient to introduce a new set of symbols $\mathcal{I}_{\mathbf{n}}^p$, with $p \in \mathbf{N}$ and $\mathbf{n} \in \mathbf{N}^2$.

Let $\widehat{\mathbb{B}}$ be the collection of ***rooted decorated trees*** with node decorations $\{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^2}$ and $\{\zeta_l\}_{l\in\{1,2,3\}}$, and edge decorations $\{\mathcal{I}_{\mathbf{n}}^p\}_{p\in\mathbf{N},\, \mathbf{n}\in\mathbf{N}^2}$. An example of elements of $\widehat{\mathbb{B}}$ is



.

Define inductively the projection map

$$\pi : \overline{\mathbb{B}} \to \widehat{\mathbb{B}}$$

by the identity

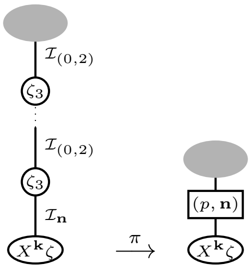
$$\pi\Big(\mathcal{I}_{\mathbf{n}}(\zeta_3 \mathcal{I}_{(0,2)})^p(\tau)\Big) = \mathcal{I}_{\mathbf{n}}^p\big(\pi(\tau)\big),$$

at each branches of the tree τ, for $\tau \in \overline{\mathbb{B}} \setminus \zeta_3 \mathcal{I}_{(0,2)}(\overline{\mathbb{B}}_\circ)$. Define

$$\mathbb{B} := \pi(\overline{\mathbb{B}})$$

as the image of π.

The letter $\mathbb{B}$ is chosen for 'basis'. Each element of $\mathbb{B}$ is then a rooted decorated tree with a further edge decoration $\boldsymbol{p} : E_\tau \to \mathbf{N}$, in addition to usual its two decorations $\mathfrak{n} : N_\tau \to \mathbf{N}^2$ and $\mathfrak{e} : E_\tau \to \mathbf{N}^2$ considered in Bruned, Hairer & Zambotti's work [10]. The decoration $\boldsymbol{p}$ represents the number of consecutive operators $\zeta_3\mathcal{I}_{(0,2)}$ in each edge.



We write $\tau^{\boldsymbol{p}}$ for a generic element of $\mathbb{B}$ when we want to emphasize its $\boldsymbol{p}$ decoration. Since $|\tau^{\boldsymbol{p}}| = |\tau^{\mathbf{0}}|$, an infinite number of trees in $\mathbb{B}$ have the same homogeneity. We will use the quantity

$$|\boldsymbol{p}| := \sum_{e\in E_\tau} \boldsymbol{p}(e)$$

to define the topology on the linear space spanned by $\mathbb{B}$. Set

$$\mathbb{B}_\circ := \pi(\overline{\mathbb{B}}_\circ).$$

The following subfamilies of elements of $\mathbb{B}$ will be useful in this and the next section.

$$\mathbb{B}_\circ^- := \big\{\tau^{\boldsymbol{p}} \in \mathbb{B}_\circ \,;\, |\tau^{\boldsymbol{p}}| < 0\big\},$$
$$\mathbb{B}^0 := \big\{\tau^{\mathbf{0}} \,;\, \tau^{\boldsymbol{p}} \in \mathbb{B}\big\},$$
$$\mathbb{B}_\circ^{-0} := \mathbb{B}_\circ^- \cap \mathbb{B}^0,$$
$$\mathbb{U} := \{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^2} \cup \{\mathcal{I}(\tau^{\boldsymbol{p}})\}_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ}.$$

The set $\mathbb{B}_\circ^{-0}$, resp. $\mathbb{B}_\circ^{-0}$, is the index set in formula (1.5), resp. (1.6), for the counterterm in the renormalized equation. We denote by

$$\mathbb{B}_\beta := \Big\{\tau^{\boldsymbol{p}} \in \mathbb{B} \;;\; |\tau^{\boldsymbol{p}}| = \beta\Big\}$$

the set of elements of $\mathbb{B}$ of homogeneity β. It is elementary to see the following properties.

7 – Proposition. The following properties hold for the set $\mathbb{B}$.

- *The set* $A := \big\{|\tau^{\boldsymbol{p}}| \;;\; \tau^{\boldsymbol{p}} \in \mathbb{B}\big\}$ *is locally finite and* $\min A = \alpha_0 - 2$.
- *The set* $\mathbb{B}_\beta \cap \mathbb{B}^0$ *is finite for each* $\beta \in A$.
- $\mathbb{B}_0 = \big\{X^{\mathbf{0}}, X^{\mathbf{0}}\zeta_2, X^{\mathbf{0}}\zeta_3\big\}$.

Moreover, we assume that

$$|\mathbb{B}_\circ^-| \cap \mathbb{Z} = \emptyset \tag{2.8}$$

through this paper. This assumption will be used in the proof of Theorem *13*.

To complete the construction of a regularity structure we consider the collection $\mathbb{B}^+$ of all the elements

$$X_+^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}^{+,q_i}(\tau_i^{\boldsymbol{p}_i})$$

with $\mathbf{k} \in \mathbf{N}^2$, $n \in \mathbf{N}$, $\tau_i^{\boldsymbol{p}_i} \in \mathbb{B}_\circ$, $q_i \in \mathbf{N}$, and $\mathbf{n}_i \in \mathbf{N}^2$ such that $|\tau_i| + 2 - |\mathbf{n}_i| > 0$ for each i. We use the label '+' to distinguish the elements of $\mathbb{B}^+$ from the elements of $\mathbb{B}_\bullet$. We define the homogeneity map $|\cdot| : \mathbb{B}^+ \to \mathbf{R}_+$ by setting

$$\Big|X_+^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}^{+,q_i}(\tau_i^{\boldsymbol{p}_i})\Big| := |\mathbf{k}|_{\mathfrak{s}} + \sum_{i=1}^n \big(|\tau_i| + 2 - |\mathbf{n}_i|_{\mathfrak{s}}\big).$$

We define Banach norms on the linear spaces spanned by $\mathbb{B}$ and $\mathbb{B}^+$. Picking a positive parameter m we define $T_\beta^{(m)}$ for each $\beta \in A$ as the completion of the linear space spanned by $\mathbb{B}_\beta$ under the norm defined by

$$\Big\| \sum_{\tau^{\boldsymbol{p}} \in \mathbb{B}_\beta} c_{\tau^{\boldsymbol{p}}} \tau^{\boldsymbol{p}} \Big\|_{\beta,m}^2 := \sum_{\tau^{\boldsymbol{p}} \in \mathbb{B}_\beta} |c_{\tau^{\boldsymbol{p}}}|^2 \, m^{2|\boldsymbol{p}|}.$$

We also define

$$T^{(m)} := \bigoplus_{\beta \in A} T_\beta^{(m)}$$

as the algebraic sum. Similarly we define the space

$$T^{(m),+} := \bigoplus_{\beta \geq 0} T_\beta^{(m),+}$$

from the set $\mathbb{B}^+$, using the same notation $\| \cdot \|_{\beta,m}$ for the norms on $T^{(m)}$ and $T^{(m),+}$. By definition $T^{(m),+}$ is an algebra.

We finally complete the construction of the regularity structure. We define the two continuous linear operators

$$\Delta : T^{(m)} \to T^{(m)} \otimes T^{(m),+}$$

and

$$\Delta^+ : T^{(m),+} \to T^{(m),+} \otimes T^{(m),+}$$

by the identities

$$\begin{aligned} \Delta \zeta_l &= \zeta_l \otimes X_+^{\mathbf{0}}, \\ \Delta^{(+)} X_{(+)}^{\mathbf{k}} &= \sum_{\mathbf{k}' \leq \mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} X_{(+)}^{\mathbf{k}'} \otimes X_+^{\mathbf{k}-\mathbf{k}'}, \\ \Delta^{(+)} \mathcal{I}_{\mathbf{n}}^{(+)} \tau &= \big(\mathcal{I}_{\mathbf{n}}^{(+)} \otimes \mathrm{Id}\big) \Delta \tau + \sum_{|\mathbf{k}|_{\mathfrak{s}} < |\tau| + 2 - |\mathbf{n}|_{\mathfrak{s}}} \frac{X_{(+)}^{\mathbf{k}}}{\mathbf{k}!} \otimes \mathcal{I}_{\mathbf{n}+\mathbf{k}}^+ \tau \end{aligned} \tag{2.9}$$

and the multiplicativity $\Delta^{(+)}(\tau_1 \cdots \tau_n) = \prod_{i=1}^n \Delta^{(+)} \tau_i$. (Recall from the Notation paragraph at the end of Section 1 our use of the notation $\Delta^{(+)}$.)

In the third identity of (2.9), we extend the symbol $\mathcal{I}_{\mathbf{n}}$ as a linear operator by imposing

$$\mathcal{I}_{\mathbf{n}}(X^{\mathbf{k}} \zeta_l) = 0, \qquad l \in \{2, 3\}.$$

For $\mathbf{k} = \mathbf{0}$ this reflects the identity

$$K^{a(v)} 1 = -\int_0^1 (\partial_t + \partial_x^2) \mathcal{Q}_\theta^{a(v)} 1 d\theta = -\int_0^1 (\partial_t + \partial_x^2) e^{-c\theta} d\theta = 0.$$

We do not assume that $K^{a(v)} x^{\mathbf{k}} = 0$ for $\mathbf{k} \neq \mathbf{0}$, but it does not matter because we do not use the symbols $\mathcal{I}_{\mathbf{n}}(X^{\mathbf{k}} \zeta_l)$ with $\mathbf{k} \neq \mathbf{0}$ and $l \in \{2, 3\}$ to solve the equation (2.7) in the space $\mathcal{D}_m^{\gamma,\eta}$ for $\gamma < 2$. See Theorem 16 for the details.

One has similar identities for the operators $\mathcal{I}_{\mathbf{n}}^{(+),p}$

$$\Delta^{(+)} \mathcal{I}_{\mathbf{n}}^{(+),p} \tau = \big(\mathcal{I}_{\mathbf{n}}^{(+),p} \otimes \mathrm{Id}\big) \Delta \tau + \sum_{\mathbf{k}} \frac{X_{(+)}^{\mathbf{k}}}{\mathbf{k}!} \otimes \mathcal{I}_{\mathbf{n}+\mathbf{k}}^{+,p} \tau,$$

for $\tau \in \mathbb{B}_\circ^-$, since $\Delta \mathcal{I}_{(0,2)} \tau = (\mathcal{I}_{(0,2)} \otimes \mathrm{Id}) \Delta \tau$ for τ with negative homogeneity. This definition of $\Delta^{(+)}$ turns it into an extension of the BHZ regularity structure for the semilinear generalized (KPZ) equation. The pair

$$\mathscr{T}^{(m)} := \big((T^{(m),+}, \Delta^+), (T^{(m)}, \Delta) \big)$$

is a concrete regularity structure in the sense of [4]. Denote by $G^{(m),+}$ the set of all continuous algebra maps $g : T^{(m),+} \to \mathbf{R}$, that is, g is multiplicative with respect to the tree product and with respect to the product with polynomials. Then $G^{(m),+}$ is a topological group with respect to the convolution product $g * h := (g \otimes h)\Delta^+$.

2.3 – Models and modelled distributions. We define in this section the notions of admissible model, modelled distribution, and state or prove two fundamental results about these objects: the reconstruction theorem and the lift in a space of modelled distributions of the inverse heat operator. We define Q_β as the canonical projection from $T^{(m)}$ to the subspace $T^{(m)}_\beta$, and define $\mathsf{Q}_{<\gamma} := \sum_{\beta\in A,\,\beta<\gamma} \mathsf{Q}_\beta$.

8 – *Definition. Given a positive parameter m, a pair $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi})$ made up of a map*

$$\mathsf{g} : \mathbf{R}^d \to G^{(m),+}$$

and a linear map

$$\mathsf{\Pi} : T^{(m)} \to \mathcal{C}^{-2}_{\mathfrak{s}}(a(v))$$

*is called **a model on** $\mathscr{T}^{(m)}$ if one has*

$$\big|\mathsf{g}_{z'z}(\tau^{\boldsymbol{p}})\big| \lesssim m^{|\boldsymbol{p}|} \|z' - z\|_{\mathfrak{s}}^{|\tau|} \qquad (\mathsf{g}_{z'z} := \mathsf{g}_{z'} * \mathsf{g}_z^{-1}),$$

for all $\tau^{\boldsymbol{p}} \in \mathbb{B}^+$ and $z, z' \in \mathbf{R}^2$, and

$$\big|\mathcal{Q}^{a(v)}_\theta\big(\mathsf{\Pi}^{\mathsf{g}}_z \sigma^{\boldsymbol{p}}\big)(z)\big| \lesssim m^{|\boldsymbol{p}|}\, \theta^{|\sigma|/4} \qquad (\mathsf{\Pi}^{\mathsf{g}}_z := (\mathsf{\Pi} \otimes \mathsf{g}_z^{-1})\Delta),$$

*for all $\sigma^{\boldsymbol{p}} \in \mathbb{B}$, $z \in \mathbf{R}^2$ and $\theta \in (0,1]$. The model M is said to be **spatially periodic** if*

$$\mathsf{g}_{(z'+(0,1))\,(z+(0,1))} = \mathsf{g}_{z'z}, \qquad \mathcal{Q}^{a(v)}_\theta\big(\mathsf{\Pi}^{\mathsf{g}}_{z+(0,1)}(\cdot)\big)(z + (0,1)) = \mathcal{Q}^{a(v)}_\theta\big(\mathsf{\Pi}^{\mathsf{g}}_z(\cdot)\big)(z)$$

for any $z, z' \in \mathbf{R}^2$.

These conditions ensure that $\mathsf{g}_{z'z}$ and $\mathsf{\Pi}^{\mathsf{g}}_z$ are continuous on the metric spaces $T^{(m)}_\beta$ and $T^{+,(m)}_\beta$ respectively under the norm $\|\cdot\|_{\beta,m}$, so the same analytical arguments as in [4] work to prove the results stated in this section. We record here for later use a straightforward adaptation of Proposition *2* and Lemma *12* in [4]. Recall from (2.6) the decomposition

$$\big(\partial_t - L^{a(v)} + c\big)^{-1} = K^{a(v)} + R^{a(v)}.$$

9 – *Lemma. For any $\sigma^{\boldsymbol{p}} \in \mathbb{B}$, $z \in \mathbf{R}^2$, $\theta \in (0,1]$, and $\mathbf{k} \in \mathbf{N}^2$ such that $|\mathbf{k}|_{\mathfrak{s}} \leq 4$, one has*

$$\Big|\big(\partial^{\mathbf{k}}_z \mathcal{Q}^{a(v)}_\theta\big)\big(\mathsf{\Pi}^{\mathsf{g}}_z \sigma^{\boldsymbol{p}}\big)(z)\Big| \lesssim m^{|\boldsymbol{p}|}\, \theta^{(|\sigma| - |\mathbf{k}|_{\mathfrak{s}})/4}.$$

Therefore for any $\sigma^{\boldsymbol{p}} \in \mathbb{B}^-_\circ$ and $\mathbf{k} \in \mathbf{N}^2$ such that $|\mathbf{k}|_{\mathfrak{s}} < |\sigma| + 2$ the integral

$$\big(\partial^{\mathbf{k}}_z K^{a(v)}\big)\big(\mathsf{\Pi}^{\mathsf{g}}_z \sigma^{\boldsymbol{p}}\big)(z) = \int_0^1 \big(\partial^{\mathbf{k}}_z K^{a(v)}_\theta\big)\big(\mathsf{\Pi}^{\mathsf{g}}_z \sigma^{\boldsymbol{p}}\big)(z)\, d\theta$$

converges for all $z \in \mathbf{R}^2$ and satisfies

$$\Big|\big(\partial^{\mathbf{k}}_z K^{a(v)}\big)\big(\mathsf{\Pi}^{\mathsf{g}}_z \sigma^{\boldsymbol{p}}\big)(z)\Big| \lesssim m^{|\boldsymbol{p}|}.$$

10 – *Definition. Pick $\eta \leq \gamma$. We denote by $\mathcal{D}^{\gamma,\eta}_m = \mathcal{D}^{\gamma,\eta}(T^{(m)}; \mathsf{g})$ the set of functions*

$$\boldsymbol{u} : \mathbf{R}^2 \to T^{(m)}_{<\gamma} := \mathsf{Q}_{<\gamma}(T^{(m)})$$

such that

$$(\!|\boldsymbol{u}|\!)_{\mathcal{D}^{\gamma,\eta}_m} := \max_{\beta<\gamma} \sup_{s>0} \Big\{ (s\wedge 1)^{\{(\beta-\eta)\vee 0\}/2} \sup_{|t|\geq s} \|\boldsymbol{u}(z)\|_{\beta,m} \Big\} < \infty,$$

$$\|\boldsymbol{u}\|_{\mathcal{D}^{\gamma,\eta}_m} := \max_{\beta<\gamma}\sup_{s>0}\left\{(s\wedge 1)^{(\gamma-\eta)/2}\sup_{|t|,|t'|\geq s,\, tt'>0}\frac{\left\|\boldsymbol{u}(z')-\widehat{\mathsf{g}_{z'z}}\boldsymbol{u}(z)\right\|_{\beta,m}}{\|z'-z\|_{\mathfrak{s}}^{\gamma-\beta}}\right\}<\infty,$$

where t and t' represent the time variable part of z and z' respectively. Equipped with the norm

$$|\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_m} := (\!|\boldsymbol{u}|\!)_{\mathcal{D}^{\gamma,\eta}_m} + \|\boldsymbol{u}\|_{\mathcal{D}^{\gamma,\eta}_m},$$

the space $\mathcal{D}^{\gamma,\eta}_m$ is a Banach space. Moreover $\boldsymbol{u}$ is said to be spatially periodic if

$$\boldsymbol{u}(z+(0,1)) = \boldsymbol{u}(z)$$

for any $z\in\mathbf{R}^2$.

Instead of $(\!|\boldsymbol{u}|\!)_{\mathcal{D}^{\gamma,\eta}_m}$ and $|\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_m}$, it will be convenient to consider the seminorms

$$(\!|\boldsymbol{u}|\!)'_{\mathcal{D}^{\gamma,\eta}_m} := \max_{\beta<\gamma}\sup_{s>0}\left\{(s\wedge 1)^{(\beta-\eta)/2}\sup_{|t|\geq s}\|\boldsymbol{u}(z)\|_{\beta,m}\right\}$$

and $|\!|\!|\boldsymbol{u}|\!|\!|'_{\mathcal{D}^{\gamma,\eta}_m} := (\!|\boldsymbol{u}|\!)'_{\mathcal{D}^{\gamma,\eta}_m} + \|\boldsymbol{u}\|_{\mathcal{D}^{\gamma,\eta}_m}$. In general $|\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_m} \leq |\!|\!|\boldsymbol{u}|\!|\!|'_{\mathcal{D}^{\gamma,\eta}_m}$ but the reverse inequality fails. However for any $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}_m$ such that

$$\lim_{t\to 0}\mathsf{Q}_\beta\boldsymbol{u}(t,x) = 0$$

for any $\beta<\eta$ the following properties hold.

- $|\!|\!|\boldsymbol{u}|\!|\!|'_{\mathcal{D}^{\gamma,\eta}_m} \lesssim |\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_m}$ (Lemma 6.5 of [23]).
- $|\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}^{\gamma',\eta}_m} \lesssim |\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}^{\gamma,\eta}_m}$, for any $\gamma'\leq\gamma$ (Lemma 6.6 of [23]).

Since g and $\mathsf{\Pi}$ are bounded linear operators on the spaces $T^{(m)}_\beta$ and $T^{(m),+}_\beta$ respectively we can prove the reconstruction theorem for $\mathcal{D}^{\gamma,\eta}_m$ similarly to [4, Theorem 20] and [25, Theorem 4.1].

11 – *Theorem. Let $\eta\leq\gamma$ and $\gamma>0$. Let M be a model on $\mathscr{T}$ of growth factor $m>0$. There exists a unique continuous linear operator*

$$\mathsf{R}^{\mathsf{M}}:\mathcal{D}^{\gamma,\eta}(T^{(m)};\mathsf{g})\to\mathcal{C}^{\eta\wedge(\alpha_0-2)}_{\mathfrak{s}}(a(v))$$

such that the bound

$$\left|\mathcal{Q}^{a(v)}_\theta\big(\mathsf{R}^{\mathsf{M}}\boldsymbol{v}-\mathsf{\Pi}^{\mathsf{g}}_z\boldsymbol{v}(z)\big)(z)\right| \lesssim \big(|t|^{1/2}\vee\theta^{1/4}\big)^{\eta\wedge(\alpha_0-2)-\gamma}\,\theta^{\gamma/4}$$

holds uniformly over for any $\boldsymbol{v}\in\mathcal{D}^{\gamma,\eta}_m$ with unit norm and $z=(t,x)\in\mathbf{R}^2$. Moreover if M and $\boldsymbol{v}$ are spatially periodic then $\mathsf{R}^{\mathsf{M}}\boldsymbol{v}$ is a spatially periodic distribution.

We say that a vector space $S=\bigoplus_{\beta\in A}S_\beta$ is a **sector** if each vector space S_β is a closed subspace of $T^{(m)}_\beta$ and $\Delta(S)\subset S\otimes T^{(m),+}$. Then

$$\beta_0 := \min\big\{\beta\in A\,;\,S_\beta\neq\{0\}\big\}$$

is called a *regularity of S*. Given a sector S we denote by $\mathcal{D}^{\gamma,\eta}_m(S)$ the set of the elements $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}_m$ taking values in S. We will use in particular the sectors

$$U \text{ and } T_\circ$$

spanned by $\mathbb{U}$ and $\mathbb{B}_\circ$, respectively. Since $\alpha=\min\big\{|\tau^{\boldsymbol{p}}|\,;\,\tau^{\boldsymbol{p}}\in\mathbb{U}\setminus\{X^{\mathbf{k}}\}_{\mathbf{k}\in\mathbf{N}^2}\big\}>0$, for any $\boldsymbol{u}\in\mathcal{D}^{\gamma,\eta}_m(U)$ the reconstruction $\mathsf{R}^{\mathsf{M}}\boldsymbol{u}$ of $\boldsymbol{u}$ coincides with the $X^{\mathbf{0}}$-component of $\boldsymbol{u}$ and belongs to $\mathcal{C}^\alpha_{\mathfrak{s}}$ on any compact subset of $(0,\infty)\times\mathbf{R}$. (This claim is the content of Proposition 3.28 of [23].)

The proper notion of admissible model in the present setting is captured by the following definition.

12 – *Definition. An **admissible model on** $\mathscr{T}^{(m)}$ is a model $(\mathsf{g},\mathsf{\Pi})$ such that*

$$\mathsf{g}_z(X^{\mathbf{k}}_+) = z^{\mathbf{k}},\qquad \mathsf{\Pi}(X^{\mathbf{k}}\tau)(z) = z^{\mathbf{k}}(\mathsf{\Pi}\tau)(z),\qquad \mathsf{\Pi}\zeta_l = 1\quad(l\in\{2,3\})$$

and one has for all $\tau^{\boldsymbol{p}} \in \mathbb{B}_\circ^-$,

$$\Pi(\mathcal{I}\tau^{\boldsymbol{p}}) = K^{a(v)}(\Pi\tau^{\boldsymbol{p}}).$$

An admissible model satisfies the identity

$$\mathsf{g}_z^{-1}\big(\mathcal{I}_{\mathbf{n}}^+\tau\big) = -\sum_{|\mathbf{k}|_{\mathfrak{s}}<|\tau|+2-|\mathbf{n}|_{\mathfrak{s}}} \frac{(-z)^{\mathbf{k}}}{\mathbf{k}!}\Big(\big(\partial_z^{\mathbf{n}+\mathbf{k}}K^{a(v)}\big)\big(\Pi_z\tau\big)\Big)(z)$$

for any $\tau \in \mathbb{B}_\circ^-$ – see e.g. Proposition *15* of [4]. The proof of the multi-level Schauder estimates can be done along the same lines as in Hairer's original statement, Theorem *5.12* of [23], but we need a slight modification because the kernel $K_\theta(z,w)$ is only twice differentiable with respect to the first variable. See [25, Theorem 5.12] for the proof in more general settings. The fact that the quantity $\mathcal{J}^{a(v)}(z)\tau^{\boldsymbol{p}}$ below is well-defined is a consequence of Lemma *9*. (We stated it explicitly to make that point clear.)

Recall that $\alpha \in (0,\alpha_0)$ is the regularity of the initial value u_0 – see Section 2*.*1.

13 – Theorem. Let M *stand for an admissible model on* $\mathscr{T}^{(m)}$*. For any* $\tau^{\boldsymbol{p}} \in \mathbb{B}_\circ^- \cup \{X^{\mathbf{0}}\}$ *set*

$$\mathcal{J}^{a(v)}(z)\tau^{\boldsymbol{p}} := \sum_{|\mathbf{k}|_{\mathfrak{s}}<|\tau^{\boldsymbol{p}}|+2} \frac{X^{\mathbf{k}}}{\mathbf{k}!}\,\partial_z^{\mathbf{k}}K^{a(v)}(\Pi_z^{\mathsf{g}}\tau^{\boldsymbol{p}})(z).$$

For $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(T_\circ,\mathsf{g})$ *with* $\gamma > 0$*, set*

$$\big(\mathcal{N}^{a(v)}\boldsymbol{u}\big)(z) := \sum_{|\mathbf{k}|_{\mathfrak{s}}<\gamma+2} \frac{X^{\mathbf{k}}}{\mathbf{k}!}\,\partial_z^{\mathbf{k}}K^{a(v)}\Big(\mathsf{R}^{\mathsf{M}}\boldsymbol{u} - \Pi_z^{\mathsf{g}}\boldsymbol{u}(z)\Big)(z).$$

For $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(T_\circ,\mathsf{g})$ *with* $\gamma > 0$ *set*

$$\big(\mathcal{K}^{a(v),\mathsf{M}}\boldsymbol{u}\big)(z) := \mathsf{Q}_{<\gamma+2}\Big\{\big(\mathcal{I} + \mathcal{J}^{a(v)}(z)\big)\boldsymbol{u}(z) + \big(\mathcal{N}^{a(v)}\boldsymbol{u}\big)(z)\Big\}.$$

If $-2 < \eta$ *and* $\gamma \in (0,\alpha)$*, the map* $\mathcal{K}^{a(v),\mathsf{M}}$ *sends continuously* $\mathcal{D}_m^{\gamma,\eta}(T_\circ)$ *into* $\mathcal{D}_m^{\gamma+2,(\eta+2)\wedge\alpha_0}(U)$*. Moreover, it holds that* $\mathsf{R}^{\mathsf{M}}\mathcal{K}^{a(v),\mathsf{M}}\boldsymbol{u} = K(\mathsf{R}^{\mathsf{M}}\boldsymbol{u})$ *for any* $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(T_\circ)$*.*

Define $\mathcal{D}_m^{\gamma,\eta}(0,T)$ as the space of modelled distributions defined on $(0,T)\times\mathbf{T}$; its norm is defined as in Definition *10* with functions $\boldsymbol{u}$ only defined on $(0,T)\times\mathbf{T}$. The set of elements of $\mathcal{D}_m^{\gamma,\eta}(0,T)$ taking values in a sector S is denoted by $\mathcal{D}_m^{\gamma,\eta}(0,T;S)$.

Recall that we denote by $\mathsf{P}_{<\gamma}$ the operator that lifts a smooth function on $(0,\infty)\times\mathbf{T}$ into the polynomial part of T of homogeneity strictly smaller than γ, so

$$\big(\mathsf{P}_{<\gamma}f\big)(z) = \sum_{|\mathbf{k}|_{\mathfrak{s}}<\gamma} \big(\partial_z^{\mathbf{k}}f\big)(z)\,\frac{X^{\mathbf{k}}}{\mathbf{k}!}.$$

Define

$$\mathcal{R}_\gamma^{a(v),\mathsf{M}}\boldsymbol{u} := \mathsf{P}_{<\gamma}\big(R^{a(v)}(\mathsf{R}^{\mathsf{M}}\boldsymbol{u})\big)$$

and

$$\mathsf{K}_\gamma^{a(v),\mathsf{M}} := \mathsf{Q}_{<\gamma}\mathcal{K}^{a(v),\mathsf{M}} + \mathcal{R}_\gamma^{a(v),\mathsf{M}}.$$

We obtain the following estimates via an extension of $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(0,T)$ into $\widetilde{\boldsymbol{u}} \in \mathcal{D}_m^{\gamma,\eta}(\mathbf{R})$ such that $\boldsymbol{u}(t,\cdot) = 0$ for $t < 0$. Because of the non-anticipative character of the kernel K the value of the modelled distribution $\mathsf{K}_{\gamma'}^{a(v),\mathsf{M}}(\widetilde{\boldsymbol{u}})$ on $(0,T)$ is uniquely determined independently to the choice of extension $\widetilde{\boldsymbol{u}}$ – see Section 4.3 of [4] for details.

14 – Theorem. Pick $\gamma \in (0,\alpha)$ *and* $\eta \in (\alpha-2,\gamma]$*. Then for any* $\kappa > 0$ *and* $\gamma' \le \gamma+2$*, we have*

$$\big\|\!\big|\mathsf{K}_{\gamma'}^{a(v),\mathsf{M}}(\boldsymbol{u})\big|\!\big\|_{\mathcal{D}_m^{\gamma',(\eta+2)\wedge\alpha_0-\kappa}(0,T)} \lesssim T^{\kappa/2}\,\|\!|\boldsymbol{u}|\!\|_{\mathcal{D}_m^{\gamma,\eta}(0,T)}.$$

Moreover it holds that $\mathsf{R}^{\mathsf{M}}\mathsf{K}^{a(v),\mathsf{M}}\boldsymbol{u} = (\partial_t - L^{a(v)} + c)^{-1}(\mathsf{R}^{\mathsf{M}}\boldsymbol{u})$ *for any* $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(T_\circ)$*.*

Proof – We know that $\mathcal{K}^{a(v),\mathsf{M}}\boldsymbol{u} \in \mathcal{D}_m^{\gamma+2,(\eta+2)\wedge\alpha_0}$ from Theorem *13*. Since $R^{a(v)}$ sends $\mathsf{R}^{\mathsf{M}}\boldsymbol{u} \in \mathcal{C}_{\mathfrak{s}}^{\eta\wedge(\alpha_0-2)}(a(v))$ into $\mathcal{C}_{\mathfrak{s}}^{\alpha+2-} \subset \mathcal{C}_{\mathfrak{s}}^{\gamma+2}$ by Theorem *6*, we have $\mathcal{R}_{\gamma+2}^{a(v),\mathsf{M}}\boldsymbol{u} \in \mathcal{D}_m^{\gamma+2,\eta'}$ for any $\eta' \in \mathbf{R}$. Hence $\mathsf{K}_{\gamma+2}^{a(v),\mathsf{M}}\boldsymbol{u} \in \mathcal{D}_m^{\gamma+2,(\eta+2)\wedge\alpha_0}$. Since $\beta \in A$ satisfying $\beta < (\eta+2)\wedge\alpha_0$ is only $\beta = 0$ and

$$\lim_{t\downarrow 0} \mathsf{Q}_0 \mathsf{K}_{\gamma+2}^{a(v),\mathsf{M}}\boldsymbol{u}(t,x) = \lim_{t\downarrow 0}(\partial_t - L^{a(v)} + c)^{-1}(\mathsf{R}^{\mathsf{M}}\boldsymbol{u})(t,x) = 0,$$

the norms $|\!|\!| \cdot |\!|\!|$ and $|\!|\!| \cdot |\!|\!|'$ are equivalent for $\mathsf{K}_{\gamma+2}^{a(v),\mathsf{M}}\boldsymbol{u}$ and we have

$$\begin{aligned}
&|\!|\!|\mathsf{K}_{\gamma'}^{a(v),\mathsf{M}}(\boldsymbol{u})|\!|\!|_{\mathcal{D}_m^{\gamma',(\eta+2)\wedge\alpha_0-\kappa}(0,T)} \lesssim |\!|\!|\mathsf{K}_{\gamma'}^{a(v),\mathsf{M}}(\boldsymbol{u})|\!|\!|'_{\mathcal{D}_m^{\gamma',(\eta+2)\wedge\alpha_0-\kappa}(0,T)} \\
&\lesssim T^{\kappa/2}|\!|\!|\mathsf{K}_{\gamma+2}^{a(v),\mathsf{M}}(\boldsymbol{u})|\!|\!|'_{\mathcal{D}_m^{\gamma+2,(\eta+2)\wedge\alpha_0}(0,T)} \lesssim T^{\kappa/2}|\!|\!|\mathsf{K}_{\gamma+2}^{a(v),\mathsf{M}}(\boldsymbol{u})|\!|\!|_{\mathcal{D}_m^{\gamma+2,(\eta+2)\wedge\alpha_0}(0,T)} \\
&\lesssim T^{\kappa/2}\,|\!|\!|\boldsymbol{u}|\!|\!|_{\mathcal{D}_m^{\gamma,\eta}(0,T)}.
\end{aligned}$$

▷

We end this section by mentioning some continuity results for some operations on modelled distributions. Below the product $\tau\sigma$ of elements τ, σ in T is defined by the linear extension of tree product, as long as it belongs to T. The following results are variants of [23, Propositions *6.12*, *6.13*, *6.15* and *6.16*] so we omit the proofs here.

– Let S_1 and S_2 are sectors of regularities α_1 and α_2 respectively, and such that the product $S_1 \times S_2 \to T^{(m)}$ is defined. Then for any $\boldsymbol{u}_i \in \mathcal{D}_m^{\gamma_i,\eta_i}(S_i)$ $(i=1,2)$, we have

$$\mathsf{Q}_{<\gamma}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_2) \in \mathcal{D}_m^{\gamma,\eta}$$

with $\gamma = (\gamma_1+\alpha_2)\wedge(\gamma_2+\alpha_1)$ and $\eta = (\eta_1+\alpha_2)\wedge(\eta_2+\alpha_1)\wedge(\eta_1+\eta_2)$. Moreover, the mapping $(\boldsymbol{u}_1,\boldsymbol{u}_2) \mapsto \mathsf{Q}_{<\gamma}(\boldsymbol{u}_1\cdot\boldsymbol{u}_2)$ is locally Lipschitz continuous.

– For any $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(U)$ and a function $h \in C^\kappa(\mathbf{R})$ with $\kappa \geq \max\{\gamma/\alpha, 1\}$, we define

$$H(\boldsymbol{u}) := \mathsf{Q}_{<\gamma}\left(\sum_{n=0}^{\infty} \frac{h^{(n)}(u_0)}{n!}(\boldsymbol{u} - u_0 X^{\mathbf{0}})^n\right),$$

where u_0 denotes the $X^{\mathbf{0}}$-component of $\boldsymbol{u}$. Then $H(\boldsymbol{u}) \in \mathcal{D}_m^{\gamma,\eta}$, and the mapping $\boldsymbol{u} \mapsto H(\boldsymbol{u})$ is locally Lipschitz continuous.

– Define $\boldsymbol{D}$ as a linear operator on T such that

$$\boldsymbol{D}X^{(k_1,k_2)} := k_2 X^{(k_1,k_2-1)}\mathbf{1}_{k_2>0}, \qquad \boldsymbol{D}\mathcal{I}_{\mathbf{n}}(\tau) := \mathcal{I}_{\mathbf{n}+(0,1)}(\tau).$$

Let $n \in \{1,2\}$. If $\gamma > n$, then the map $\mathcal{D}_m^{\gamma,\eta}(U) \ni \boldsymbol{u} \mapsto \boldsymbol{D}^n\boldsymbol{u} \in \mathcal{D}_m^{\gamma-n,\eta-n}$ is continuous and satisfies $\mathsf{R}^{\mathsf{M}}\boldsymbol{D}^n\boldsymbol{u} = \partial_x^n \mathsf{R}^{\mathsf{M}}\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathcal{D}_m^{\gamma,\eta}(U)$.

3 – Local well-posedness

We prove in this section that the regularity structure formulation (1.4) of the quasilinear equation (1.1) is locally well-posed in time. We emphasize some elementary facts before stating and proving the well-posedness result in Theorem *16*. They follow from Definition *5* and Lemma *41*.

15 – *Lemma.* *Let $\alpha \in (0,1)$. For any $f \in V^\alpha(0,T)$, the function $\mathsf{P}_{<2}f$ belongs to $\mathcal{D}_m^{\gamma,\eta}(0,T)$ for any $\gamma \in (1,2)$ and $\eta \leq \alpha$ and has the estimate*

$$|\!|\!|\mathsf{P}_{<2}f|\!|\!|_{\mathcal{D}_m^{\gamma,\eta}(0,T)} \lesssim \|f\|_{L^\infty((0,T)\times\mathbf{T})} + T^{(\alpha-\eta)/2}\|f\|_{V^\alpha(0,T)}.$$

Consequently, the following estimates hold for any $u_0 \in C^\alpha(\mathbf{T})$.

(i) Denote by Q_t either of $e^{t\partial_x^2}$ or $Q_t^{a(v),c}$ with $c > 0$. For any $\gamma \in (1,2)$ and $\eta \leq \alpha$,

$$|\!|\!|\mathsf{P}_{<2}(Q.u_0)|\!|\!|_{\mathcal{D}_m^{\gamma,\eta}(0,T)} \lesssim \|u_0\|_{C^\alpha(\mathbf{T})}.$$

Moreover, for any $\gamma \in (2, 2+\alpha)$ and $\eta \le \alpha$,

$$\|\mathsf{P}_{\le 2}(Q.u_0)\|_{\mathcal{D}_m^{\gamma,\eta}(0,T)} \lesssim \|u_0\|_{C^\alpha(\mathbf{T})}.$$

(ii) For any $\gamma \in (1,2)$ and $\eta < \alpha$,

$$\Big\|\mathsf{P}_{<2}\big\{(Q_t^{a(v),c} - e^{t\partial_x^2})u_0(x)\big\}\Big\|_{\mathcal{D}_m^{\gamma,\eta}(0,T)} \lesssim T^{(\alpha-\eta)/2}\,\|u_0\|_{C^\alpha(\mathbf{T})}.$$

16 – Theorem. Let $\alpha \in (0,\alpha_0)$. For any $u_0 \in C^\alpha(\mathbf{T})$, we choose $v \in V^\alpha(0,T)$ satisfying (2.5) for sufficiently small $\delta > 0$ (depending only on $\|u_0\|_{C^\alpha(\mathbf{T})}$). Then for any admissible model M, there exists sufficiently small $t_0 \in (0,T]$ such that equation (2.7) has a unique solution $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ in the class

$$\mathcal{D}_m^{\gamma,\alpha}(0,t_0;U) \times \mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0;T_\circ) \times \mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0;T_\circ) \tag{3.1}$$

for any $\gamma \in (2-\alpha_0, 2-\alpha_0+\alpha)$. The time t_0 can be chosen to be a lower semicontinuous function of M and u_0.

Proof – We find a solution by the Picard iteration. Let $\boldsymbol{v}_0 = \boldsymbol{w}_0 = 0$ and

$$\begin{aligned} \boldsymbol{u}_n &= \mathsf{P}_{<2}\big(Q^{a(v)}u_0\big) + \mathsf{K}_\gamma^{a(v),\mathsf{M}}(\boldsymbol{v}_n + \boldsymbol{w}_n),\\ \boldsymbol{v}_{n+1} &= \mathsf{Q}_{\le 0}\Big\{F(\boldsymbol{u}_n)\zeta_1 + \big\{G(\boldsymbol{u}_n)(\boldsymbol{D}\boldsymbol{u}_n)^2 + c\boldsymbol{u}\big\}\zeta_2\Big\},\\ \boldsymbol{w}_{n+1} &= \mathsf{Q}_{\le 0}\Big\{\big\{A(\boldsymbol{u}_n) - A(\mathsf{P}_{<2}(v))\big\}(\boldsymbol{D}^2\mathsf{P}_{\le 2}Q^{a(v)}u_0 + \boldsymbol{D}^2\mathsf{K}_{\gamma+\alpha_0}^{a(v),\mathsf{M}}(\boldsymbol{v}_n+\boldsymbol{w}_n))\zeta_3\Big\} \end{aligned} \tag{3.2}$$

In what follows C means a constant which is independent to t_0, u_0 and $(\boldsymbol{u}_n,\boldsymbol{v}_n,\boldsymbol{w}_n)$. Similarly $P(x)$ means a polynomial of a variable x whose coefficients are independent to t_0, u_0 and $(\boldsymbol{u}_n,\boldsymbol{v}_n,\boldsymbol{w}_n)$. The value of C may change from one occurrence to the others. By the multi-level Schauder estimate from Theorem *14* we have

$$\begin{aligned} &\|\boldsymbol{u}_{n+1}\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)}\\ &\le \Big\|\mathsf{P}_{<2}\big(Q^{a(v)}u_0\big)\Big\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)} + C\big(t_0^{\kappa/2}\,\|\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)} + \|\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\big)\\ &\le C\Big(\|u_0\|_{C^\alpha(\mathbf{T})} + t_0^{\kappa/2}\,\|\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)} + \|\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\Big), \end{aligned} \tag{3.3}$$

where $\kappa = \alpha \wedge (\alpha_0 - \alpha) > 0$. Next we consider $\boldsymbol{v}_{n+1}$. Since $\boldsymbol{u}_n$ takes values in the sector U, all $F(\boldsymbol{u}_n), G(\boldsymbol{u}_n), A(\boldsymbol{u}_n)$ are well-defined elements of $\mathcal{D}_m^{\gamma,\alpha}$. Since ζ has a homogeneity $\alpha_0 - 2$,

$$F(\boldsymbol{u}_n)\zeta \in \mathcal{D}_m^{\gamma+\alpha_0-2,\alpha+\alpha_0-2}(T_\circ).$$

Since $\boldsymbol{D}\boldsymbol{u}_n \in \mathcal{D}_m^{\gamma-1,\alpha-1}$ is in a sector of regularity $\alpha_0 - 1$,

$$(\boldsymbol{D}\boldsymbol{u}_n)^2 \in \mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(T_\circ),$$

and thus

$$G(\boldsymbol{u}_n)(\boldsymbol{D}\boldsymbol{u}_n)^2 \in \mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(T_\circ).$$

Therefore,

$$\|\boldsymbol{v}_{n+1}\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)} \le P\big(\|\boldsymbol{u}_n\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)}\big). \tag{3.4}$$

Finally we consider $\boldsymbol{w}_{n+1}$. Since $\boldsymbol{D}^2\mathsf{K}_{\gamma+\alpha_0}^{a(v),\mathsf{M}}$ maps $\mathcal{D}_m^{\gamma+\alpha_0-2,\eta-2}(T_\circ)$ into $\mathcal{D}_m^{\gamma+\alpha_0-2,\eta\wedge\alpha_0-2}$ continuously, one has

$$\begin{aligned} &\|\boldsymbol{w}_{n+1}\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\\ &\le C\big\|A(\boldsymbol{u}_n) - A(\mathsf{P}_{<2}(v))\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}\,\big\|\boldsymbol{D}^2\mathsf{P}_{\le 2}Q^{a(v)}u_0 + \boldsymbol{D}^2\mathsf{K}_{\gamma+\alpha_0}^{a(v),\mathsf{M}}(\boldsymbol{v}_n+\boldsymbol{w}_n))\zeta_3\big\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\\ &\le C\big\|A(\boldsymbol{u}_n) - A(\mathsf{P}_{<2}(v))\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}\,\big(\|u_0\|_{C^\alpha(\mathbf{T})} + \|\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)} + \|\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\big), \end{aligned}$$

where η is a positive constant such that $\eta < \alpha$. To obtain a small factor from the second term of the right hand side, we decompose

$$A(\boldsymbol{u}_n) - A\big(\mathsf{P}_{<2}(v)\big) = \big\{A(\boldsymbol{u}_n) - A\big(\mathsf{P}_{<2}(Q^{a(v)}u_0)\big)\big\} + \big\{A\big(\mathsf{P}_{<2}(Q^{a(v)}u_0)\big) - A\big(\mathsf{P}_{<2}(e^{t\partial_x^2}u_0)\big)\big\}$$

$$+\left\{A\big(\mathsf{P}_{<2}(e^{t\partial_x^2}u_0)\big)-A\big(\mathsf{P}_{<2}(v)\big)\right\}.$$

For the first part, since A is locally Lipschitz as a mapping from $\mathcal{D}_m^{\gamma,\eta}$ to itself,

$$\begin{aligned}
&\big\|A(\boldsymbol{u}_n)-A\big(\mathsf{P}_{<2}(Q^{a(v)}u_0)\big)\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}\\
&\leq P\big(\|\boldsymbol{u}_n\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)},\|u_0\|_{C^\alpha(\mathbf{T})}\big)\big\|\boldsymbol{u}_n-\mathsf{P}_{<2}(Q^{a(v)}u_0)\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}\\
&\leq P\big(\|\boldsymbol{u}_n\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)},\|u_0\|_{C^\alpha(\mathbf{T})}\big)\|\mathsf{K}_\gamma^{a(v),\mathsf{M}}(\boldsymbol{v}_n+\boldsymbol{w}_n)\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}\\
&\leq P\big(\|\boldsymbol{u}_n\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)},\|u_0\|_{C^\alpha(\mathbf{T})}\big)t_0^{(\alpha-\eta)/2}\big(\|\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)}+\|\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\big).
\end{aligned}$$

For the second and third parts, we use Lemma *15* to have the estimate

$$\begin{aligned}
\big\|A\big(\mathsf{P}_{<2}(Q^{a(v)}u_0)\big)-A\big(\mathsf{P}_{<2}e^{t\partial_x^2}u_0)\big)\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}&\leq P\big(\|u_0\|_{C^\alpha(\mathbf{T})}\big)\big\|\mathsf{P}_{<2}\big(Q^{a(v)}u_0-e^{t\partial_x^2}u_0\big)\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}\\
&\leq P\big(\|u_0\|_{C^\alpha(\mathbf{T})}\big)t_0^{(\alpha-\eta)/2}\,\|u_0\|_{C^\alpha(\mathbf{T})}
\end{aligned}$$

and from the assumption (2.5),

$$\begin{aligned}
\big\|A\big(\mathsf{P}_{<2}(e^{t\partial_x^2}u_0)\big)-A\big(\mathsf{P}_{<2}(v)\big)\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}&\leq P\big(\|u_0\|_{C^\alpha(\mathbf{T})}\big)\big\|\mathsf{P}_{<2}(e^{t\partial_x^2}u_0-v)\big\|_{\mathcal{D}_m^{\gamma,\eta}(0,t_0)}\\
&\leq P\big(\|u_0\|_{C^\alpha(\mathbf{T})}\big)\big(\delta+t_0^{(\alpha-\eta)/2}\big)\|u_0\|_{C^\alpha(\mathbf{T})}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\|\boldsymbol{w}_{n+1}\|_{\mathcal{D}_m^{\gamma-2,\alpha-2}(0,t_0)}\\
&\leq\delta P\big(\|u_0\|_{C^\alpha(\mathbf{T})}\big)\big(\|u_0\|_{C^\alpha(\mathbf{T})}+\|\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)}+\|\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\big)\\
&\quad+t_0^{(\alpha-\eta)/2}P\big(\|u_0\|_{C^\alpha(\mathbf{T})},\|\boldsymbol{u}_n\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)}\|\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)},\|\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\big).
\end{aligned}\tag{3.5}$$

By (3.3), (3.4), and (3.5), by choosing sufficiently small $\delta,t_0>0$, we can find large constants $M_1,M_2,M_3>0$ such that

$$\|\boldsymbol{u}_n\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)}\leq M_1,\quad\|\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)}\leq M_2,\quad\|\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\leq M_3$$

for any $n\in\mathbf{N}$. Note that $\delta>0$ is chosen as $\delta P(\|u_0\|_{C^\alpha(\mathbf{T})})\ll 1$, so it is independent of M_1,M_2 and M_3. By the local Lipschitz estimates of the operations in (3.2) (product, composition with smooth function, differentiation, and integration) we have the similar estimate

$$\begin{aligned}
&\|\boldsymbol{u}_{n+1}-\boldsymbol{u}_n\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)}+\|\boldsymbol{v}_{n+1}-\boldsymbol{v}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)}+\|\boldsymbol{w}_{n+1}-\boldsymbol{w}_n\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,\alpha-2}(0,t_0)}\\
&\leq P(M_1,M_2,M_3)\,t_0^\delta\,\Big(\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}\|_{\mathcal{D}_m^{\gamma,\alpha}(0,t_0)}+\|\boldsymbol{v}_n-\boldsymbol{v}_{n-1}\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)}\\
&\qquad\qquad+\|\boldsymbol{w}_n-\boldsymbol{w}_{n-1}\|_{\mathcal{D}_m^{\gamma+\alpha_0-2,2\alpha-2}(0,t_0)}\Big)
\end{aligned}$$

for a small exponent $\delta>0$. Hence we can choose t_0 smaller such that $(\boldsymbol{u}_n,\boldsymbol{v}_n,\boldsymbol{w}_n)$ is a Cauchy sequence. The limit solves equation (2.7). Uniqueness also holds because of the local Lipschitz estimates. ▷

Otto, Sauer, Smith & Weber [29] and Linares, Otto & Tempelmayr [26] set up an analytic and an algebraic framework to deal with the quasilinear equation (1.2) with additive forcing, i.e. $f=1$ and $g=0$. They use in particular a greedy index set for their local expansions and prove an a priori bound for the solutions to a renormalized form of their equation driven by a smooth noise. Their result holds in the full sub-critical regime but they do not prove a well-posedness result for their equation. The a priori result entails a compactness statement that ensure the existence of some converging subsequence when the regularizing parameter in the noise is sent to 0. The analysis of the present section shows that one can run the analysis of the general equation (3.2) within the variant of the usual regularity structure for the generalized (KPZ) equation described in Section 2. The present section can also be seen as a simple alternative to the somewhat convoluted approach of Gerencsér & Hairer [20]. The interest of this reformulation of (1.1) will be clear in the next section. The formulation of [20] does not lend itself to an easy formulation of a renormalized equation for (1.1). At the level of generality of [20] the counterterm in their renormalized equation is a priori a nonlocal functional of the solution. Our main result, Theorem 1 in Section 1, shows that there is, in the full subcritical regime, a renormalized equation whose counterterm is a local functional of its solution. (Recall there is not a unique renormalized equation.)

4 – Renormalization matters

This section is dedicated to the analysis of the equation satisfied by the reconstruction of the solution $\boldsymbol{u}$ obtained in Theorem *16* – the so called renormalized equation. The first systematic treatment of this equation in a semilinear setting was done by Bruned, Chandra, Chevyrev & Hairer in [9]. They relied on a morphism property satisfied by the coefficients u_τ of generic solutions to semilinear singular SPDEs, for some multi-pre-Lie structures. A deeper structure on the elements of BHZ regularity structures was unveiled by Bruned & Manchon in [11] and applied by Bailleul & Bruned in [2] to simplify a lot the analysis of the renormalized equation. This structure is encoded in the $\star$ product introduced in Section 4.2. Its importance in the analysis of equation (1.4) is emphasized by Proposition *21*; it provides a basic morphism property – the counterpart here of the multi-pre-Lie morphism property used in [9]. We introduce in Section 4.3 the class of preparation maps – special linear maps from $T^{(m)}$ into itself, and their associated admissible models. A preliminary form of Theorem *1* follows from their properties in Proposition *23*. A special class of preparation maps is associated with the set of characters on $\mathbb{B}_\circ^-$. We show in Section 4.4 that working with the preparation map associated with the analogue in our setting of the BHZ character leads to Theorem *1*.

4.1 – Notations. We first fix some notations. In this section, we consider the set of *all* decorated trees

$$\overline{\mathbb{T}} = \overline{\mathbb{T}}_\bullet \cup \overline{\mathbb{T}}_\circ,$$

since the operators which we define below may not be closed in the smaller set $\mathbb{B}$. The sets $\overline{\mathbb{T}}_\bullet$ and $\overline{\mathbb{T}}_\circ$ are defined by the smallest sets satisfying that

$$\overline{\mathbb{T}}_\bullet = \Big\{ X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i) \ ; \ \mathbf{k} \in \mathbf{N}^2, \ n \in \mathbf{N}, \ \mathbf{n}_i \in \mathbf{N}^2, \ \tau_i \in \overline{\mathbb{T}}_\circ \Big\}$$

and

$$\overline{\mathbb{T}}_\circ = \Big\{ \zeta_l \, \sigma \ ; \ l \in \{1,2,3\}, \ \sigma \in \overline{\mathbb{T}}_\bullet \Big\}.$$

Similarly to Section 2.2 we assume that the tree product is commutative. For convenience we denote a generic element of $\overline{\mathbb{T}}$ by

$$X^{\mathbf{k}} \zeta_l \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i)$$

for $l \in \{1,2,3,4\}$ with the convention

$$\zeta_4 := X^{\mathbf{0}}.$$

The combinatorial symmetry factor $S(\tau)$ of the tree

$$\tau = X^{\mathbf{k}} \zeta_l \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i)^{\beta_i}$$

with $(\mathbf{n}_i, \tau_i) \neq (\mathbf{n}_j, \tau_j)$ for any $i \neq j$ is inductively defined by

$$S(\tau) := \mathbf{k}! \bigg(\prod_{i=1}^n S(\tau_i)^{\beta_i} \beta_i! \bigg).$$

We also define the map π similarly to what was done in Section 2.2 to introduce a further edge decoration $\boldsymbol{p}$ and set

$$\mathbb{T} := \pi(\overline{\mathbb{T}}).$$

The $\boldsymbol{p}$ decoration is used to deal with infinite sums. However it will also be convenient to use the set $\overline{\mathbb{T}}$ to deal with some operators defined similarly as in [9, 1]. The following identity will be useful later.

17 – Lemma. Let $\mathbb{S}$ be a finite set of $\mathbb{T}$ such that $\tau^{\boldsymbol{0}} \in \mathbb{S}$ if $\tau^{\boldsymbol{p}} \in \mathbb{S}$ and let $\{c_{\tau^{\boldsymbol{p}}}\}_{\tau^{\boldsymbol{p}}\in\mathbb{S}}$ be a family of real numbers. Then one has the identity

$$\sum_{\tau^{\boldsymbol{p}}\in\mathbb{S}} \frac{c_{\tau^{\boldsymbol{p}}}}{S(\tau^{\boldsymbol{p}})}\tau^{\boldsymbol{p}} = \sum_{\tau^{\boldsymbol{0}}\in\mathbb{S}} \frac{1}{S(\tau^{\boldsymbol{0}})} \sum_{\boldsymbol{p}\in\mathbf{N}^{E_\tau}} c_{\tau^{\boldsymbol{p}}}\tau^{\boldsymbol{p}}.$$

Note that $S(\tau^{\boldsymbol{p}})$ is smaller than or equal to $S(\tau^{\boldsymbol{0}})$ in general. The above identity comes from the order of the sums for trees and decorations. In the left hand side each $\tau^{\boldsymbol{p}}$ is considered as a non-planar tree. In the right hand side however, we fix a tree τ first and put a decoration $\boldsymbol{p}$ later, so $\tau^{\boldsymbol{p}}$ is rather considered as a planar tree. For example the tree $\tau^{p,q} := \mathcal{I}^p(\zeta_1)\mathcal{I}^q(\zeta_1)$ is the same as $\tau^{q,p}$ in the set $\mathbb{T}$, and we have

$$\sum_{\tau^{p,q}=\mathcal{I}^p(\zeta_1)\mathcal{I}^q(\zeta_1)\in\mathbb{T}} \frac{c_{\tau^{p,q}}}{S(\tau^{p,q})}\tau^{p,q} = \sum_{p\in\mathbf{N}} \frac{c_{\tau^{p,p}}}{2}\tau^{p,p} + \sum_{p<q\in\mathbf{N}} c_{\tau^{p,q}}\tau^{p,q} = \frac{1}{S(\tau^{0,0})} \sum_{p,q\in\mathbf{N}} c_{\tau^{p,q}}\tau^{p,q}.$$

We denote by T the linear space spanned by $\mathbb{T}$, and by T^* its algebraic dual. For a fixed $m > 0$ and any $\tau^{\boldsymbol{0}} \in \mathbb{T}$ we define $\mathsf{T}_\tau^{(m)}$ as the completion of the linear space spanned by non-planar trees $\{\tau^{\boldsymbol{p}}\}_{\boldsymbol{p}}$ under the norm

$$\Big\| \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}\tau^{\boldsymbol{p}} \Big\|_m^2 := \sum_{\boldsymbol{p}} |c_{\boldsymbol{p}}|^2 m^{2|\boldsymbol{p}|}.$$

We define

$$\mathsf{T}^{(m)} = \bigoplus_{\tau^{\boldsymbol{0}}\in\mathbb{T}} \mathsf{T}_\tau^{(m)}$$

as the algebraic sum. Setting

$$\big\langle \tau^{\boldsymbol{p}}, (\sigma^{\boldsymbol{q}})^* \big\rangle := S(\tau^{\boldsymbol{p}})\,\mathbf{1}_{\tau^{\boldsymbol{p}}=\sigma^{\boldsymbol{q}}}$$

for $\tau^{\boldsymbol{p}} \in \mathbb{T}$ and the dual element $(\sigma^{\boldsymbol{q}})^*$ of $\sigma^{\boldsymbol{q}} \in \mathbb{T}$ we can extend the duality relation between $\mathsf{T}^{(m)}$ and $\mathsf{T}^{*,(1/m)}$ to the completion of T^* under the norm $\|\cdot\|_{1/m}$.

4.2 – Coherence and morphism property for the $\star$ product. We write $\overline{\tau}$ to mean a generic element of $\overline{\mathbb{T}}$. We denote by $\overline{\mathsf{T}}_{(\cdot)}$ the linear space spanned by $\overline{\mathbb{T}}_{(\cdot)}$ with $(\cdot) \in \{\emptyset, \circ, \bullet\}$, and by $\overline{\mathsf{T}}^*_{(\cdot)}$ its algebraic dual.

4.2.1 – Coherence property. Let $\mathsf{c} = (\mathsf{c}_{\mathbf{k}})_{\mathbf{k}\in\mathbf{N}^2}$ and $\mathsf{c}' = (\mathsf{c}'_{\mathbf{k}})_{\mathbf{k}\in\mathbf{N}^2}$ be abstract variables. We introduce the differential operators $D'_{\mathbf{n}} := \partial_{\mathsf{c}'_{\mathbf{n}}}$, for $\mathbf{n} \in \mathbf{N}^2$, and set, for $\mathbf{k}_0 \in \big\{(1,0),(0,1)\big\}$ in the canonical basis of $\mathbf{N}^2$,

$$\partial^{\mathbf{k}_0} := \sum_{\mathbf{n}\in\mathbf{N}^2} \Big(\mathsf{c}_{\mathbf{n}+\mathbf{k}_0} D_{\mathbf{n}} + \mathsf{c}'_{\mathbf{n}+\mathbf{k}_0} D'_{\mathbf{n}} \Big).$$

The vector fields $\partial^{(1,0)}$ and $\partial^{(0,1)}$ commute, so one defines unambiguously for $\mathbf{k} = (k_1,k_2) \in \mathbf{N}^2$ a $|\mathbf{k}|$-th order differential operator on functions of finitely many components of c and c' setting

$$\partial^{\mathbf{k}} := (\partial^{(1,0)})^{k_1} (\partial^{(0,1)})^{k_2}.$$

The following elementary relation is of crucial use in the proof of Proposition *21* below; its elementary proof is left to the reader.

18 – Lemma. For any $(\mathbf{k}_1,\dots,\mathbf{k}_n)\in(\mathbf{N}^2)^n$ *and* $\mathbf{m}\in\mathbf{N}^2$ *one has*

$$\sum_{\substack{(\mathbf{l}_1,\dots,\mathbf{l}_n)\in(\mathbf{N}^2)^n,\\ \mathbf{l}_1+\cdots+\mathbf{l}_n\le\mathbf{m}}}\binom{\mathbf{m}}{\mathbf{l}_1,\dots,\mathbf{l}_n}\prod_{j=1}^n\partial^{\mathbf{m}-\mathbf{l}_j}D_{\mathbf{k}_j-\mathbf{l}_j}=\Big(\prod_{j=1}^n D_{\mathbf{k}_j}\Big)\partial^{\mathbf{m}},\tag{4.1}$$

where

$$\binom{\mathbf{m}}{\mathbf{l}_1,\dots,\mathbf{l}_n}:=\frac{\mathbf{m}!}{\mathbf{l}_1!\cdots\mathbf{l}_n!}.$$

For any $\overline{\tau}\in\overline{\mathbb{T}}$ we define the function $\mathfrak{F}^a(\overline{\tau}^*)$ of the variables (c,c') as follows. Set

$$h(\mathsf{c}_{\mathbf{0}},\mathsf{c}'_{\mathbf{0}}):=a(\mathsf{c}_{\mathbf{0}})-a(\mathsf{c}'_{\mathbf{0}})$$

and

$$\begin{aligned}\mathfrak{F}^a(\zeta_1^*)(\mathsf{c},\mathsf{c}')&:=f(\mathsf{c}_{\mathbf{0}}),\\ \mathfrak{F}^a(\zeta_2^*)(\mathsf{c},\mathsf{c}')&:=g(\mathsf{c}_{\mathbf{0}})\,\mathsf{c}_{(0,1)}^2+c\,\mathsf{c}_{\mathbf{0}},\\ \mathfrak{F}^a(\zeta_3^*)(\mathsf{c},\mathsf{c}')&:=h(\mathsf{c}_{\mathbf{0}},\mathsf{c}'_{\mathbf{0}})\,\mathsf{c}_{(0,2)},\end{aligned}\tag{4.2}$$

and

$$\mathfrak{F}^a(\zeta_4^*)(\mathsf{c},\mathsf{c}'):=0,$$

and for $\overline{\tau}=X^{\mathbf{k}}\zeta_l\prod_{i=1}^n\mathcal{I}_{\mathbf{n}_i}(\overline{\tau}_i)\in\overline{\mathbb{T}}$ set

$$\mathfrak{F}^a(\overline{\tau}^*)(\mathsf{c},\mathsf{c}'):=\bigg(\Big\{\partial^{\mathbf{k}}\Big(\prod_{i=1}^n D_{\mathbf{n}_i}\Big)\mathfrak{F}^a(\zeta_l^*)\Big\}\prod_{i=1}^n\mathfrak{F}^a(\overline{\tau}_i^*)\bigg)(\mathsf{c},\mathsf{c}').\tag{4.3}$$

With

$$\tau_1:=\mathcal{I}(\zeta_1)\mathcal{I}_{(0,1)}(\zeta_1)^2\zeta_2,\quad \tau_2:=\mathcal{I}(\zeta_1)^2\mathcal{I}_{(0,2)}(\zeta_1)\zeta_3$$

one has for instance

$$\begin{aligned}\mathfrak{F}^a(\tau_1^*)(\mathsf{c},\mathsf{c}')&=\Big\{D_{\mathbf{0}}D_{(0,1)}^2\mathfrak{F}^a(\zeta_2^*)\Big\}\mathfrak{F}^a(\zeta_1^*)^3(\mathsf{c},\mathsf{c}')=2g'(\mathsf{c}_{\mathbf{0}})f(\mathsf{c}_{\mathbf{0}})^3,\\ \mathfrak{F}^a(\tau_2^*)(\mathsf{c},\mathsf{c}')&=\Big\{D_{\mathbf{0}}^2D_{(0,2)}\mathfrak{F}^a(\zeta_3^*)\Big\}\mathfrak{F}^a(\zeta_1^*)^3(\mathsf{c},\mathsf{c}')=a^{(2)}(\mathsf{c}_{\mathbf{0}})f(\mathsf{c}_{\mathbf{0}})^3.\end{aligned}$$

We see on these definitions that $\mathsf{c}_{\mathbf{0}}$ and $\mathsf{c}'_{\mathbf{0}}$ are placeholders for u and v in equation (1.3). The function $\mathfrak{F}^a$ vanishes outside $\overline{\mathbb{B}}$. Actually if $\overline{\tau}\in\overline{\mathbb{T}}\setminus\overline{\mathbb{B}}$ then it has a node $v\in N_\tau$ such that a collection of all edges leaving from v contains either an edge $\mathcal{I}_{\mathbf{k}}$ with $\mathbf{k}\neq\mathbf{0},(0,1),(0,2)$, or more than two edges $\mathcal{I}_{(0,1)}$, or more than one edges $\mathcal{I}_{(0,2)}$, then $\mathfrak{F}^a(\overline{\tau}^*)$ vanishes at v. By a similar argument it is easy to check that $\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)(\mathsf{c},\mathsf{c}')$ with $\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-$ are functions of $(\mathsf{c}_{\mathbf{0}},\mathsf{c}_{(0,1)})$ and $(\mathsf{c}'_{\mathbf{0}},\mathsf{c}'_{(0,1)})$ only. Furthermore, since the equality

$$\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)=h^{|\boldsymbol{p}|}\,\mathfrak{F}^a\big((\tau^{\mathbf{0}})^*\big)$$

follows from the definition, we have that

$$\bigg\|\mathfrak{F}^a\Big(\sum_{\boldsymbol{p}}c_{\boldsymbol{p}}(\tau^{\boldsymbol{p}})^*\Big)\bigg\|_{L^\infty(\mathcal{O}\times\mathbf{R}^2)}\le\big\|\mathfrak{F}^a\big((\tau^{\mathbf{0}})^*\big)\big\|_{L^\infty(\mathcal{O}\times\mathbf{R}^2)}\sum_{\boldsymbol{p}}\|h\|_{L^\infty(\mathcal{O})}^{|\boldsymbol{p}|}\,|c_{\boldsymbol{p}}|$$

for any domain $\mathcal{O}$ of $\mathbf{R}^2$. This means that $\mathfrak{F}^a$ maps $\mathsf{T}^{*,(m')}$ into $C_b(\mathcal{O}\times\mathbf{R}^2)$ if $\|h\|_{L^\infty(\mathcal{O})}<m'$.

19 – Proposition. The solution $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ *to equation* (2.7) *in the space* (3.1) *satisfies*

$$\begin{aligned}\boldsymbol{v}&=\sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-\cap\mathbb{B}^2}u_{\tau^{\boldsymbol{p}}}\tau^{\boldsymbol{p}}+v_{\zeta_2}\zeta_2,\qquad \boldsymbol{w}=\sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-\cap\mathbb{B}^3}u_{\tau^{\boldsymbol{p}}}\tau^{\boldsymbol{p}}+w_{\zeta_2}\zeta_2\\ \boldsymbol{u}&=\sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-}u_{\tau^{\boldsymbol{p}}}\mathcal{I}(\tau^{\boldsymbol{p}})+u_{\mathbf{0}}X^{\mathbf{0}}+u_{(0,1)}X^{(0,1)},\\ u_{\tau^{\boldsymbol{p}}}&=\frac{1}{S(\tau^{\boldsymbol{p}})}\,\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u_{\mathbf{0}},u_{(0,1)},v,\partial_xv\big).\end{aligned}\tag{4.4}$$

Proof – Given the definitions of the nonlinearities of $\boldsymbol{u}$ and $\mathsf{P}_{<2}(v)$ identity (4.4) is a direct encoding of the fixed point relation

$$\begin{cases} \boldsymbol{u} \in \mathsf{Q}_{<2}\big\{\mathcal{I}(\boldsymbol{v}+\boldsymbol{w}) + T_X\big\}, \\ \boldsymbol{v} \in \mathsf{Q}_{\leq 0}\Big\{F(\boldsymbol{u})\zeta_1 + \big\{G(\boldsymbol{u})(\boldsymbol{D}\boldsymbol{u})^2 + c\boldsymbol{u}\big\}\zeta_2\Big\}, \\ \boldsymbol{w} \in \mathsf{Q}_{\leq 0}\Big\{\big\{A(\boldsymbol{u}) - A(\mathsf{P}_{<2}(v))\big\}(\mathcal{I}_{(0,2)}(\boldsymbol{v}+\boldsymbol{w}) + T_X)\zeta_3\Big\} \end{cases}$$

satisfied by $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$, where $T_X := \mathrm{span}\{X^{\mathbf{k}}\}$. ▷

The analogue of identity (4.4) in the usual regularity structure setting was named '*coherence*' in [9].

4.2.2 – Star product. Following [1, Section 2] we introduce some bilinear operators on $\overline{\mathsf{T}}$. Let $\uparrow_v^{\mathbf{n}} \overline{\tau}$ be the derivation of $\overline{\tau}$ given by adding to $\overline{\tau}$ the polynomial decoration $X^{\mathbf{n}}$ at the vertex v. For $\overline{\sigma} \in \overline{\mathbb{T}}_\circ$, $\overline{\tau} \in \overline{\mathbb{T}}$, and $\mathbf{n} \in \mathbf{N}^2$, set

$$\overline{\sigma} \curvearrowright_{\mathbf{n}} \overline{\tau} := \sum_{v \in N_\tau} \sum_{\mathbf{m} \in \mathbf{N}^2} \binom{\mathfrak{n}_v}{\mathbf{m}} \overline{\sigma} \curvearrowright_{\mathbf{n}-\mathbf{m}}^{v} (\uparrow_v^{-\mathbf{m}} \overline{\tau}),$$

where $\mathfrak{n}_v$ is the polynomial decoration at the node v, and $\curvearrowright_{\mathbf{n}-\mathbf{m}}^{v}$ grafts $\overline{\sigma}$ onto $\overline{\tau}$ at the node v with an edge of type $\mathcal{I}_{\mathbf{n}-\mathbf{m}}$. One has the following analogue of the Chapoton-Livernet universality result.

20 – *Proposition. The space* $\overline{\mathsf{T}}_\circ$ *is freely generated by the symbols* $\big(X^{\mathbf{k}}\zeta_l\big)_{\mathbf{k}\in\mathbf{N}^2, 1\leq l\leq 3}$ *and the family of operations* $(\curvearrowright_{\mathbf{n}})_{\mathbf{n}\in\mathbf{N}^2}$*.*

We define the $\star$ product by following [1, Section 2]. First define for $\overline{\tau} \in \overline{\mathbb{T}}$ and $B \subset N_{\overline{\tau}}$, the derivation map $\uparrow_B^{\mathbf{k}}$ by

$$\uparrow_B^{\mathbf{k}} \overline{\tau} = \sum_{\sum_{v\in B} \mathbf{k}_v = \mathbf{k}} \prod_{v \in B} \uparrow_v^{\mathbf{k}_v} \overline{\tau}.$$

Also we define

$$\mathcal{I}_{\mathbf{n}}(\overline{\sigma}) \curvearrowright \overline{\tau} := \overline{\sigma} \curvearrowright_{\mathbf{n}} \overline{\tau},$$

and

$$\Big(\prod_i \mathcal{I}_{\mathbf{n}_i}(\overline{\sigma}_i)\Big) \curvearrowright \overline{\tau}$$

by grafting each tree $\overline{\sigma}_i$ on $\overline{\tau}$ along the grafting operator corresponding to $\mathbf{n}_i$, independently of the others. Set finally for all $\overline{\sigma} = X^{\mathbf{k}} \prod \mathcal{I}_{\mathbf{n}_i}(\overline{\sigma}_i) \in \overline{\mathbb{T}}_\bullet$ and $\tau \in \overline{\mathbb{T}}$

$$\overline{\sigma} \star \overline{\tau} := \uparrow_{N_\tau}^{\mathbf{k}} \left(\prod_i \mathcal{I}_{\mathbf{n}_i}(\overline{\sigma}_i) \curvearrowright \overline{\tau} \right).$$

One has for instance

$$X^{\mathbf{k}} \zeta_l \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\overline{\sigma}_i) = \left(X^{\mathbf{k}} \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\overline{\sigma}_i) \right) \star \zeta_l. \tag{4.5}$$

One proves as in Section 3.3 of [11] that the $\star$ product is associative in the sense that

$$\overline{\tau} \star (\overline{\sigma} \star \overline{\eta}) = (\overline{\tau} \star \overline{\sigma}) \star \overline{\eta}$$

for any $\overline{\tau}, \overline{\sigma} \in \overline{\mathbb{T}}_\bullet$ and $\overline{\eta} \in \overline{\mathbb{T}}$. We also define the $\star$ operation on $\overline{\mathsf{T}}_\bullet^* \times \overline{\mathsf{T}}^*$ setting

$$\overline{\sigma}^* \star \overline{\tau}^* := (\overline{\sigma} \star \overline{\tau})^*.$$

The following morphism property of $\mathfrak{F}^a$ with respect to the $\star$ product plays the crucial role in our argument, instead of pre-Lie morphisms applied in the original approach of Bruned, Chandra, Chevyrev & Hairer [9]. The morphism property is proved similarly to the proof of Proposition *2* in [1] based on identity (4.1).

21 – Proposition. One has

$$\mathfrak{F}^a\bigg(\Big\{X^{\mathbf{k}}\prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\overline{\sigma}_i)\Big\}^* \star \overline{\tau}^*\bigg)(\mathsf{c},\mathsf{c}') = \bigg(\Big\{\partial^{\mathbf{k}} D_{\mathbf{n}_1}...D_{\mathbf{n}_n}\mathfrak{F}^a(\overline{\tau}^*)\Big\}\prod_{i=1}^n \mathfrak{F}^a(\overline{\sigma}_i^*)\bigg)(\mathsf{c},\mathsf{c}'). \tag{4.6}$$

Note that the expansion formula (4.4) in our case only involves the $\mathbf{k}\in\{\mathbf{0},(0,1)\}$ case of the general formula (4.6). We see on (4.5) that formula (4.6) is a generalization of the defining identity (4.3). The interest of formula (4.6) will appear below in Proposition *22* when we will look for a recursive formula for some quantities of the form $\mathfrak{F}^a\big(R(z)^*(\tau^{\boldsymbol{p}})^*\big)$, for a (spacetime dependent) linear map R^* on $\overline{\mathsf{T}}^*$.

4.3 – Strong preparation maps and their associated models. The objects introduced in this section are the building blocks of an inductive construction of a renormalization process.

4.3.1 – Preparation maps. For $\tau^{\boldsymbol{p}}\in\mathbb{T}$ denote by $|\tau|_{\zeta_1}$ the number of noise symbols ζ_1 that appear in τ. Recall from Bruned's work [8] that a ***preparation map*** is a linear map $R:\mathsf{T}\to\mathsf{T}$ such that for each basis vector $\tau^{\boldsymbol{p}}\in\mathbb{T}$ one has

$$\begin{aligned} &R(\zeta_l)=\zeta_l, \qquad R(X^{\mathbf{k}}\tau^{\boldsymbol{p}}) = X^{\mathbf{k}}R(\tau^{\boldsymbol{p}}) \text{ for all } \mathbf{k}\in\mathbf{N}^2,\\ &R(\mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}})) = \mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}}) \text{ for all } \mathbf{n}\in\mathbf{N}^2 \text{ and } q\in\mathbf{N}, \end{aligned} \tag{4.7}$$

and there exist *finitely many* $\tau_i^{\boldsymbol{p}_i}\in T$ and constants λ_i such that

$$R\tau^{\boldsymbol{p}} = \tau^{\boldsymbol{p}} + \sum_i \lambda_i\tau_i^{\boldsymbol{p}_i}, \quad \text{with} \quad |\tau_i^{\boldsymbol{p}_i}|\geq|\tau^{\boldsymbol{p}}| \quad \text{and} \quad |\tau_i^{\boldsymbol{p}_i}|_{\zeta_1} < |\tau^{\boldsymbol{p}}|_{\zeta_1},$$

and R is closed in $\mathbb{B}$ and satisfies the 'commutation' relation

$$(R\otimes\mathrm{Id})\,\Delta = \Delta R. \tag{4.8}$$

The role of R is to provide a definition of the product of two trees that have already been renormalized. Its use in Section 4.3.2 in the recursive definition of the actual analytical objects associated with decorated trees will make that point clear; see in particular (4.13). Accordingly the second and third identities of (4.7) account for the fact that there is no need, in the induction process that builds an admissible model, to 'renormalize' elements of the form $X^{\mathbf{k}}\tau^{\boldsymbol{p}}$ and $\mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}})$ if the element $\tau^{\boldsymbol{p}}$ has already been renormalized. We can think of a preparation map as generating a renormalization process in the same way as a vector fields generates a flow.

Denote by R^* the algebraic dual of the map R; it is defined by the identity

$$\big\langle R\sigma^{\boldsymbol{q}}, (\tau^{\boldsymbol{p}})^*\big\rangle = \big\langle \sigma^{\boldsymbol{q}}, R^*(\tau^{\boldsymbol{p}})^*\big\rangle.$$

It is elementary to see that identity (4.8) is equivalent to having the right derivation identity

$$R^*\left((\sigma^{\boldsymbol{q}})^*\star(\tau^{\boldsymbol{p}})^*\right) = (\sigma^{\boldsymbol{q}})^*\star\left(R^*(\tau^{\boldsymbol{p}})^*\right) \tag{4.9}$$

for all $\sigma^{\boldsymbol{q}}\in\mathbb{B}^+$ ($\mathbb{B}^+$ is regarded as a subset of $\mathbb{T}_\bullet$) and $\tau^{\boldsymbol{p}}\in\mathbb{B}$ – see e.g. Proposition *3* in [1]. A ***strong preparation map*** is defined by a preparation map satisfying identity (4.9) *for all* $\sigma^{\boldsymbol{q}}\in\mathbb{T}_\bullet$ *and* $\tau^{\boldsymbol{p}}\in\mathbb{T}$ – and not only for $\sigma^{\boldsymbol{p}}\in\mathbb{B}^+$ and $\tau^{\boldsymbol{p}}\in\mathbb{B}$.

Definition – (a) *A **spacetime dependent strong preparation map on** $\mathsf{T}^{(m)}$ is a continuous map*

$$R:\big(\mathbf{R}_+\times\mathbf{T}\big)\times\mathsf{T}^{(m)}\to\mathsf{T}^{(m)}$$

satisfying the following properties for any fixed $z\in\mathbf{R}_+\times\mathbf{T}$.

- *The map* $R(z,\cdot):\mathsf{T}^{(m)}\to\mathsf{T}^{(m)}$ *is linear, closed in* $T^{(m)}$*, and satisfies* (4.7).
- *For any* $\tau^{\mathbf{0}}\in\mathbb{T}$ *there exist finitely many* $\sigma_1^{\mathbf{0}},\dots,\sigma_n^{\mathbf{0}}\in\mathbb{T}$ *such that* $|\sigma_i|>|\tau|$, $|\sigma_i|_{\zeta_1}<|\tau|_{\zeta_1}$*, and*
$$\big(R(z,\cdot)-\mathrm{Id}\big)\mathsf{T}^{(m)}_\tau\subset\bigoplus_{i=1}^n\mathsf{T}^{(m)}_{\sigma_i}.$$
- *The map* $R(z,\cdot)^*$ *satisfies* (4.9) *for any* $\sigma^{\boldsymbol{q}}\in\mathbb{T}_\bullet$ *and* $\tau^{\boldsymbol{p}}\in\mathbb{T}$.

(b) *A* ***spacetime dependent renormalization character on*** $\mathbb{B}_\circ^-$***, of growth factor*** $m'>0$***,*** *is a map*
$$\ell : \big(\mathbf{R}_+\times\mathbf{T}\big)\times\mathbb{B}_\circ^-\to\mathbf{R}$$
which is continuous in $\mathbf{R}_+\times\mathbf{T}$ *and vanishes on the elements of the form*
$$X^{\mathbf{k}}\tau^{\boldsymbol{p}}\quad(\mathbf{k}\neq 0),\qquad \mathcal{I}^q_{\mathbf{n}}(\tau^{\boldsymbol{p}}),$$
and such that for any $\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}$ *there exists a constant* $C(\tau)$ *such that*
$$\big|\ell(z,\tau^{\boldsymbol{p}})\big|\leq C(\tau)(m')^{|\boldsymbol{p}|}\tag{4.10}$$
for any $\boldsymbol{p}\in\mathbf{N}^{E_\tau}$ *and* $z\in\mathbf{R}_+\times\mathbf{T}$*.*

One associates to a spacetime dependent character $\ell(z,\cdot)$ of growth factor m' the linear map
$$R_\ell(z)^*\big((\tau^{\boldsymbol{p}})^*\big):=(\tau^{\boldsymbol{p}})^*+\sum_{\sigma^{\boldsymbol{q}}\in\mathbb{B}_\circ^-}\frac{\ell(z,\sigma^{\boldsymbol{q}})}{S(\sigma^{\boldsymbol{q}})}\,(\tau^{\boldsymbol{p}})^*\star(\sigma^{\boldsymbol{q}})^*,\qquad(\tau^{\boldsymbol{p}}\in\mathbb{T}_\bullet)\tag{4.11}$$
and
$$R_\ell(z)^*((\tau^{\boldsymbol{p}})^*)=(\tau^{\boldsymbol{p}})^*,\qquad(\tau^{\boldsymbol{p}}\in\mathbb{T}_\circ).$$
It can be easily checked that R_ℓ is a strong preparation map on $T^{(m)}$ with $m>m'$. The definition above corresponds to the usual definition of its dual described by the contraction of trees as in Corollary *4.5* of [8]. So, $R_\ell(z)$ is closed in $\mathbb{B}$. For the commutation relation (4.9) we use the associativity of the $\star$ product as in Proposition *4* of [1]. It remains to show that R_ℓ is bounded in $\mathsf{T}^{(m)}$. Actually since
$$\left\|\big(R_\ell(z)^*-\mathrm{Id}\big)\Big(\sum_{\boldsymbol{p}}c_{\boldsymbol{p}}(\tau^{\boldsymbol{p}})^*\Big)\right\|_{1/m}\leq\sum_{\boldsymbol{p},\boldsymbol{q},\sigma}\frac{|\ell(z,\sigma^{\boldsymbol{q}})|}{S(\sigma^{\boldsymbol{q}})}\,|c_{\boldsymbol{p}}|\,\big\|\tau^{\boldsymbol{p}}\star\sigma^{\boldsymbol{q}}\big\|_{1/m}$$
$$\lesssim\sum_{\boldsymbol{p},\boldsymbol{q},\sigma}\frac{|\ell(z,\sigma^{\boldsymbol{q}})|}{S(\sigma^{\boldsymbol{q}})}\,|c_{\boldsymbol{p}}|\,|\boldsymbol{q}|\,m^{-|\boldsymbol{p}|-|\boldsymbol{q}|}\leq\bigg(\sum_{\boldsymbol{q},\sigma}\frac{|\ell(z,\sigma^{\boldsymbol{q}})|^2}{S(\sigma^{\boldsymbol{q}})^2}\,|\boldsymbol{q}|^2\,m^{-2|\boldsymbol{q}|}\bigg)^{1/2}\bigg\|\sum_{\boldsymbol{p}}c_{\boldsymbol{p}}\tau^{\boldsymbol{p}}\bigg\|_{1/m},$$
the map $R_\ell(z)^*:\mathsf{T}^{*,(1/m)}\to\mathsf{T}^{*,(1/m)}$ is continuous because of (4.10). So, $R_\ell(z)$ sends continuously $\mathsf{T}^{(m)}$ into itself. The next proposition follows from Proposition *21* and identity (4.11).

22 – Proposition. Let $\mathcal{O}$ *be a domain in* $\mathbf{R}^2$ *and let* $\|h\|_{L^\infty(\mathcal{O})}<1/m$*. Let* R *be a spacetime dependent strong preparation map on* $\mathsf{T}^{(m)}$*. For every* $z\in\mathbf{R}_+\times\mathbf{T}$ *and* $(\mathsf{c}_0,\mathsf{c}_0',\mathsf{c}_{(0,1)},\mathsf{c}_{(0,1)}')\in\mathcal{O}\times\mathbf{R}^2$ *one has*
$$\mathfrak{F}^a\bigg(R(z)^*\Big(X^{\mathbf{k}}\zeta_l\prod_{i=1}^n\mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)^*\bigg)(\mathsf{c},\mathsf{c}')$$
$$=\bigg(\Big\{\partial^{\mathbf{k}}D_{\mathbf{n}_1}...D_{\mathbf{n}_n}\mathfrak{F}^a\big(R(z)^*\zeta_l^*\big)\Big\}\prod_{i=1}^n\mathfrak{F}^a\big((\tau_i^{\boldsymbol{p}_i})^*\big)\bigg)(\mathsf{c},\mathsf{c}').$$

Proof – By writing
$$X^{\mathbf{k}}\zeta_l\prod_{i=1}^n\mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})=\Big(X^{\mathbf{k}}\prod_{i=1}^n\mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)\star\zeta_l$$
and using the right derivation property (4.9) – here we use the fact that the preparation map is 'strong' – one gets
$$R^*(z)\left(\Big(X^{\mathbf{k}}\prod_{i=1}^n\mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)\star\zeta_l\right)^*=\Big(X^{\mathbf{k}}\prod_{i=1}^n\mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)^*\star\big(R(z)^*\zeta_l^*\big).\tag{4.12}$$
Identity (4.6) in Proposition *21* then yields the identity of the statement. ▷

4.3.2 – Admissible model associated with a preparation map. Fix a regularization parameter ε and denote by

$$\xi^\varepsilon =: \xi_1 \in C^\infty(\mathbf{R}\times\mathbf{T})$$

a regularized version of the spacetime white noise ξ and set

$$\xi_2 = \xi_3 = 1.$$

For any spacetime dependent strong preparation map R on $\mathsf{T}^{(m)}$ we define inductively the maps $\mathsf{\Pi}^{R,a(v)}$ and $\Pi^{R,a(v),\times}$ as follows. For $1\le l\le 3$, set

$$\mathsf{\Pi}^{R,a(v)}\zeta_l = \Pi^{R,a(v),\times}\zeta_l := \xi_l,$$

and define

$$\begin{aligned}\mathsf{\Pi}^{R,a(v)} &= \Pi^{R,a(v),\times}\circ R, \quad \Pi^{R,a(v),\times}(\tau_1\tau_2) = \big(\Pi^{R,a(v),\times}\tau_1\big)\big(\Pi^{R,a(v),\times}\tau_2\big),\\ \Pi^{R,a(v),\times}(\mathcal{I}^p_{\mathbf{n}}\tau) &= \partial_z^{\mathbf{n}}\big(K^{a(v)}\circ\big(\partial_x^2K^{a(v)}\big)^{\circ p}\big)\big(\mathsf{\Pi}^{R,a(v)}\tau\big),\end{aligned} \tag{4.13}$$

where the symbol $\circ$ stands for the composition operator and the notation $\mathcal{A}^{\circ p}$ stands for the p-fold iteration of an operator $\mathcal{A}$. The operator $\partial_x^2K^{a(v)}$ makes sense here, because $\mathsf{\Pi}^{R,a(v)}\tau$ constructed as above belongs to $\mathcal{C}^{0+}_{\mathfrak{s}}$ (Note that $K^{a(v)}$ maps $\mathcal{C}^{0+}_{\mathfrak{s}}$ into $\mathcal{C}^{2+}_{\mathfrak{s}}$ – see Theorem *6*). As R is spacetime dependent the first identity in (4.13) reads

$$\big(\mathsf{\Pi}^{R,a(v)}\tau\big)(z) = \Pi^{R,a(v),\times}\big(R(z)\tau\big)(z),$$

for all z and all τ. It follows from this definition and the fact that we work with preparation maps R leaving fixed the elements of T of the form $\mathcal{I}^p_{\mathbf{n}}(\tau)$ that the map $\mathsf{\Pi}^{R,a(v)}$ satisfies the admissibility condition

$$\mathsf{\Pi}^{R,a(v)}(\mathcal{I}^p_{\mathbf{n}}\tau) = \partial_z^{\mathbf{n}}\Big(K^{a(v)}\circ(\partial_x^2K^{a(v)})^{\circ p}\Big)(\mathsf{\Pi}^{R,a(v)}\tau).$$

Define as well $\mathsf{g}^{R,a(v)}$ inductively from the identity

$$(\mathsf{g}_z^{R,a(v)})^{-1}\big(\mathcal{I}^{+,p}_{\mathbf{n}}\tau\big) = -\sum_{|\mathbf{k}|_{\mathfrak{s}}<|\tau|+2-|\mathbf{n}|_{\mathfrak{s}}}\frac{(-z)^{\mathbf{k}}}{\mathbf{k}!}\Big(\partial_z^{\mathbf{n}+\mathbf{k}}\big(K^{a(v)}\circ(\partial_x^2K^{a(v)})^{\circ p}\big)(\mathsf{\Pi}_z^{R,a(v)}\tau)\Big)(z).$$

One can follow verbatim Section 7.1 of [2] and see that $\big(\mathsf{\Pi}^{R,a(v)},\mathsf{g}^{R,a(v)}\big)$ *is a smooth admissible model* on $\mathscr{T}^{(m)}$ with a constant m coming from the operator norm of $\partial_x^2K^{a(v)}$.

Among the renormalization characters, we are interested in the one $\ell^\varepsilon_{a(v)}(z,\tau^{\boldsymbol{p}})$ defined by the similar way to Section 6.3 of [10]. We denote by $R^\varepsilon_{a(v)}$ the strong preparation map defined by (4.11) with ℓ replaced by $\ell^\varepsilon_{a(v)}$. The associated model $M^\varepsilon_{a(v)}$ is called the BPHZ model. Note that, when $\ell^\varepsilon_{a(v)}$ has a growth factor $m'<m$, the BPHZ model $\mathsf{M}^\varepsilon_{a(v)}$ is a model on $\mathscr{T}^{(m)}$.

1 – ***Assumption.*** *There exists a character $\ell^\varepsilon_{a(v)}$ of growth factor $m'\in(0,m)$ for each $\varepsilon\in(0,1]$ (the constant $C(\tau)$ in* (4.10) *may be ε-dependent) and the BPHZ renormalized model $\mathsf{M}^\varepsilon_{a(v)}$ is convergent as $\varepsilon>0$ goes to 0.*

We conjecture that ***Assumption 1*** holds true in the full subcritical regime, but we do not discuss it in this paper. Such a convergence result was proved in several works (cf. [12, 24, 5]) in semilinear settings, but we need slight modifications. For instance, we cannot directly use [12] because the kernel $\partial_x^2K^{a(v)}$ is too singular to be integrable around the origin. We would be able to solve this difficulty by considering $K^{a(v)}\circ(\partial_x^2K^{a(v)})^{\circ p}$ as one integrable kernel – see Proposition *27* below. The inductive proofs in [24, 5] are also not directly applied because the integral operator $K^{a(v)}$ is not homogeneous. However, [5, Lemma 9] implies that the convergence of the model is reduced to the ε-uniform boundedness of the expectations

$$\big|\mathbb{E}\big[\mathcal{Q}_\theta((\mathsf{\Pi}^\varepsilon)^{\mathsf{g}}_z\tau^{\boldsymbol{p}})(z)\big]\big| \lesssim m^{|\boldsymbol{p}|}\theta^{(|\tau|-|\mathbf{k}|_{\mathfrak{s}})/4}$$

for any τ with $|\tau| \leq 0$ and their convergences as $\varepsilon \to 0$. This fact would reduce the effort for the proof significantly, because this expectation vanishes for any τ with an odd number of ζ_1 symbols by the property of Gaussian noise.

4.4 – Renormalized equation. Denote by $\mathsf{R}^\varepsilon_{a(v)}$ the reconstruction map associated with $\mathsf{M}^\varepsilon_{a(v)}$. The proof of Theorem *10* in [2] works verbatim and gives in our setting the following result.

23 – *Proposition. Let* $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{w}^\varepsilon) \in \mathcal{D}^{\gamma;\alpha}_m(0,t_0;U) \times \mathcal{D}^{\gamma+\alpha_0-2,2\alpha-2}_m(0,t_0;T_\circ) \times \mathcal{D}^{\gamma+\alpha_0-2,\alpha-2}_m(0,t_0)$ *stand for the modelled distribution solution of* (2.7) *with respect to the model* $\mathsf{M}^\varepsilon_{a(v)}$. *Then one can choose* $t_0' < t_0$ *and* $\varepsilon_0 > 0$ *both small enough for*

$$u^\varepsilon := \mathsf{R}^\varepsilon_{a(v)}(\boldsymbol{u}^\varepsilon)$$

to satisfy the bound

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \sup_{t\in(0,t_0')} \big\| a(u^\varepsilon) - a(v) \big\|_{L^\infty(\mathbf{T})} < \frac{1}{m}$$

and solve the 'renormalized' equation

$$\big(\partial_t - a(u^\varepsilon)\partial_x^2\big)u^\varepsilon = f(u^\varepsilon)\xi^\varepsilon + g(u^\varepsilon)(\partial_x u^\varepsilon)^2 + \sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-} \frac{\ell^\varepsilon_{a(v)}(\cdot,\tau^{\boldsymbol{p}})}{S(\tau^{\boldsymbol{p}})} \mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon, \partial_x u^\varepsilon, v\big) \tag{4.14}$$

on $(0,t_0') \times \mathbf{T}$, *with initial condition* u_0. *The last term of* (4.14) *has a growth that is at most linear with respect to* $\partial_x u^\varepsilon$.

Proof – Denote by $R^{\varepsilon,*}_{a(v)}$ the dual of $R^\varepsilon_{a(v)}$. Theorem *9* of [1] yields that u^ε solves the equation

$$\begin{aligned}\big(\partial_t - L^{a(v)}\big)u^\varepsilon = f(u)\xi^\varepsilon + g(u)(\partial_x u^\varepsilon)^2 + \big(a(u^\varepsilon)-a(v)\big)\partial_x^2 u^\varepsilon \\ + \sum_{l=1}^4 \mathfrak{F}^a\Big((R^{\varepsilon,*}_{a(v)} - \mathrm{Id})\zeta_l^*\Big)\big(u^\varepsilon, \partial_x u^\varepsilon, v, \partial_x v\big).\end{aligned}$$

Since $R^{\varepsilon,*}_{a(v)}\zeta_l^* \in \mathsf{T}^{*,(1/m)}$ by duality the term $\mathfrak{F}^a\Big((R^{\varepsilon,*}_{a(v)} - \mathrm{Id})\zeta_l^*\Big)$ is actually convergent in $C_b((0,t_0') \times \mathbf{T})$ by the remark before Proposition *19*. We have the right hand side of (4.14) from the definition of $R^{\varepsilon,*}_{a(v)}$. To see that this counterterm does not depend on $\partial_x v$ note that any renormalization character satisfies

$$\ell\big(X^{(0,1)}\sigma^{\boldsymbol{p}}\big) = 0 \tag{4.15}$$

by definition. Since the only functions $\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)(\mathsf{c},\mathsf{c}')$ that depend on $\mathsf{c}'_{(0,1)}$ correspond to $\tau^{\boldsymbol{p}}$ of the form $X^{(0,1)}\sigma^{\boldsymbol{p}}$, the corresponding counterterms are null.
It remains to check the last statement of the proposition. If some function $\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)(\mathsf{c},\mathsf{c}')$ were of degree greater than 1 with respect to $\mathsf{c}_{(0,1)}$ then $\tau^{\boldsymbol{p}}$ would have at least two ζ_2-type nodes from where exactly one edge $\mathcal{I}_{(0,1)}$ would leave. Since the minimal homogeneity among the trees

$$X^{\mathbf{k}}\zeta_2\mathcal{I}_{(0,1)}(\sigma)\prod_{i=1}^n \mathcal{I}(\sigma_i)$$

is $\big|\zeta_2\mathcal{I}(\zeta_1)\mathcal{I}_{(0,1)}(\zeta_1)\big| = 2\alpha - 1 > -1$, such $\tau^{\boldsymbol{p}}$ cannot have negative homogeneity. ▷

Next we reduce the v-dependence of the counterterm of (4.14). Define inductively the function $\chi^a(\tau^{\boldsymbol{p}})(\mathsf{c}_0)$ by the relations

$$\begin{aligned}&\chi^a\big(\zeta_3\,\mathcal{I}_{(0,2)}(\tau^{\boldsymbol{p}})\big) = \chi^a(\tau^{\boldsymbol{p}}), \\ &\chi^a\Big(\zeta_3\,\mathcal{I}_{(0,2)}(\tau^{\boldsymbol{p}})\prod_{i=1}^n \mathcal{I}(\tau_i^{\boldsymbol{p}_i})\Big)(\mathsf{c}_0) = a^{(n)}(\mathsf{c}_0)\prod_{i=1}^n \chi^a\big(\tau_i^{\boldsymbol{p}_i}\big)(\mathsf{c}_0), \qquad \text{for } n \geq 1,\end{aligned} \tag{4.16}$$

and for $l \in \{1,2\}$

$$\chi^a\Big(\zeta_l \prod_{i=1}^n \mathcal{I}_{\mathbf{n}_i}(\tau_i^{\boldsymbol{p}_i})\Big)(\mathsf{c}_{\mathbf{0}}) = \prod_{i=1}^n \chi^a\big(\tau_i^{\boldsymbol{p}_i}\big)(\mathsf{c}_{\mathbf{0}}). \tag{4.17}$$

We see on this definition that χ^a is a polynomial function of a and its derivatives. It is important to note that $\chi^a(\tau^{\boldsymbol{p}})$ does not depend on the $\boldsymbol{p}$-decoration of τ – rather it depends on the location of the ζ_3 vertices within $\tau^{\boldsymbol{p}}$. We denote by τ the non-$\boldsymbol{p}$-decorated tree associated with $\tau^{\boldsymbol{p}}$, so the symbols τ and $\tau^{\mathbf{0}}$ are used here interchangeably. As a shorthand notation we write

$$\chi^a_\tau := \chi^a(\tau).$$

With

$$\tau_1 := \mathcal{I}(\zeta_1)\mathcal{I}_{(0,1)}(\zeta_1)^2\zeta_2, \quad \tau_2 := \mathcal{I}(\zeta_1)^2\mathcal{I}_{(0,2)}(\zeta_1)\zeta_3$$

one has for instance

$$\chi^a(\tau_1)(\mathsf{c}_{\mathbf{0}}) = 1, \quad \chi^a(\tau_2)(\mathsf{c}_{\mathbf{0}}) = a^{(2)}(\mathsf{c}_{\mathbf{0}})\chi^a(\zeta_1)^2(\mathsf{c}_{\mathbf{0}}) = a^{(2)}(\mathsf{c}_{\mathbf{0}})$$

We also define the functions $\mathfrak{F}(\tau^*)$ for $\tau \in \mathbb{B}^0$ by the same inductive relations as the functions $\mathfrak{F}^a(\tau^*)$ by replacing $\mathsf{c}_{\mathbf{0}}$-derivatives of $h(\mathsf{c}_{\mathbf{0}}, \mathsf{c}'_{\mathbf{0}})$ of any order by the constant function equal to 1. Then the functions $\mathfrak{F}(\tau^*)$ depend only on $\mathsf{c}_{\mathbf{0}}$. It is elementary to obtain the following identity by induction.

24 – Lemma. One has

$$\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(\mathsf{c}_{\mathbf{0}}, \mathsf{c}_{(0,1)}, \mathsf{c}'_{\mathbf{0}}\big) = \chi^a_\tau(\mathsf{c}_{\mathbf{0}})\,\big(a(\mathsf{c}_{\mathbf{0}}) - a(\mathsf{c}'_{\mathbf{0}})\big)^{|\boldsymbol{p}|}\,\mathfrak{F}(\tau^*)(\mathsf{c}_{\mathbf{0}}). \tag{4.18}$$

For a positive parameter λ we denote by

$$Z^\lambda_t(x) = Z^\lambda(t,x) = \mathbf{1}_{t>0}\frac{e^{-ct}}{\sqrt{4\pi\lambda t}}\exp\bigg(-\frac{|x|^2}{4\lambda t}\bigg)$$

the fundamental solution built from the constant coefficient parabolic operator $\partial_t - \lambda\partial_x^2 - c$. The naive admissible model on $\mathscr{T}$ associated with Z^λ and the smooth noise ξ^ε is the unique multiplicative model such that

$$\mathsf{\Pi}^\varepsilon_\lambda\zeta_1 = \xi^\varepsilon, \qquad \mathsf{\Pi}^\varepsilon_\lambda\zeta_l = 1 \quad (l \in \{2,3\}), \qquad \big(\mathsf{\Pi}^\varepsilon_\lambda X^{\mathbf{k}}\big)(z) = z^{\mathbf{k}},$$

and

$$\mathsf{\Pi}^\varepsilon_\lambda\big(\mathcal{I}^p_{\mathbf{n}}(\tau^{\boldsymbol{p}})\big) = \Big(\partial_z^{\mathbf{n}}Z^\lambda * (\partial_x^2 Z^\lambda)^{*p}\Big) * \mathsf{\Pi}^\varepsilon_\lambda\tau^{\boldsymbol{p}}.$$

The *BHZ character* $l^\varepsilon_\lambda(\cdot)$ on $\mathbb{B}^-_\circ$ is defined in that setting as

$$l^\varepsilon_\lambda(\tau^{\boldsymbol{p}}) := h^\varepsilon_\lambda(S'_-\tau^{\boldsymbol{p}}), \qquad h^\varepsilon_\lambda(\tau^{\boldsymbol{p}}) := \mathbb{E}\big[\mathsf{\Pi}^\varepsilon_\lambda\tau^{\boldsymbol{p}}(0)\big],$$

where $S'_- : T^- \to \mathbf{R}[T]$ is the natural extension to our setting of the negative twisted antipode – see Proposition *6.6* in [10] or Section 7 of [4] for its definition in the usual BHZ setting.

2 – Assumption. *For any $\tau^{\mathbf{0}} \in \mathbb{B}^{-0}_\circ$ there exist a constant $m > 0$ and an ε-independent constant $C(\tau)$ such that*

$$\Big|\ell^\varepsilon_{a(v)}(z,\tau^{\boldsymbol{p}}) - l^\varepsilon_{a(v(z))}(\tau^{\boldsymbol{p}})\Big| \leq C(\tau)\,m^{|\boldsymbol{p}|}$$

for any $\boldsymbol{p} \in \mathbf{N}^{E_\tau}$ and $z \in \mathbf{R}_+ \times \mathbf{T}$.

We check that ***Assumption 2*** holds for some examples in the next section. The next statement is the core fact to get the renormalized equation under the form (1.6) stated in Theorem *2*, though it is elementary.

25 – Lemma. For any $\tau^{\mathbf{0}} \in \mathbb{B}^{-0}_\circ$ the function

$$\lambda \mapsto l^\varepsilon_\lambda(\tau^{\mathbf{0}})$$

is analytic in any given bounded interval of $\mathbf{R}$ *whose closure does not contain the point* 0 *and*

$$\frac{1}{n!}\partial_\lambda^n l_\lambda^\varepsilon(\tau^{\mathbf{0}}) = \sum_{\boldsymbol{p}\in\mathbf{N}^{E_\tau},\,|\boldsymbol{p}|=n} l_\lambda^\varepsilon(\tau^{\boldsymbol{p}}).$$

Proof – By an elementary computation we have

$$\partial_\lambda Z^\lambda(t,x) = t\partial_x^2 Z^1(\lambda t,x) = \int_0^t \int_{\mathbf{R}} Z^\lambda(t-s,x-y)\partial_x^2 Z^\lambda(s,y)\,dy\,ds. \tag{4.19}$$

Therefore once ∂_λ applies to one edge to which the kernel $\partial^{\mathbf{k}} Z^\lambda$ is associated then this kernel turns into a spacetime convolution $\partial^{\mathbf{k}} Z^\lambda * \partial_x^2 Z^\lambda$. ▷

Proof of Theorem 2 – It follows from Lemma *17*, Lemma *25* and (4.18) that the counterterm in the renormalized equation (4.14) equals to the following simple form up to an ε-uniform remainder term:

$$\begin{aligned}
&\sum_{\tau^{\boldsymbol{p}}\in\mathbb{B}_\circ^-} \frac{l^\varepsilon_{a(v(\cdot))}(\tau^{\boldsymbol{p}})}{S(\tau^{\boldsymbol{p}})}\,\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon,\partial_x u^\varepsilon,v\big) \\
&\quad= \sum_{\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}} \frac{1}{S(\tau^{\mathbf{0}})} \sum_{\boldsymbol{p}\in\mathbf{N}^{E_\tau}} l^\varepsilon_{a(v(\cdot))}(\tau^{\boldsymbol{p}})\,\mathfrak{F}^a\big((\tau^{\boldsymbol{p}})^*\big)\big(u^\varepsilon,\partial_x u^\varepsilon,v\big) \\
&\quad= \sum_{\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}} \frac{1}{S(\tau^{\mathbf{0}})}\,\chi_\tau^a(u^\varepsilon)\,\mathfrak{F}(\tau^*)\big(u^\varepsilon,\partial_x u^\varepsilon\big)\sum_{n=0}^\infty \big(a(u^\varepsilon)-a(v)\big)^n \sum_{|\boldsymbol{p}|=n} l^\varepsilon_{a(v(\cdot))}(\tau^{\boldsymbol{p}}) \\
&\quad= \sum_{\tau^{\mathbf{0}}\in\mathbb{B}_\circ^{-0}} \frac{1}{S(\tau^{\mathbf{0}})}\,\chi_\tau^a(u^\varepsilon)\,\mathfrak{F}(\tau^*)\big(u^\varepsilon,\partial_x u^\varepsilon\big)\,l^\varepsilon_{a(u^\varepsilon(\cdot))}(\tau^{\mathbf{0}}).
\end{aligned}$$

This completes the proof of Theorem *2*. ▷

We finish this section by showing that the a priori diverging term $l^\varepsilon_{a(u^\varepsilon(\cdot))}$ in the counterterm takes a particularly nice form under the condition that the noise is Gaussian and regularized only in the spatial variable by symmetric mollifiers. To avoid the situation where temporally regularization is necessary, we consider only spatial noise or spacetime noise that is white in time with $f = 1$. Recall that $|\tau|_{\zeta_1}$ denotes the number of ζ_1-type nodes that appear in τ.

26 – Proposition. Assume that ξ *is a stationary centered Gaussian noise and define*

$$\xi^\varepsilon(t,x) = \big(\xi(t,\cdot)*\rho_\varepsilon\big)(x)$$

with an even mollifier ρ_ε*, for the spatial convolution operator* $*$*. Then* $h_\lambda^\varepsilon(\tau^{\mathbf{0}}) = 0$ *if* $|\tau|_{\zeta_1}$ *is odd, otherwise*

$$h_\lambda^\varepsilon(\tau^{\mathbf{0}}) = \begin{cases} \lambda^{-\sharp N_\tau+1} h_1^\varepsilon(\tau^{\mathbf{0}}), & \textit{if } \xi(x) \textit{ depends on only space}, \\ \lambda^{|\tau|_{\zeta_1}/2-\sharp N_\tau+1} h_1^\varepsilon(\tau^{\mathbf{0}}), & \textit{if } \xi(t,x) \textit{ is white in time}. \end{cases}$$

Proof – The former part holds because ξ^ε is centered Gaussian. Let $|\tau^{\mathbf{0}}|_{\zeta_1} = 2a$ be an even number and let b be the number of ζ_2,ζ_3-type nodes. If the root of $\tau^{\mathbf{0}}$ is not a ζ_1-type node, then the expectation of $\big(\mathsf{\Pi}_\lambda^\varepsilon \tau^{\mathbf{0}}\big)(0)$ is given by an integral of the form

$$\int C_\varepsilon(z^1-z^2)\cdots C_\varepsilon(z^{2a-1}-z^{2a})(z^1)^{n_1}\cdots(z^{2a})^{n_{2a}} A^\lambda\big(z^1,\dots,z^{2a}\big)\,dz^1\cdots dz^{2a},$$

where

$$C_\varepsilon(z) := \mathbb{E}\big[\xi_\varepsilon(z)\xi_\varepsilon(0)\big]$$

and

$$A^\lambda(z^1,\dots,z^{2a}) = \int \widetilde{A}^\lambda\big(z^1,\dots,z^{2a},w^1,\dots,w^{b-1}\big)\,dw^1\cdots dw^{b-1}$$

with a product $\widetilde{A}$ of polynomials $(w^i)^{m_i}$ and kernels $\partial^{e_{ij}} Z^\lambda$. Because of the form of equation and the restriction $\alpha_0 \in (0,1)$, the $\mathfrak{n}$-decorations m_i and $\mathfrak{e}$-decorations e_{ij} are $\mathbf{0}$ or $(0,1)$. So, they are independent of the change of variables $z \mapsto z_\lambda := (\lambda t, x)$. Hence

$$A^\lambda\big(z^1,\dots,z^{2a}\big) = \lambda^{-b+1} A^1\big(z^1_\lambda,\dots,z^{2a}_\lambda\big)$$

by a scaling argument. If $\xi(x)$ is t-independent then we have

$$\big(\Pi^\varepsilon_\lambda \tau\big)(0) = \lambda^{-2a-b+1}\big(\Pi^\varepsilon_1 \tau\big)(0),$$

since $C_\varepsilon(x)$ does not depend on time. If $\xi(t,x)$ is white in time, since $C_\varepsilon(z) = \delta(t) C'_\varepsilon(x)$ for some function C'_ε, which reduces the number of time components $t^1,\dots,t^{2a}$ of $z^1,\dots,z^{2a}$ by a half and yields

$$\big(\Pi^\varepsilon_\lambda \tau\big)(0) = \lambda^{-a-b+1}\big(\Pi^\varepsilon_1 \tau\big)(0).$$

We can perform similar computations when $N_\circ$ contains the root. $\rhd$

In the setting of Lemma *26* the counterterm is of the form

$$\sum_{\tau\in\mathbb{B}_\circ^{-0}} \frac{1}{S(\tau)} \bigg(\sum_\sigma s(\tau,\sigma)\frac{h^\varepsilon_1(\sigma)}{\lambda^{\theta(\sigma)}}\bigg)\chi^a_\tau(u^\varepsilon)\mathfrak{F}(\tau^*)(u^\varepsilon,\partial_x u^\varepsilon), \tag{4.20}$$

where $S'_-\tau = \sum_{\sigma\in\mathbb{B}_\circ^{-0}} s(\tau,\sigma)\sigma$ and the exponent $\theta(\sigma)$ is given in the statement of Proposition *26*. This situation applies in the example of equation (1.2) with a linear additive forcing, and more generally in situations where $f = 1$, so that all the terms in the regularized and renormalized equation

$$\big(\partial_t - a(u^\varepsilon)\partial_x^2\big)u^\varepsilon = \xi^\varepsilon + g(u^\varepsilon)(\partial_x u^\varepsilon)^2 + \sum_{\tau\in\mathbb{B}_\circ^{-0}} \frac{l^\varepsilon_{a(u^\varepsilon)}(\tau)}{S(\tau)}\,\chi^a_\tau(u^\varepsilon)\,\mathfrak{F}(\tau^*)(u^\varepsilon,\partial_x u^\varepsilon)$$

make sense.

4.5 – Examples. We consider in this section some examples satisfying ***Assumption 2***. Recall from [23] that we can associate to each character $\ell^\varepsilon_{a(v)}(\tau^{\boldsymbol{p}})$ a directed graph called Feynman diagram whose edges are related with kernels $K^{a(v)}$ and estimate it by using the singularity of each kernel around the origin. To estimate the difference between $\ell^\varepsilon_{a(v)}(\bullet,\tau^{\boldsymbol{p}})$ and $l^\varepsilon_{a(v(\bullet))}(\tau^{\boldsymbol{p}})$, we show that the difference between $K^{a(v)}$ and $Z^{a(v(\bullet))}$ whose coefficient is frozen at the root is sufficiently regular. First recall from Theorem *6* that the integration operator $K^{a(v)}((t,x),(s,y))$ can be replaced with to the operator

$$\big(\partial_t - L^{a(v)} + c\big)^{-1}(\cdot) = \int_{-\infty}^t Q^{a(v)}_{t,s}(\cdot)ds$$

up to the cost of the good operator R sending $\mathcal{C}^{-2+}_{\mathfrak{s}}(a(v))$ into $\mathcal{C}^{2+}_{\mathfrak{s}}$. Furthermore by Proposition *27* below we can replace $Q^{a(v)}_{t,s}(x,y)$ with $Z^{a(v(\bullet))}_{t-s}(x-y)$ up to the cost of a less singular kernel.

The next two propositions play an important role in Section 4.5.1 and Section 4.5.2. We defer their proof to Appendix A.6. For simplicity we write

$$b = a(v)$$

in what follows. We use the notations from Appendix A.1 where we consider some properties of Gaussian-like kernels. This type of kernels appear in the construction of the fundamental solution Q^b of the operator $\partial_t - L^b + c$ or more general operators, as described in Section A.2. Recall from Appendix A.1 the definition of the class $\mathsf{G}^\beta(x)$ with $d = 1$, $\mathfrak{s} = 1$ and $N = 2$. Note also that Z^λ is analytic with respect to λ and

$$\partial_\lambda^n \partial_x^k Z^\lambda_t = t^n \partial_x^{2n+k} Z^\lambda_t \in \mathsf{G}^{-k}$$

by (4.19). In what follows, an element in G^β is denoted by the symbol $(\mathsf{G}^\beta)_{t,s}(x,y)$, if its explicit form is not important. The functions represented by such symbols can be different from line to line.

27 – Proposition. If a, hence b, is an element of $\mathcal{C}^\alpha_{\mathfrak{s}}$ with $\alpha \in (0,1]$ then for any $k \in \{0,1,2\}$ and $p \in \mathbf{N}$ one has

$$(\partial_x^k Q^b) * (\partial_x^2 Q^b)^{*p}_{t,s}(x,y) = (\partial_\lambda^p \partial_x^k Z^\lambda_{t-s})|_{\lambda=b(t,x)}(x-y) + (\mathsf{G}^{\alpha-k})_{t,s}(x,y). \tag{4.21}$$

We need a more detailed expansion in Section 4.5.3. By choosing a t-independent function $v \in C^2(\mathbf{T})$ – one choice is $v(x) = e^{\delta\partial_x^2}$ for sufficiently small $\delta > 0$ as given after Definition 5, we have the following estimate. Note that the fundamental solution Q^b_t is t-homogeneous in this case.

28 – Proposition. Suppose that $v \in C^2(\mathbf{T})$ is independent to t. Then for any $k \in \{0,1,2\}$ and $p \in \mathbf{N}$ one has

$$(\partial_x^k Q^b_t) * (\partial_x^2 Q^b)^{*p}_t(x,y) = (\partial_\lambda^p \partial^k Z^\lambda_t)(x-y)_{\lambda=b(x)} + b'(x) Y_t^{k,p,b(x)}(x-y) + (\mathsf{G}^{2-k})_{t,0}(x,y) \tag{4.22}$$

where $Y_t^{k,p,\lambda}(\cdot)$ is a function indexed by a constant $\lambda > 0$ belonging to G^{1-k} locally uniformly over λ. When k is even, respectively odd, the function $Y_t^{k,p,\lambda}(\cdot)$ is odd, respectively even.

Equipped with the previous estimates we can now look at three examples where ***Assumption 2*** is satisfied. For simplicity we consider only trees with vanishing $\boldsymbol{p}$-decoration.

4.5.1 – Two dimensional parabolic Anderson model. In the slightly singular setting of the quasilinear parabolic Anderson model equation

$$\partial_t u - a(u)\Delta u = f(u)\xi$$

on a two dimensional torus, with space white noise ξ. In this case one can choose $2/3 < \alpha_0 < 1$. The only elements $\tau \in \mathbb{B}_\circ^{-0}$ with an even number of ζ_1 noises are the trees

$\tau_1 = \zeta_1 \mathcal{I}(\zeta_1) =$  , $\qquad \tau_2 = \zeta_3 \mathcal{I}(\zeta_1)\mathcal{I}_{(0,2)}(\zeta_1) =$  .

Here the thick line denotes the operator $\mathcal{I}$ and the double line denotes $\mathcal{I}_{(0,2)}$. The noise symbol ζ_1 is denoted by a white circle, while ζ_3 is denoted by a circled dot. The corresponding characters are

$$\ell^\varepsilon_{a(v)}((t,x),\tau_1) = \int_{(-\infty,t)\times\mathbf{R}} Q^{a(v)}_{t,s}(x,y)\, C^\varepsilon(x-y)\, dsdy,$$

$$\ell^\varepsilon_{a(v)}((t,x),\tau_2) = \int_{\{(-\infty,t)\times\mathbf{R}\}^2} Q^{a(v)}_{t,s}(x,y)\, \partial_x^2 Q^{a(v)}_{t,s'}(x,y')\, C^\varepsilon(y-y')\, dsds'dydy',$$

where

$$C^\varepsilon(x) := \mathbb{E}[\xi_\varepsilon(x)\xi_\varepsilon(0)].$$

By (4.21) of Proposition *27* we can replace $Q^{a(v)}_{t,s}(x,y)$ above by $Z^{a(v(t,x))}_{t-s}(x-y)$ up to integrable kernels. Indeed one has for the difference

$$\int_{(-\infty,t)\times\mathbf{R}} (\mathsf{G}^\alpha)_{t,s}(x,y)\, C^\varepsilon(x-y)\, dsdy \sim \int_{-\infty}^t (\mathsf{G}^\alpha)_{t,s}(x,x)\, ds \lesssim \int_{-\infty}^t \frac{e^{\gamma(t-s)}}{(t-s)^{\alpha/2}}\, ds < \infty,$$

where $a \sim b$ means that a is equal to b up to an ε-uniform remainder term, and we can choose a negative γ for sufficiently large $c > 0$. A similar estimate holds for $\ell^\varepsilon_{a(v)}((t,x),\tau_2)$. One thus has

$$l^\varepsilon_\lambda(\tau_1) = \int_{(-\infty,t)\times\mathbf{R}} Z^\lambda_{t-s}(x-y)\, C^\varepsilon(x-y)\, dsdy,$$

$$\sim -\frac{1}{2\pi\lambda}\int_{\mathbf{R}} \log|y|\, C^\varepsilon(y)\, dy$$

$$\begin{aligned} l^\varepsilon_\lambda(\tau_2) &= \int_{\{(-\infty,t)\times\mathbf{R}\}^2} Z^\lambda_{t-s}(x-y)\,\partial_x^2 Z^\lambda_{t-s'}(x-y')\,C^\varepsilon(y-y')\,dsds'dydy' \\ &= -\frac{1}{\lambda}\int_{(-\infty,t)\times\mathbf{R}\times\mathbf{R}} Z^\lambda_{t-s}(x-y)\,\delta_0(x-y')\,C^\varepsilon(y-y')\,dsdydy' \\ &\sim \frac{1}{2\pi\lambda^2}\int_{\mathbf{R}} \log|y|\,C^\varepsilon(y)\,dy. \end{aligned}$$

The action of the characters acting on trees with nonzero $\boldsymbol{p}$-decorations can be estimated similarly using Proposition *27*, showing that ***Assumption 2*** holds in that case. Then formula (1.6) takes the form

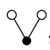
$$\Big(l^\varepsilon_{a(\cdot)}(\tau_1)f'f + l^\varepsilon_{a(\cdot)}(\tau_2)a'f^2\Big)(u^\varepsilon) = c^\varepsilon\left(\frac{f'f}{a} - \frac{a'f^2}{a^2}\right)(u^\varepsilon)$$

with a constant $c^\varepsilon = -\frac{1}{2\pi}\int_{\mathbf{R}}\log|y|\,C^\varepsilon(y)dy$. This matches the previous works on the subject by Bailleul, Debussche & Hofmanová [3], Furlan & Gubinelli [18] and Otto & Weber [30].

4.5.2 – Quasilinear generalized (KPZ) equation with regularized noise. We work in this paragraph in the one dimensional space torus. Let ξ be the mildly singular case of a spacetime Gaussian noise of parabolic regularity $\alpha_0 - 2$ with $1/2 < \alpha_0 < 2/3$ and consider the quasilinear equation

$$\partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x^2 u).$$

Then the only elements $\tau \in \mathbb{B}_\circ^{-0}$ with an even number of noise symbols ζ_1 are the trees

$\tau_1 = \zeta_1\,\mathcal{I}(\zeta_1) =$ , $\tau_2 = \zeta_3\,\mathcal{I}(\zeta_1)\mathcal{I}_{(0,2)}(\zeta_1) =$ , $\tau_3 = \zeta_2\,\mathcal{I}_{(0,1)}(\zeta_1)^2 =$ , (4.23)

where the thin line denotes the operator $\mathcal{I}_{(0,1)}$ and the black dot denotes the symbol ζ_2. Since all of them have homogeneity $2\alpha_0 - 2 > -1$, we can replace the kernel $Q^{a(v)}$ by $Z^{a(v)}$ up to integrable kernel G^α by Proposition *27*. Thus they satisfy ***Assumption 2*** and the counterterm takes the form

$$\Big(l^\varepsilon_{a(\cdot)}(\tau_1)\,f'f + l^\varepsilon_{a(\cdot)}(\tau_2)\,gf^2 + l^\varepsilon_{a(\cdot)}(\tau_3)\,a'f\Big)(u^\varepsilon).$$

As mentioned in Gerencsér & Hairer's work [20] the renormalization constants are cancelled as follows. We assume that the function

$$C^\varepsilon(z) := \mathbb{E}\big[\xi^\varepsilon(z)\xi^\varepsilon(0)\big]$$

is an even function. By explicit calculation,

$$\begin{aligned} l^\varepsilon_\lambda(\tau_1) &= -\int_{\mathbf{R}^2} C^\varepsilon(z)Z^\lambda(z)\,dz, \\ l^\varepsilon_\lambda(\tau_2) &= -\int_{(\mathbf{R}^2)^2} C^\varepsilon(z-z')\,\partial_x Z^\lambda(z)\partial_x Z^\lambda(z')\,dzdz' = -\int_{\mathbf{R}^2} C^\varepsilon(z)\,\overline{\partial_x Z^\lambda} * \partial_x Z^\lambda(z)\,dz, \\ l^\varepsilon_\lambda(\tau_3) &= -\int_{(\mathbf{R}^2)^2} C^\varepsilon(z-z')Z^\lambda(z)\,\partial_x^2 Z^\lambda(z')\,dzdz' = -\int_{\mathbf{R}^2} C^\varepsilon(z)\,\overline{Z^\lambda} * \partial_x^2 Z^\lambda(z)\,dz, \end{aligned}$$

where $\overline{f}(z) := f(-z)$ for any function f. As we see from the identity

$$\overline{\partial_x Z^\lambda} * \partial_x Z^\lambda(z) = -\overline{Z^\lambda} * \partial_x^2 Z^\lambda(z) = \frac{1}{2\lambda}Z^\lambda(|t|,x) + O(1),$$

that we have

$$\lambda l^\varepsilon_\lambda(\tau_2) = -\lambda l^\varepsilon_\lambda(\tau_3) = l^\varepsilon_\lambda(\tau_1) + O(1),$$

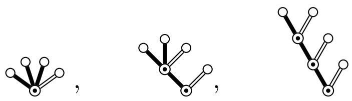
our formula matches with Gerencsér & Hairer's formula [20]

$$l^\varepsilon_{a(u^\varepsilon)}(\tau_1)\left(f'f + \frac{gf^2}{a} - \frac{a'f}{a}\right)(u^\varepsilon).$$

4.5.3 – Quasilinear generalized (KPZ) equation with space-time white noise. Let ξ be a spacetime Gaussian noise of parabolic regularity $\alpha_0 - 2$ with $2/5 < \alpha_0 < 1/2$ and consider the stochastic heat equation

$$\partial_t u - a(u)\partial_x^2 u = \xi.$$

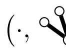
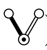
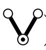
Then the only elements $\tau \in \mathbb{B}_\circ^{-0}$ with an even number of noise symbols ζ_1 are the trees τ_1, τ_2, τ_3 from (4.23) together with the trees



(4.24)

Since the last two trees have homogeneity $(4\alpha_0 - 2)$, we can replace the kernel $Q^{a(v)}$ by $Z^{a(v)}$ in the same way as before. However, ***Assumption 2*** is not ensured at this stage since on the edge e whose lower node (associated with the spacetime variable (s, y)) is not the root (associated with (t, x)), the kernel $Q_{s,\cdot}^{a(v)}(y,\cdot)$ is replaced by $Z_{s-\cdot}^{a(v(s,y))}(y-\cdot)$, not $Z_{s-\cdot}^{a(v(t,x))}(y-\cdot)$. To show ***Assumption 2***, by using the analyticity of Z^λ, we have to replace $Z_{s-\cdot}^{a(v(s,y))}(y-\cdot)$ with $Z_{s-\cdot}^{a(v(t,x))}(y-\cdot)$. The remainder is of size $|t-s|^{1/2} + |x-y|$, which also smears the singularity of the Feynman diagram by α. Thus ***Assumption 2*** holds for the trees in (4.24).

It turns out that the first tree of (4.24) is not involved in equation (1.6) because ξ is a centered Gaussian. Indeed, by decomposing

$l_{a(v)}^\varepsilon(\cdot,$  $) = -h_{a(v)}^\varepsilon(\cdot,$  $) + 3h_{a(v)}^\varepsilon(\cdot,$  $)h_{a(v)}^\varepsilon(\cdot,$  $),$

we see that the right hand side is zero because of Wick theorem for Gaussian random variables.

It remains to consider the tree τ_2. Since it has a homogeneity $2\alpha_0 - 2 < -1$, it is not sufficient to replace $Q^{a(v)}$ by $Z^{a(v)}$ with the kernel of singularity $\alpha \le 1$. One sees however from (4.22) of Proposition *28* that if we use a t-independent function $v \in C^2(\mathbf{T})$ then we have

$\ell_{a(v)}^\varepsilon\big((t,x),$  $\big) =$

$$\int_{(\mathbf{R}^2)^2} \Big\{ Z_{t-s}^{a(v(x))}(x-y)(\partial_x^2 Z_{t-s'})^{a(v(x))}(x-y') + (\star) + \sum_{\beta_1+\beta_2\ge 0} (\mathsf{G}^{(\beta_1)})_{t,s}(x,y)(\mathsf{G}^{(\beta_2)})_{t,s'}(x,y')\Big\}\, C^\varepsilon(s-s', y-y')\, ds ds' dy dy', \tag{4.25}$$

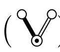
where $(\star)$ is of the form

$$a'(v(x))v'(x)\big\{Y_{t-s}^{0,0,a(v(x))}(x-y)(\partial_x^2 Z_{t-s'})^{a(v(x))}(x-y') + Z_{t-s}^{a(v(x))}(x-y)Y_{t-s'}^{2,0,a(v(x))}(x-y')\big\}.$$

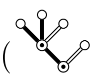
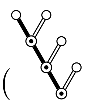
The last term in (4.25) does not matter because one has the ε-uniform estimate

$$\begin{aligned}&\int_{(\mathbf{R}^2)^2} (\mathsf{G}^{(\beta_1)})_{t,s}(x,y)(\mathsf{G}^{(\beta_2)})_{t,s'}(x,y')\, C^\varepsilon(s-s', y-y')\, ds ds' dy dy' \\ &\lesssim \int_{\mathbf{R}^2} (\mathsf{G})_{t-s}^{(\beta_1+\beta_2-1)}(x,y)\, ds dy \lesssim \int_{-\infty}^t \frac{e^{\gamma(t-s)}}{(t-s)^{(1-(\beta_1+\beta_2))/2}} ds < \infty.\end{aligned}$$

Although the $(\star)$ term in (4.25) is not estimated as above, if we assume that the mollifier ρ_ε is an even function, then C^ε is also an even function of its space argument. So, the (y, y')-integral $\int_{\mathbf{R}^2}(\star)C^\varepsilon(s-s', y-y')dydy'$ vanishes because of the parity of the functions Y. In the end only the first term of (4.25) survives and ***Assumption 2*** is satisfied with

$l_\lambda^\varepsilon($  $) = \int_{(\mathbf{R}^2)^2} Z_{t-s}^\lambda(x-y)\,(\partial_x^2 Z_{t-s'}^\lambda)(x-y')\, C^\varepsilon(s-s', y-y')\, ds ds' dy dy'.$

Eventually the counterterm takes the form

$\Big\{ l_{a(\cdot)}^\varepsilon($  $)\, a' + l_{a(\cdot)}^\varepsilon($  $)\, a'a'' + l_{a(\cdot)}^\varepsilon($  $)\, (a')^3 \Big\}(u^\varepsilon),$

which matches Gerencsér's formula in Theorem *1.1* of [19]. We see on this formula the rule (4.16)-(4.17) in action.

In the case of the quasilinear generalized (KPZ) equation (1.1)

$$\partial_t u - a(u)\partial_x^2 u = f(u)\xi + g(u)(\partial_x u)^2$$

driven by a one dimensional spacetime white noise ξ on the torus, the list of trees $\tau \in \mathbb{B}_\circ^{-0}$ with an even number of noise symbols ζ_1 contains, in addition to the previous trees, the trees τ_1, τ_3 from (4.23) and a number of other trees of homogeneity $4\alpha_0 - 2$. That ***Assumption 2*** holds true for all the trees of homogeneity $4\alpha_0 - 2$ can be seen as for the trees of (4.24). The counterterms corresponding to τ_1 and τ_3 can be seen to satisfy ***Assumption 2*** by a similar computation as in (4.25).

We note that the present analysis of equation (1.1) holds for a large class of Gaussian spacetime noises of parabolic regularity $\alpha_0 - 2$, up to $\alpha_0 > 1/3$, because there are no other trees of homogeneity strictly smaller than -1 except those considered above.

A – Appendix

In this appendix we prove some technical properties of the fundamental solutions of anisotropic parabolic operators following the arguments in [16, 15]. We believe that the results given here are known but we could not find any suitable references. For the sake of generality for them we work on the space $\mathbf{R}^d$ and an anisotropic scaling $\mathfrak{s} = (\mathfrak{s}_j)_{j=1}^d \in \mathbf{N}^d$. Set

$|\mathfrak{s}| := \sum_{j=1}^d \mathfrak{s}_j$,
$|k|_\mathfrak{s} := \sum_{j=1}^d \mathfrak{s}_j k_j$, for $k = (k_j)_{j=1}^d \in \mathbf{N}^d$,
$\|x\|_\mathfrak{s} := \sum_{j=1}^d |x_j|^{1/\mathfrak{s}_j}$, for $x = (x_j)_{j=1}^d \in \mathbf{R}^d$,
$\partial_x^k := \prod_{j=1}^d \partial_{x_j}^{k_j}$, for $k = (k_j)_{j=1}^d \in \mathbf{N}^d$.

Through this appendix, we consider the anisotropic parabolic operator

$$\partial_t - P(t,x,\partial_x) := \partial_t - \sum_{|k|_\mathfrak{s} \le N} a_k(t,x)\partial_x^k \tag{A.1}$$

with coefficients $a_k(t,x)$ defined in a domain $D = (a,b) \times \mathbf{R}^d$, where $-\infty \le a < b \le \infty$. In addition, N is an integer satisfying $N > \max_j \mathfrak{s}_j$.

29 – *Definition. We call a function $Q_{t,s}(x,y)$ defined on $a < s < t < b$ and $x, y \in \mathbf{R}^d$* ***a fundamental solution*** *of the operator* (A.1) *if for any $f \in C_b(\mathbf{R}^d)$ the function*

$$F(t,x;s) := \int_{\mathbf{R}^d} Q_{t,s}(x,y)f(y)dy$$

satisfies the properties

$$\big(\partial_t - P(t,x,\partial_x)\big)F(t,x;s) = 0, \qquad t > s,\ x \in \mathbf{R}^d, \tag{A.2}$$

$$\lim_{t\downarrow s} F(t,x;s) = f(x), \qquad x \in \mathbf{R}^d \tag{A.3}$$

for any fixed $s \in (a,b)$.

We prove the existence of the fundamental solution and Gaussian estimates for it (Theorem *34*) in Appendix A.2, and prove uniqueness (Theorem *39*) in Appendix A.3. Appendix A.1 is devoted to giving preliminary results. In Appendix A.4 we consider temporally homogeneous operators. The estimates of anisotropic Taylor remainders (Corollary *43*) given in Appendix A.5 has an important role in the proof of Theorem *13*. In Appendix A.6 we give the proof of Propositions *27* and *28*, which are used in Section 4.5.

A.1 – Gaussian kernels. In this section, we prove some technical properties of exponential functions. For $c>0$ and $\beta\in\mathbf{R}$, we define the function

$$\mathsf{G}_t^{(c,\beta)}(x) := t^{(\beta-|\mathfrak{s}|)/N}\exp\Big\{-c\sum_{j=1}^d\Big(\frac{|x_j|^{N/\mathfrak{s}_j}}{t}\Big)^{\mathfrak{s}_j/(N-\mathfrak{s}_j)}\Big\}\qquad t>0,\ x\in\mathbf{R}^d.$$

30 – Lemma. Let $\beta,\beta_1,\beta_2\in\mathbf{R}$ and $c,c_1,c_2>0$.

(i) For any $\alpha>0$ and $c'\in(0,c)$, one has

$$(t^{1/N}+\|x\|_{\mathfrak{s}})^\alpha\,\mathsf{G}_t^{(c,\beta)}(x)\le C\mathsf{G}_t^{(c',\alpha+\beta)}(x).$$

(ii) For any $\|h\|_{\mathfrak{s}}\le t^{1/N}$ and $c'\in(0,c)$, one has

$$\mathsf{G}_t^{(c,\beta)}(x+h)\le C\mathsf{G}_t^{(c',\beta)}(x).$$

(iii) If $c_1<c_2$ and $0<s<t$, one has

$$\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c_2,\beta_1)}(x-y)\,\mathsf{G}_s^{(c_2,\beta_2)}(y)\,dy\le C(t-s)^{\beta_1/N}s^{\beta_2/N}\,\mathsf{G}_t^{(c_1,0)}(x).$$

(iv) If $c_1<c_2$ and $\beta_1,\beta_2>-N$, one has

$$\int_0^t\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c_2,\beta_1)}(x-y)\,\mathsf{G}_s^{(c_2,\beta_2)}(y)\,dyds\le C\,\frac{\Gamma(\frac{\beta_1+N}{N})\Gamma(\frac{\beta_2+N}{N})}{\Gamma(\frac{\beta_1+\beta_2+N}{N})}\,\mathsf{G}_t^{(c_1,\beta_1+\beta_2+N)}(x).$$

(v) If $c_1<c_2$, $\beta_1>-N+|\mathfrak{s}|$, and $\beta_2>-N$, one has

$$\int_0^t\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c_1,\beta_1)}(x-y)\,\mathsf{G}_s^{(c_2,\beta_2)}(y)\,dyds\le C\,\frac{\Gamma(\frac{\beta_1-|\mathfrak{s}|+N}{N})\Gamma(\frac{\beta_2+N}{N})}{\Gamma(\frac{\beta_1+\beta_2-|\mathfrak{s}|+N}{N})}\,\mathsf{G}_t^{(c_1,\beta_1+\beta_2+N)}(x).$$

—The constants C are independent to $t,x,\beta,\beta_1,\beta_2$.

Proof – The proofs of the statements *(i)* and *(ii)* are elementary and left to the readers. For *(iii)* note that the elementary inequality

$$\mathsf{G}_{t-s}^{(c,0)}(x-y)\mathsf{G}_s^{(c,0)}(y)\le t^{|\mathfrak{s}|/N}(t-s)^{-|\mathfrak{s}|/N}s^{-|\mathfrak{s}|/N}\mathsf{G}_t^{(c,0)}(x)\tag{A.4}$$

holds. This inequality reduces to

$$|x_j|\le t^{\mathfrak{s}_j/N}\Big\{\Big(\frac{|x_j-y_j|}{(t-s)^{\mathfrak{s}_j/N}}\Big)^{N/(N-\mathfrak{s}_j)}+\Big(\frac{|y_j|}{s^{\mathfrak{s}_j/N}}\Big)^{N/(N-\mathfrak{s}_j)}\Big\}^{(N-\mathfrak{s}_j)/N},$$

which follows from the Hölder's inequality. By integration we have

$$\begin{aligned}&\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c_2,\beta_1)}(x-y)\mathsf{G}_s^{(c_2,\beta_2)}(y)dy\\&=(t-s)^{(\beta_1+|\mathfrak{s}|)/N}s^{(\beta_2+|\mathfrak{s}|)/N}\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c_1,0)}(x-y)\mathsf{G}_{t-s}^{(c_2-c_1,0)}(x-y)\mathsf{G}_s^{(c_1,0)}(y)\mathsf{G}_s^{(c_2-c_1,0)}(y)dy\\&\le t^{|\mathfrak{s}|/N}(t-s)^{\beta_1/N}s^{\beta_2/N}\mathsf{G}_t^{(c_1,0)}(x)\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c_2-c_1,0)}(x-y)\mathsf{G}_s^{(c_2-c_1,0)}(y)dy.\end{aligned}$$

Since $\mathsf{G}_t^{(c,0)}(x)\le t^{-|\mathfrak{s}|/N}$ and

$$C_c:=\int_{\mathbf{R}^d}\mathsf{G}_t^{(c,0)}(x)dx=\int_{\mathbf{R}^d}\mathsf{G}_1^{(c,0)}(x)dx$$

is t-independent we have

$$\begin{aligned}&\int_{\mathbf{R}^d}\mathsf{G}_{t-s}^{(c_2-c_1,0)}(x-y)\mathsf{G}_s^{(c_2-c_1,0)}(y)dy\\&\le\min\{(t-s)^{-|\mathfrak{s}|/N},s^{-|\mathfrak{s}|/N}\}\int_{\mathbf{R}^d}\mathsf{G}_1^{(c_2-c_1,0)}(y)dy\le C_{c_2-c_1}(t/2)^{-|\mathfrak{s}|/N}.\end{aligned}$$

By integrating *(iii)*, we immediately have *(iv)*. For *(v)*, we use (A.4) again and have

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} \mathsf{G}^{(c_1,\beta_1)}_{t-s}(x-y)\mathsf{G}^{(c_2,\beta_2)}_s(y)\,dsdy \\
&\qquad = \int_0^t (t-s)^{\beta_1/N} s^{(\beta_2+|\mathfrak{s}|)/N} \int_{\mathbf{R}^d} \mathsf{G}^{(c_1,0)}_{t-s}(x-y)\mathsf{G}^{(c_1,0)}_s(y)\mathsf{G}^{(c_2-c_1,0)}_s(y)dyds \\
&\qquad \le t^{|\mathfrak{s}|/N} \int_0^t (t-s)^{(\beta_1-|\mathfrak{s}|)/N} s^{\beta_2/N} \mathsf{G}^{(c_1,0)}_t(x) \int_{\mathbf{R}^d} \mathsf{G}^{(c_2-c_1,0)}_s(y)dyds \\
&\qquad \le C_{c_2-c_1} t^{(\beta_1+\beta_2+N)/N} \frac{\Gamma(\frac{\beta_1-|\mathfrak{s}|+N}{N})\Gamma(\frac{\beta_2+N}{N})}{\Gamma(\frac{\beta_1+\beta_2-|\mathfrak{s}|+N}{N})} \mathsf{G}^{(c_1,0)}_t(x).
\end{aligned}$$

▷

31 – *Definition. For $\beta \in \mathbf{R}$, denote by G^β the class of functions $A = A_{t,s}(x,y)$ defined on $a < s < t < b$ and $x, y \in \mathbf{R}^d$ such that*

$$|A_{t,s}(x,y)| \le Ce^{c_0(t-s)}\,\mathsf{G}^{(c_1,\beta)}_{t-s}(x-y)$$

for some positive constants C, c_0, c_1. Moreover, for any $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^d \in \prod_{i=1}^d [0, \mathfrak{s}_i]$, denote by $\mathsf{G}^\beta_{\boldsymbol{\alpha},0}$ the class of functions $A \in \mathsf{G}^\beta$ satisfying

$$\big|A_{t,s}(x+he_i,y) - A_{t,s}(x,y)\big| \le Ce^{c_0(t-s)}\,|h|^{\alpha_i/\mathfrak{s}_i}\,\mathsf{G}^{(c_1,\beta-\alpha_i)}_{t-s}(x-y)$$

for any $i \in \{1,\dots,d\}$ and $|h|^{1/\mathfrak{s}_i} \le (t-s)^{1/N}$ (or equivalently,

$$\begin{aligned}
&\big|A_{t,s}(x+he_i,y) - A_{t,s}(x,y)\big| \\
&\le Ce^{c_0(t-s)}\,|h|^{\alpha_i/\mathfrak{s}_i}\big\{\mathsf{G}^{(c_1,\beta-\alpha_i)}_{t-s}(x+he_i-y) + \mathsf{G}^{(c_1,\beta-\alpha_i)}_{t-s}(x-y)\big\}
\end{aligned} \tag{A.5}$$

for any $h \in \mathbf{R}$). We also define $\mathsf{G}^\beta_{0,\boldsymbol{\alpha}}$ as the set of functions $A \in \mathsf{G}^\beta$ such that $\widetilde{A}_{t,s}(x,y) := A_{t,s}(y,x)$ is in the class $\mathsf{G}^\beta_{\boldsymbol{\alpha},0}$. Finally, define $\mathsf{G}^\beta_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} := \mathsf{G}^\beta_{\boldsymbol{\alpha},0} \cap \mathsf{G}^\beta_{0,\boldsymbol{\alpha}'}$.

For any functions $A_{t,s}(x,y)$ and $B_{t,s}(x,y)$, we define the spacetime convolution

$$(A * B)_{t,s}(x,y) := \int_{(s,t)\times\mathbf{R}^d} A_{t,u}(x,z)B_{u,s}(z,y)\,dudz$$

if it exists.

32 – *Lemma. Let $\beta, \beta' \in \mathbf{R}$, $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \prod_{i=1}^d [0,\mathfrak{s}_i]$, $A \in \mathsf{G}^\beta_{\boldsymbol{\alpha},0}$, and $B \in \mathsf{G}^{\beta'}_{0,\boldsymbol{\alpha}'}$.*

(i) Suppose that $\beta, \beta' > -N$, $\max_i \alpha_i < \beta + N$, and $\max_i \alpha'_i < \beta' + N$. Then $A * B \in \mathsf{G}^{\beta+\beta'+N}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$.

(ii) Suppose that $\beta \ge -N$, $\beta' > -N$, $\max_i \alpha'_i < \beta' + N$, and $B \in \mathsf{G}^{\beta'}_{(\delta,\dots,\delta),\boldsymbol{\alpha}'}$ for some $\delta > 0$. If

$$\left|\int_{\mathbf{R}^d} A_{t,s}(x,y)dy\right| \lesssim e^{c_0(t-s)}(t-s)^{(\beta+\delta)/N}, \tag{A.6}$$

then $A * B \in \mathsf{G}^{\beta+\beta'+N}_{0,\boldsymbol{\alpha}'}$. If in addition to (A.6)*, we assume $\max_i \alpha_i < \beta + N + \delta$, $\partial_{x_i} A_{t,s}(x,y) \in \mathsf{G}^{\beta-\mathfrak{s}_i}$, and*

$$\left|\int_{\mathbf{R}^d} \partial_{x_i} A_{t,s}(x,y)dy\right| \lesssim e^{c_0(t-s)}(t-s)^{(\beta-\mathfrak{s}_i+\delta)/N} \tag{A.7}$$

for any $i \in \{1,\dots,d\}$, then $A * B \in \mathsf{G}^{\beta+\beta'+N}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$.

(iii) A similar statements to (ii) hold with the roles of first and second variables reversed.

Proof – Item *(i)* follows from Lemma *30-(iv)*. To show item *(ii)* we decompose

$$(A*B)_{t,s}(x,y) = \int_{(t+s)/2}^{t} du \int_{\mathbf{R}^d} A_{t,u}(x,z)B_{u,s}(z,y)\,dz + \int_{s}^{(t+s)/2} du \int_{\mathbf{R}^d} A_{t,u}(x,z)B_{u,s}(z,y)\,dz =: I_{t,s}(x,y) + J_{t,s}(x,y).$$

We can prove that $J_{t,s}(x,y) \in \mathsf{G}^{\beta+\beta'+N}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$ in the same way as *(i)*, where we do not need to assume $\max_i \alpha_i < \beta + N$ because $(t-u)^{(\beta-\alpha_i)/N}$ is integrable on $u \in [s, \frac{t+s}{2}]$ for any α_i. For $I_{t,s}(x,y)$, we set

$$I_{t,s}(x,y) = \int_s^t C_{t,u,s}(x,y)du, \qquad C_{t,u,s}(x,y) := \int_{\mathbf{R}^d} A_{t,u}(x,z)B_{u,s}(z,y)dz.$$

If (A.6) holds then we can decompose $C_{t,u,s}$ into

$$\begin{aligned}
&|C_{t,u,s}(x,y)| \\
&\le \left| \int_{\mathbf{R}^d} A_{t,u}(x,z)\,dz \right| |B_{u,s}(x,y)| + \left| \int_{\mathbf{R}^d} A_{t,u}(x,z)\big(B_{u,s}(z,y) - B_{u,s}(x,y)\big)\,dz \right| \\
&\lesssim e^{c_0(t-s)} \Big\{ (t-u)^{(\beta+\delta)/N} \mathsf{G}^{(c_1,\beta')}_{u-s}(x-y) \\
&\qquad + \int_{\mathbf{R}^d} \mathsf{G}^{(c_1,\beta)}_{t-u}(x-z) \|z-x\|_{\mathfrak{s}}^{\delta} \big(\mathsf{G}^{(c_1,\beta'-\delta)}_{u-s}(z-y) + \mathsf{G}^{(c_1,\beta'-\delta)}_{u-s}(x-y) \big)\,dz \Big\} \\
&\lesssim e^{c_0'(t-s)} \Big\{ (t-u)^{(\beta+\delta)/N} \mathsf{G}^{(c_1,\beta'-\delta)}_{t-s}(x-y) \\
&\qquad + (t-u)^{(\beta+\delta)/N} (u-s)^{(\beta'-\delta)/N} \mathsf{G}^{(c_1',0)}_{t-s}(x-y) \Big\}
\end{aligned} \tag{A.8}$$

for any $c_i' \in (0, c_i)$ for $i \in \{0,1\}$, where we use that (A.5) also holds for multidimensional shifts in the second inequality, and that $\mathsf{G}^{(c,\beta)}_{t'}(x) \lesssim \mathsf{G}^{(c,\beta)}_{t}(x)$ if $t/2 \le t' \le t$ in the third inequality. Since $\beta + \delta > -N$, the integral in u is finite and we have

$$\begin{aligned}
|I_{t,s}(x,y)| &\lesssim e^{c_0'(t-s)} \Big\{ (t-s)^{(\beta+\delta+N)/N} \mathsf{G}^{(c_1,\beta'-\delta)}_{t-s}(x-y) + (t-s)^{(\beta+\beta'+N)/N} \mathsf{G}^{(c_1',0)}_{t-s}(x-y) \Big\} \\
&\lesssim e^{c_0'(t-s)}\, \mathsf{G}^{(c_1',\beta+\beta'+N)}_{t-s}(x-y).
\end{aligned}$$

For the proof of the Hölder estimate of $A*B$ with respect to the first variable, it is sufficient to consider the x_1-shift by $h \in \mathbf{R}$ such that $|h|^{1/\mathfrak{s}_1} \le (t-s)^{1/N}$. We write $x_h = x + he_1$ and decompose

$$C_{t,u,s}(x_h,y) - C_{t,u,s}(x,y) = h \int_0^1 \int_{\mathbf{R}^d} \partial_{x_1} A_{t,u}(x_{\theta h}, z) B_{u,s}(z,y) dz d\theta$$

and have as above

$$\begin{aligned}
|C_{t,u,s}(x_h,y) - C_{t,u,s}(x,y)| \lesssim e^{c_0'(t-s)} |h| \Big\{ &(t-u)^{(\beta-\mathfrak{s}_1+\delta)/N} \mathsf{G}^{(c_1,\beta'-\delta)}_{t-s}(x-y) \\
&+ (t-u)^{(\beta-\mathfrak{s}_1+\delta)/N} (u-s)^{(\beta'-\delta)/N} \mathsf{G}^{(c_1',0)}_{t-s}(x-y) \Big\}
\end{aligned}$$

by Lemma *30-(ii)*. By interpolation between it and (A.8), we have

$$\begin{aligned}
|C_{t,u,s}(x_h,y) - C_{t,u,s}(x,y)| \lesssim e^{c_0'(t-s)} |h|^{\alpha_1/\mathfrak{s}_1} \Big\{ &(t-u)^{(\beta-\alpha_1+\delta)/N} \mathsf{G}^{(c_1,\beta'-\delta)}_{t-s}(x-y) \\
&+ (t-u)^{(\beta-\alpha_1+\delta)/N} (u-s)^{(\beta'-\delta)/N} \mathsf{G}^{(c_1',0)}_{t-s}(x-y) \Big\}
\end{aligned}$$

for any $\alpha_1 \in [0, \mathfrak{s}_1]$. If $\beta - \alpha_1 + \delta > -N$, the integral in u is finite and we have

$$|I_{t,s}(x_h,y) - I_{t,s}(x,y)| \lesssim e^{c_0'(t-s)} |h|^{\alpha_1/\mathfrak{s}_1} \mathsf{G}^{(c_1'',\beta_1+\beta_2+N-\alpha_1)}_{t-s}(x-y),$$

which completes the proof. ▷

A.2 – Existence of the fundamental solution. First we consider the parabolic operator (A.1) when the coefficients a_k are constants. Then we write

$$\partial_t - P(\partial_x) := \partial_t - \sum_{|k|_{\mathfrak{s}} \le N} a_k \partial_x^k. \tag{A.9}$$

33 – Lemma. *Assume the existence of a constant $\delta > 0$ such that the inequality*

$$\operatorname{Re} P(i\xi) = \operatorname{Re} \sum_{|k|_{\mathfrak{s}} \le N} a_k (i\xi)^k \le -\delta \|\xi\|_{\mathfrak{s}}^N \tag{A.10}$$

holds for any $\xi \in \mathbf{R}^d$. Then for any $\varepsilon > 0$, $k \in \mathbf{N}^d$, and $n \in \mathbf{N}$, there exist positive constants C and c which depend only on $\mathfrak{s}, N, A := \max_k |a_k|, \delta, \varepsilon, k, n$ such that the fundamental solution $Z_t(x)$ of the operator (A.9) *satisfies*

$$\left|\partial_t^n \partial_x^k Z_t(x)\right| \le C e^{\varepsilon t} \mathsf{G}_t^{(c, -|k|_{\mathfrak{s}} - Nn)}(x) \tag{A.11}$$

for any $t > 0$ and $x \in \mathbf{R}^d$. When $(k, n) = (0, 0)$, the constant C depends only on δ.

Proof – By definition $Z_t(x)$ is obtained as the Fourier inverse transform of the function $e^{tP(i\xi)}$ of $\xi \in \mathbf{R}^d$. Following the arguments in [16, Chapter 9, Section 2], we consider the bound of $e^{tP(i\xi - \eta)}$ for $\eta, \xi \in \mathbf{R}^d$. By the binomial theorem, we can expand

$$P(i\xi - \eta) = P(i\xi) + R(\xi, \eta),$$

where $R(\xi, \eta)$ is a linear combination of monomials $\xi^k \eta^\ell$ with $|k + \ell|_{\mathfrak{s}} \le N$ and $\ell \neq 0$, and with coefficients depending only on $\{a_k\}$. For any $\varepsilon > 0$, by Young's inequality we have

$$\begin{aligned} |R(\xi, \eta)| &\le A \sum_{m \ge 0,\, n > 0,\, m+n \le N} \|\xi\|_{\mathfrak{s}}^m \|\eta\|_{\mathfrak{s}}^n \\ &\le \varepsilon + \frac{\delta}{2}\|\xi\|_{\mathfrak{s}}^N + c'\|\eta\|_{\mathfrak{s}}^N \le \varepsilon + \frac{\delta}{2}\|\xi\|_{\mathfrak{s}}^N + c\sum_{j=1}^d |\eta_j|^{N/\mathfrak{s}_j}, \end{aligned}$$

where c' and c are positive constants depending only on A, ε, δ. By the condition (A.10), we have

$$|e^{tP(i\xi - \eta)}| \le e^{t \operatorname{Re} P(i\xi)} e^{t|R(\xi, \eta)|} \le \exp\left\{ t\left(\varepsilon - \frac{\delta}{2}\|\xi\|_{\mathfrak{s}}^N + c\sum_{j=1}^d |\eta_j|^{N/\mathfrak{s}_j} \right) \right\}.$$

By using the Cauchy's theorem for each component, we have

$$\begin{aligned} |Z_t(x)| &= \left| \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{ix\cdot\xi} e^{tP(i\xi)} d\xi \right| = \left| \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{ix\cdot(\xi + i\eta)} e^{tP(i\xi - \eta)} d\xi \right| \\ &\le \frac{e^{\varepsilon t}}{(2\pi)^d} \exp\left(-x\cdot\eta + ct\sum_{j=1}^d |\eta_j|^{N/\mathfrak{s}_j} \right) \int_{\mathbf{R}^d} \exp\left(-\frac{\delta t}{2}\|\xi\|_{\mathfrak{s}}^N \right) d\xi \end{aligned}$$

for any $\eta \in \mathbf{R}^d$. If we choose η_j as

$$\eta_j = (\operatorname{sgn} x_j)\left(\frac{|x_j|}{cp_j t} \right)^{1/(p_j - 1)}, \tag{A.12}$$

where $p_j = N/\mathfrak{s}_j$, then

$$-x_j\eta_j + ct|\eta_j|^{p_j} = -\frac{p_j - 1}{p_j}\left(\frac{|x_j|^{p_j}}{cp_j t} \right)^{1/(p_j - 1)},$$

which becomes the argument of the exponential function in (A.11). The integral in ξ becomes $Ct^{-|\mathfrak{s}|/N}$ with some constant C depending only on δ.

For the derivatives $\partial_x^k Z_t(x)$ we can derive the required estimate by a similar way from the identity

$$\partial_x^k Z_t(x) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{ix\cdot\xi}(i\xi)^k e^{tP(i\xi)}d\xi = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} e^{ix\cdot(\xi+i\eta)}(i\xi-\eta)^k e^{tP(i\xi-\eta)}d\xi.$$

We decompose $(i\xi-\eta)^k$ into the linear combination of monomials $\xi^\ell\eta^m$ with $\ell+m=k$. The integral of $|\xi^\ell|\exp(-\frac{\delta t}{2}\|\xi\|_{\mathfrak{s}}^N)$ over ξ becomes the factor $Ct^{-(|\mathfrak{s}|+|\ell|_{\mathfrak{s}})/N}$. For the choice of η as in (A.12) we have

$$|\eta^m| = t^{-|m|_{\mathfrak{s}}/N}\prod_{j=1}^d\left(\frac{|x_j|}{cp_j t^{1/p_j}}\right)^{m_j/(p_j-1)}.$$

Since any powers of $|x_j|/t^{1/p_j}$ are absorbed in the exponential part of (A.11) and the factor $t^{-|m|_{\mathfrak{s}}/N}$ remains, we have the required estimate for $\partial_x^k Z_t(x)$. We have similar estimates for the time derivatives because $\partial_t^n Z_t(x) = (P(\partial_x))^n Z_t(x)$. ▷

Based on the above theorem we consider the operator (A.1) with variable coefficients $a_k(t,x)$.

34 – Theorem. Assume the following conditions for $a_k(t,x)$.

(a) There exists a constant $\delta>0$ such that the inequality

$$\operatorname{Re}P(t,x,i\xi) = \operatorname{Re}\sum_{|k|_{\mathfrak{s}}\le N} a_k(t,x)(i\xi)^k \le -\delta\|\xi\|_{\mathfrak{s}}^N \tag{A.13}$$

holds for any $(t,x)\in D$ and $\xi\in\mathbf{R}^d$.

(b) For some $\alpha>0$, one has

$$A := \max_{|k|_{\mathfrak{s}}\le N}\sup_{(t,x)\in D}|a_k(t,x)|<\infty,$$

$$H := \max_{|k|_{\mathfrak{s}}\le N}\sup_{(t,x),(s,y)\in D}\frac{|a_k(t,x)-a_k(s,y)|}{(|t-s|^{1/N}+\|x-y\|_{\mathfrak{s}})^\alpha}<\infty.$$

Then the fundamental solution $Q_t(x,y)$ of the operator (A.1) *exists and $\partial_x^k Q_{t,s}(x,y)$ is in the class $\mathsf{G}^{-|k|_{\mathfrak{s}}}_{\boldsymbol{\alpha},(\alpha',\dots,\alpha')}$ for any $k\in\mathbf{N}^d$ with $|k|_{\mathfrak{s}}\le N$, any $\boldsymbol{\alpha}\in\prod_{i=1}^d[0,\mathfrak{s}_i]$ such that $\alpha_i<N-|k|_{\mathfrak{s}}+\alpha$, and any $\alpha'\in(0,\alpha)$, where the positive constants C,c_0,c_1 used in Definition 31 depends only on $\mathfrak{s},N,\delta,A,H,k$.*

We prove this theorem following [16, Chapter 9]. Let

$$L_{t,s}(x,y) := Z_{t-s}^{s,y}(x-y),$$

where $Z_t^{s,y}(x)$ is the fundamental solution of $\partial_t - P(s,y,\partial_x)$ for fixed (s,y). We aim to construct the fundamental solution $Q_{t,s}(x,y)$ in the form

$$Q = L + L*\Phi \tag{A.14}$$

with some function $\Phi=\Phi_{t,s}(x,y)$. We set

$$K_{t,s}(x,y) := (P(t,x,\partial_x)-\partial_t)L_{t,s}(x,y) = \big(P(t,x,\partial_x)-P(s,y,\partial_x)\big)Z_{t-s}^{s,y}(x-y).$$

Then $Q_{t,s}(x,y)$ satisfies $\big(\partial_t - P(t,x,\partial_x)\big)Q_{t,s}(x,y)=0$ if and only if

$$\Phi = K + K*\Phi.$$

This implies that the formal solution Φ is given by the form

$$\Phi_{t,s}(x,y) = \sum_{m=1}^\infty K_{t,s}^{(m)}(x,y),\qquad K^{(m)} := K^{*m} = K^{(m-1)}*K. \tag{A.15}$$

It turns out that the series (A.15) is actually absolutely convergent and we can obtain $Q_{t,s}(x,y)$ by the formula (A.14).

35 – Lemma. $\partial_x^k L_{t,s}(x,y)$ *is in the class* $\mathsf{G}^{-|k|_{\mathfrak{s}}}_{\mathfrak{s},(\alpha,\dots,\alpha)}$ *for any* $k \in \mathbf{N}^d$.

Proof – The Gaussian estimate and the Hölder estimate for the first variable immediately follow from Lemma *33*. The Hölder estimate for the second variable comes from the same argument as Lemma 3 of [16, Chapter 9]. ▷

36 – Lemma. $K_{t,s}(x,y)$ *is in the class* $\mathsf{G}^{\alpha-N}_{(\alpha,\dots,\alpha),(\alpha,\dots,\alpha)}$.

Proof – Since $K_{t,s}(x,y) = \big(P(t,x,\partial_x) - P(s,y,\partial_x)\big)Z^{s,y}_{t-s}(x-y)$, we have

$$\begin{aligned} |K_{t,s}(x,y)| &\lesssim (|t-s|^{1/N} + \|x-y\|_{\mathfrak{s}})^\alpha e^{\varepsilon(t-s)}\mathsf{G}^{(c,-N)}_{t-s}(x-y) \\ &\lesssim e^{\varepsilon(t-s)}\mathsf{G}^{(c_1,\alpha-N)}_{t-s}(x-y) \end{aligned} \tag{A.16}$$

for some $c_1 < c$ by Lemma *30-(i)*. The Hölder estimate for both variables are obtained by a similar way. ▷

37 – Lemma. $\Phi_{t,s}(x,y)$ *is in the class* $\mathsf{G}^{\alpha-N}_{(\alpha',\dots,\alpha'),(\alpha',\dots,\alpha')}$ *for any* $\alpha' < \alpha$.

Proof – First we show the estimates

$$\big|K^{(m)}_{t,s}(x,y)\big| \le Ce^{\varepsilon(t-s)}\frac{B^m(t-s)^{m\alpha/N-1}}{\Gamma(\frac{m\alpha-|\mathfrak{s}|}{N})}\mathsf{G}^{(c',0)}_{t-s}(x-y) \tag{A.17}$$

for some constants $c', C, B > 0$ which depend only on $\mathfrak{s}, N, \delta, A, H, \varepsilon$. Let m_0 be the smallest integer m_0 such that $m_0\alpha > |\mathfrak{s}|$. Up to $m \le m_0$, (A.17) is inductively obtained by Lemma *30-(iv)*. Indeed, starting from (A.16) we have

$$\begin{aligned} |K^{(m)}_{t,s}(x,y)| &\lesssim e^{\varepsilon(t-s)}\big(\mathsf{G}^{(c_{m-1},(m-1)\alpha-N)} * \mathsf{G}^{(c_1,\alpha-N)}\big)_{t-s}(x-y) \\ &\lesssim e^{\varepsilon(t-s)}\mathsf{G}^{(c_m,m\alpha-N)}_{t-s}(x-y) \end{aligned}$$

for some $c_m < c_{m-1}$. For $m > m_0$, we use Lemma *30-(v)* to obtain

$$\begin{aligned} |K^{(m)}_{t,s}(x,y)| &\le e^{\varepsilon(t-s)}\frac{B^{m-1}}{\Gamma(\frac{(m-1)\alpha-|\mathfrak{s}|}{N})}\big(\mathsf{G}^{(c',(m-1)\alpha-N)} * \mathsf{G}^{(c_1,\alpha-N)}\big)_{t-s}(x-y) \\ &\le e^{\varepsilon(t-s)}\frac{B^{m-1}C\Gamma(\frac{\alpha}{N})}{\Gamma(\frac{m\alpha-|\mathfrak{s}|}{N})}\mathsf{G}^{(c',m\alpha-N)}_{t-s}(x-y). \end{aligned}$$

Hence (A.17) holds with $B = C\Gamma(\frac{\alpha}{N})$. Summing up (A.17) over $m \ge 1$, we have

$$|\Phi_{t,s}(x,y)| \lesssim e^{c_0(t-s)}(t-s)^{\alpha/N-1}\mathsf{G}^{(c',0)}_{t-s}(x-y) = e^{\gamma(t-s)}\mathsf{G}^{(c',\alpha-N)}_{t-s}(x-y)$$

for some c_0. The Hölder estimates are obtained by applying Lemma *32-(i)* to the formula

$$\Phi = K + K * \Phi = K + \Phi * K.$$

▷

Proof of Theorem 34 – We have the Gaussian and Hölder estimates of $\partial_x^k Q$ by applying Lemma *32-(ii)* to the formula

$$\partial_x^k Q = \partial_x^k L + \partial_x^k L * \Phi.$$

By Lemma *35* and *37*, we have

$$\partial_x^k L * \Phi \in \mathsf{G}^{\alpha-|k|_{\mathfrak{s}}}_{\boldsymbol{\alpha},(\alpha',\dots,\alpha')}$$

for any $|k|_{\mathfrak{s}} \le N$ and any $\boldsymbol{\alpha} \in \prod_{i=1}^d[0,\mathfrak{s}_i]$ such that $\max_i \alpha_i < N - |k|_{\mathfrak{s}} + \alpha$ and $\alpha' < \alpha$. Note that

$$\begin{aligned} \left|\int_{\mathbf{R}^d}\partial_x^\ell L_{t,s}(x,y)dy\right| &= \left|\int_{\mathbf{R}^d}\big(\partial_x^\ell Z^{s,y}_{t,s}(x-y) - \partial_x^\ell Z^{s,x}_{t,s}(x-y)\big)dy\right| \\ &\lesssim e^{\varepsilon(t-s)}\int_{\mathbf{R}^d}\|x-y\|^\alpha_{\mathfrak{s}}\mathsf{G}^{(c,-|\ell|_{\mathfrak{s}})}_{t-s}(x-y)\,dz \lesssim e^{\varepsilon(t-s)}(t-s)^{(\alpha-|\ell|_{\mathfrak{s}})/N} \end{aligned}$$

for any $\ell \in \mathbf{N}^d \setminus \{0\}$.
We can check that $Q_{t,s}(x,y)$ is indeed a fundamental solution by a similar way to [16]. See [16, Theorem 11, Section 6, Chapter 1] for the condition (A.2), and see [16, Chapter 9, Section 4] for the condition (A.3). Only the Gaussian and Hölder estimates of $\partial_x^k Q$ are used in the proof of them. ▷

Next theorem can be obtained from a similar argument to Theorem 9 of [16, Section 5, Chapter 1] and Property 10 of [15, Section I.3]. For $\beta > 0$, define $\mathcal{C}_{\mathfrak{s}}^{\beta}(D)$ as the classical parabolic α-Hölder space on the domain $D = (a,b) \times \mathbf{R}^d$, that is, $f \in \mathcal{C}_{\mathfrak{s}}^{\beta}$ if $\partial_t^n \partial_x^k f$ exists and is bounded for any $Nn + |k|_{\mathfrak{s}} < \beta$, and $\partial_t^n \partial_x^k f$ with $Nn + |k|_{\mathfrak{s}} = \lfloor \beta \rfloor$ is $(\beta - \lfloor \beta \rfloor)$-Hölder continuous with respect to the parabolic norm $\| \cdot \|_{\mathfrak{s}}$.

38 – *Theorem. Let $Q_{t,s}(x,y)$ be the fundamental solution of the operator* (A.1) *satisfying the assumptions of Theorem 34. When $a > -\infty$, for any $g \in \mathcal{C}_{\mathfrak{s}}^{\beta}(D)$ define*

$$G(t,x) = \int_a^t Q_{t,s}(x,y) g(s,y) ds dy.$$

Then G belongs to $\mathcal{C}_{\mathfrak{s}}^{\beta+N}(D)$ for any $\beta \in (0,\alpha)$ and satisfies

$$\big(\partial_t - P(t,x,\partial_x)\big) G(t,x) = 0.$$

Moreover, $a = -\infty$ is allowed if the constant c_0 in Definition 31 can be chosen as a strictly negative number.

A.3 – Uniqueness of the fundamental solution. We prove the uniqueness of the fundamental solution $Q_t(x,y)$ of the operator (A.1) by the same way as Theorem 4.3 of [15, Section Ⅲ.2]. See also Lemma 6.1.2 of [28]. For any time interval $I \subset (a,b)$, define $C_{\mathfrak{s}}^{1,N}(I \times \mathbf{R}^d)$ as the collection of bounded continuous functions f on $I \times \mathbf{R}^d$ such that $\partial_t f$ and $\partial_x^k f$ for any $|k|_{\mathfrak{s}} \le N$ are bounded and continuous.

39 – *Theorem. Suppose that the coefficients $a_k(t,x)$ of the operator* (A.1) *satisfies the assumptions of Theorem 34. For any fixed $s \in (a,b)$ and $f \in C_b(\mathbf{R}^d)$, the Cauchy problem* (A.2)-(A.3) *has a unique solution $F \in C_{\mathfrak{s}}^{1,N}([s,b) \times \mathbf{R}^d)$. Consequently, the fundamental solution $Q_{t,s}(x,y)$ of the operator* (A.1) *is unique (up to Lebesgue null sets in y).*

Proof – It is sufficient to show that the solution F of (A.2)-(A.3) with $f = 0$ is equal to zero. Set

$$W(t) := \sum_{|k|_{\mathfrak{s}} \le N} \int_s^t \|\partial_x^k F(r)\|_{C_b(\mathbf{R}^d)} dr.$$

We fix a point $y \in \mathbf{R}^d$ and write the equation in the form

$$\big(\partial_t - P(t,y,\partial_x)\big) F(t,x) = \big(P(t,x,\partial_x) - P(t,y,\partial_x)\big) F(t,x) =: f^y(t,x).$$

Note that the fundamental solution of the operator (A.1) is unique if the coefficients $a_k(t)$ are x-independent continuous functions of t. Thus we can write

$$F(t,x) = \int_s^t \int_{\mathbf{R}^d} Q_{t,r}^y(x-z) f^y(r,z) dz dr,$$

where $Q_{t,s}^y$ is a fundamental solution of $\partial_t - P(t,y,\partial_x)$. The derivatives of $F(t,x)$ are given by

$$\partial_x^k F(t,x) = \int_s^t dr \int_{\mathbf{R}^d} \partial_x^k Q_{t,r}^y(x-z) f^y(r,z) dz, \qquad |k|_{\mathfrak{s}} < N,$$

$$\partial_x^k F(t,x) = \int_s^t dr \int_{\mathbf{R}^d} \partial_x^k Q_{t,r}^y(x-z) f^y(r,x) dz$$

$$+\int_s^t dr \int_{\mathbf{R}^d} \partial_x^k Q_{t,r}^y(x-z)\big\{f^y(r,z)-f^y(r,x)\big\}dz, \qquad |k|_{\mathfrak{s}}=N$$

for any $y \in \mathbf{R}^d$. We estimate the derivatives of F by setting $y=x$. Since

$$|f^x(r,z)| \lesssim \|z-x\|_{\mathfrak{s}}^{\alpha} \sum_{|k|_{\mathfrak{s}}\le N} \|\partial_x^k F(r)\|_{C_b(\mathbf{R}^d)},$$

using the Gaussian estimates of $\partial^k Q$ we have

$$\|\partial_x^k f(t)\|_{C_b(\mathbf{R}^d)} \lesssim \int_s^t (t-r)^{-(|k|_{\mathfrak{s}}-\alpha)/N} \sum_{|\ell|_{\mathfrak{s}}\le N} \|\partial_x^\ell F(r)\|_{C_b(\mathbf{R}^d)} dr.$$

By integration we can conclude that there exists a constant $C>0$ such that

$$W(t) \le C(t-s)^{\alpha/N} W(t).$$

Hence it follows that $W(t)=0$ for any $s<t<s+t_0$, where $t_0 := (1/C)^{N/\alpha}>0$. Since $F(s')=0$ for some $s' \in (t_0/2, t_0)$ and the coefficients $a_k(t,x)$ are uniformly bounded and Hölder continuous, we can repeat the same argument as above with s replaced by s' and obtain that $W(t)=0$ for any $s'<t<s'+t_0$. In the end, we can establish that $W(t)=0$ for any $t>s$. ▷

A.4 – Temporally homogeneous operator. Next we consider the operator

$$\partial_t - P(x,\partial_x) = \partial_t - \sum_{|k|_{\mathfrak{s}}\le N} a_k(x)\partial_x^k \tag{A.18}$$

with t-independent coefficients $a_k(x)$. Let P be satisfy the assumptions of Theorem *34*, and denote by $Q_t(x,y)$ be its fundamental solution defined on $t\in(0,\infty)$ and $x,y\in\mathbf{R}^d$. For any $f\in C_b(\mathbf{R}^d)$, we define the integral operator

$$\big(Q_t f\big)(x) := \int_{\mathbf{R}^d} Q_t(x,y) f(y) dy.$$

It should be noted that Q_t satisfies the semigroup property $Q_tQ_sf = Q_{t+s}f$.

For any $\beta>0$, denote by $C_{\mathfrak{s}}^{\beta}$ the collection of $f\in C_b(\mathbf{R}^d)$ such that $\partial_x^k f$ is bounded and continuous for any $|k|_{\mathfrak{s}}<\beta$, and $\partial_x^k f$ with $|k|_{\mathfrak{s}}=\lfloor\beta\rfloor$ is $(\beta-\lfloor\beta\rfloor)$-Hölder continuous with respect to $\|\cdot\|_{\mathfrak{s}}$. Let then define for $\beta<0$ the space $C_{\mathfrak{s}}^{\beta}(P)$ as the completion of the set of $f\in C_b(\mathbf{R}^d)$ under the norm

$$\|f\|_{C_{\mathfrak{s}}^{\beta}(P)} := \sup_{0<t\le 1} t^{-\beta/N}\|Q_t f\|_{L^\infty(\mathbf{R}^d)}.$$

40 – Theorem. Let c_0 be a positive number given in Definition 31. For any $c>c_0$ and any $f\in C_b(\mathbf{R}^d)$, define

$$\big(c-P(x,\partial_x)\big)^{-1} f(x) := \int_0^\infty e^{-ct} Q_t f(x) dt.$$

Then the map $(c-P(x,\partial_x))^{-1}$ is continuously extended to the map from $C_{\mathfrak{s}}^{\beta}(P)$ into $C_{\mathfrak{s}}^{\beta+N}(\mathbf{R}^d)$ for any $\beta\in(-N,0)$ such that $\beta+N$ is not an integer.

Proof – We write $P=P(x,\partial_x)$ and $Q_t^c = e^{-ct}Q_t$ for simplicity. By the semigroup property and the Gaussian estimate of Q_t, we have

$$\|\partial_x^k Q_t^c f\|_{L^\infty} = \|\partial_x^k Q_{t/2}^c Q_{t/2}^c f\|_{L^\infty} \lesssim t^{-|k|_{\mathfrak{s}}/N}\|Q_{t/2}^c f\|_{L^\infty} \lesssim t^{(\beta-|k|_{\mathfrak{s}})/N}\|f\|_{C_{\mathfrak{s}}^{\beta}(P)}$$

for any $t\in(0,2]$, and

$$\|\partial_x^k Q_t^c f\|_{L^\infty} = \|\partial_x^k Q_{t-1}^c Q_1^c f\|_{L^\infty} \lesssim e^{-(c-c_0)(t-1)}\|f\|_{C_{\mathfrak{s}}^{\beta}(P)}$$

for any $t \geq 2$. By integration,

$$\|\partial_x^k (c-P)^{-1} f\|_{L^\infty} \leq \int_0^2 \|\partial_x^k Q_t^c f\|_{L^\infty}\, dt + \int_2^\infty \|\partial_x^k Q_t^c f\|_{L^\infty}\, dt \lesssim \|f\|_{C_{\mathfrak{s}}^\beta(P)}$$

for any $|k|_{\mathfrak{s}} < \beta + N$. To show the Hölder estimates of $\partial_x^k (c-P)^{-1} f$ with $|k|_{\mathfrak{s}} = \lfloor \beta + N \rfloor < N$, it is sufficient to consider the region $\|x' - x\|_{\mathfrak{s}} < 2$. We decompose

$$\partial_x^k (c-P)^{-1} f(x') - \partial_x^k (c-P)^{-1} f(x) = \int_0^\infty \int_{\mathbf{R}^d} \big\{\partial_x^k Q_{t_1}^c(x', y) - \partial_x^k Q_{t_1}^c(x, y)\big\} Q_{t_0}^c f(y) dy dt$$

as before, where $t = t_0 + t_1$ and $t_0 := \min\{t/2, 1\}$. Setting $\gamma = \beta + N - \lfloor \beta + N \rfloor$ and choosing sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
&|\partial_x^k (c-P)^{-1} f(x') - \partial_x^k (c-P)^{-1} f(x)| \\
&\leq \|x' - x\|_{\mathfrak{s}}^{\gamma - \varepsilon} \int_0^{\|x'-x\|_{\mathfrak{s}}} t^{(-|k|_{\mathfrak{s}} - (\gamma - \varepsilon) + \beta)/N} dt + \|x' - x\|_{\mathfrak{s}}^{\gamma + \varepsilon} \int_{\|x'-x\|_{\mathfrak{s}}}^2 t^{(-|k|_{\mathfrak{s}} - (\gamma + \varepsilon) + \beta)/N} dt \\
&\quad + \|x' - x\|_{\mathfrak{s}}^\gamma \int_2^\infty t^{(-|k|_{\mathfrak{s}} - \gamma)/N} e^{-(c - c_0)(t-1)} dt \lesssim \|x' - x\|_{\mathfrak{s}}^\gamma
\end{aligned}$$

by the Hölder estimates of Q_t. ▷

41 – *Lemma. Suppose that $a_0 = 0$ (the constant term of P). For any $T > 0$, there exists a constant $C > 0$ depending only on T and the constants in the Gaussian estimate of Q such that, for any $f \in C_{\mathfrak{s}}^\beta$ with $\beta \in [0, 1]$, the following estimates hold for any $0 < t \leq T$.*

$$\|\partial_t^n \partial_x^k Q_t f\|_{L^\infty(\mathbf{R}^d)} \leq \begin{cases} C\|f\|_{L^\infty(\mathbf{R}^d)}, & ((n,k) = (0,0)) \\ C t^{(\beta - Nn - |k|_{\mathfrak{s}})/N} \|f\|_{C_{\mathfrak{s}}^\beta}, & (1 \leq Nn + |k|_{\mathfrak{s}} \leq N) \end{cases} \tag{A.19}$$

$$\big\|(Q_t - \mathrm{Id}) f\big\|_{L^\infty(\mathbf{T})} \leq C t^{\beta/N} \|f\|_{C_{\mathfrak{s}}^\beta}, \tag{A.20}$$

$$\big\|Q_t f\big\|_{C_{\mathfrak{s}}^\beta} \leq C \|f\|_{C_{\mathfrak{s}}^\beta}. \tag{A.21}$$

Proof – These are immediate consequences of the Gaussian estimates of Q_t. For the latter part of (A.19) and (A.20), we decompose

$$\partial_t^n \partial_x^k Q_t f(x) = \int_{\mathbf{R}^d} \partial_t^n \partial_x^k Q_t(x, y) \big(f(y) - f(x)\big)\, dy + f(x) \int_{\mathbf{R}^d} \partial_t^n \partial_x^k Q_t(x, y)\, dy,$$

and use the α-Hölder continuity of f and the fact that $\int_{\mathbf{R}^d} Q_t(x, y) dy = 1$, which follows from $a_0 = 0$ and the uniqueness of Q_t. The estimate (A.21) follows by interpolation between (A.20) and (A.19), with $(n, k) = (0, e_i)$. ▷

A.5 – Anisotropic Taylor formula. Continuing the previous section, we consider the time-homogeneous operator (A.18). In what follows, we consider the parabolic scaling $\mathfrak{s} = (2, 1, 1, \ldots, 1)$. We denote by

$$x = (x_1, x_2, \ldots, x_d) =: (x_1, \bar{x})$$

a generic element of $\mathbf{R}^d$. The following anisotropic Taylor formula is an analogue of Proposition A.1 of [23], but here we restrict the differentiability of the function.

42 – *Proposition. For any function f on $\mathbf{R}^d$ which is k-th differentiable for any $|k|_{\mathfrak{s}} \leq n$, we have*

$$\begin{aligned}
\Big| f(y) - \sum_{|k|_{\mathfrak{s}} \leq n} \frac{(y-x)^k}{k!} \partial_x^k f(x) \Big| &\lesssim \|y - x\|_{\mathfrak{s}}^{n-1} \sup_{|k|_{\mathfrak{s}} = n-1} \sup_{(z_1, \bar{z})} \big|\partial^k f(z_1, \bar{z}) - \partial^k f(x_1, \bar{z})\big| \\
&\quad + \|y - x\|_{\mathfrak{s}}^n \sup_{|k|_{\mathfrak{s}} = n} \sup_{(z_1, \bar{z})} \big|\partial^k f(z_1, \bar{z}) - \partial^k f(x)\big|,
\end{aligned}$$

where z_1 (resp. $\bar{z}$) runs over the interval (x_1, y_1) (resp. $(\bar{x}, \bar{y})$).

Proof. Denote by $A = \{k \in \mathbf{N}^d\,;\, |k|_{\mathfrak{s}} \le n\}$ and define

$$A^\circ = \{k \in A\,;\, k + e_i \in A \text{ for all } i \in \{1, \dots, d\}\} = \{k \in \mathbf{N}^d\,;\, |k|_{\mathfrak{s}} \le n-2\}.$$

Setting $x(\theta) := (x_1(\theta), x'(\theta)) := x + \theta(y-x)$ for $\theta \in [0,1]$ and repeating the Taylor expansion of first order, we have

$$\begin{aligned}
f(y) - f(x) &= \sum_{e_i \in A^\circ} (y_i - x_i) \int_0^1 \partial_i f(x(\theta)) d\theta + \sum_{e_i \notin A^\circ} (y_i - x_i) \int_0^1 \partial_i f(x(\theta)) d\theta \\
&= \sum_{e_i \in A^\circ} (y_i - x_i) \partial_i f(x) + \sum_{e_i + e_j \in A^\circ} (y-x)^{e_{ij}} \int_0^1 (1-\theta) \partial_{ij} f(x(\theta)) d\theta \\
&\quad + \sum_{\substack{e_i \in A^\circ \\ e_i + e_j \notin A^\circ}} (y-x)^{e_{ij}} \int_0^1 (1-\theta) \partial_{ij} f(x(\theta)) d\theta + \sum_{e_i \notin A^\circ} (y_i - x_i) \int_0^1 \partial_i f(x(\theta)) d\theta \\
&= \cdots \\
&= \sum_{k \in A^\circ \setminus \{0\}} \frac{(y-x)^k}{k!} \partial^k f(x) + \sum_{\substack{k \in A^\circ \\ k + e_i \notin A^\circ}} \frac{(y-x)^{k+e_i}}{k!} \int_0^1 (1-\theta)^{|k|} \partial^{k+e_i} f(x(\theta)) d\theta
\end{aligned}$$

in the end. In other words,

$$\begin{aligned}
f(y) = \sum_{|k|_{\mathfrak{s}} \le n-2} \frac{(y-x)^k}{k!} \partial^k f(x) &+ \sum_{|k|_{\mathfrak{s}} = n-1} \frac{(y-x)^k}{k!} \int_0^1 |k| (1-\theta)^{|k|-1} \partial^k f(x(\theta)) d\theta \\
&+ \sum_{|k|_{\mathfrak{s}} = n-2} \frac{(y-x)^{k+e_1}}{k!} \int_0^1 (1-\theta)^{|k|} \partial^{k+e_1} f(x(\theta)) d\theta,
\end{aligned}$$

where $|k| := k_1 + \cdots + k_d$. We treat the last two terms by taking care the restriction of the differentiability. For $|k|_{\mathfrak{s}} = n-1$, we decompose

$$\begin{aligned}
\partial^k f(x(\theta)) = \partial^k f(x) &+ \big(\partial^k f(x_1(\theta), \bar{x}(\theta)) - \partial^k f(x_1, \bar{x}(\theta))\big) \\
&+ \theta \sum_{i=2}^d (x_i(\theta) - x_i) \int_0^1 \partial^{k+e_i} f(x_1, \bar{x}(\theta\theta')) d\theta'
\end{aligned}$$

and have

$$\begin{aligned}
&f(y) - \sum_{|k|_{\mathfrak{s}} \le n} \frac{(y-x)^k}{k!} \partial^k f(x) \\
&= \sum_{|k|_{\mathfrak{s}} = n-1} \frac{(y-x)^k}{k!} \int_0^1 |k| (1-\theta)^{|k|-1} \big\{\partial^k f(x_1(\theta), \bar{x}(\theta)) - \partial^k f(x_1, \bar{x}(\theta))\big\} d\theta \\
&\quad + \sum_{|k|_{\mathfrak{s}} = n-1} \sum_{i=2}^d \frac{(y-x)^{k+e_i}}{k!} \int_0^1 (1-\theta)^{|k|} \big\{\partial^{k+e_i} f(x_1, \bar{x}(\theta)) - \partial^{k+e_i} f(x)\big\} d\theta \\
&\quad + \sum_{|k|_{\mathfrak{s}} = n-2} \frac{(y-x)^{k+e_1}}{k!} \int_0^1 (1-\theta)^{|k|} \big\{\partial^{k+e_1} f(x(\theta)) - \partial^{k+e_1} f(x)\big\} d\theta,
\end{aligned}$$

which provides the required estimate. □

43 – *Corollary.* *Let $Q_t(x,y)$ be the fundamental solution of the operator* (A.18) *satisfying the assumptions of Theorem 34. Then for any $\delta \in (0, \alpha)$,*

$$\left| \partial_{x'}^k Q_t(x', y) - \sum_{|k+\ell|_{\mathfrak{s}} \le N} \frac{(x'-x)^\ell}{\ell!} \partial_x^{k+\ell} Q_t(x,y) \right|$$

$$\lesssim e^{c_0 t}\|x'-x\|_{\mathfrak{s}}^{N+\delta-|k|_{\mathfrak{s}}}\Big\{\mathsf{G}_t^{(c_1,-N-\delta)}(x'-y)+\mathsf{G}_t^{(c_1,-N-\delta)}(x-y)\Big\}.$$

Proof – We apply Proposition 42 to $f=Q_t(\cdot,y)$ and $n=N-|k|_{\mathfrak{s}}$. By Theorem 34,

$$\big|\partial_x^m Q_t((x_1',\bar{x}),y)-\partial_x^m Q_t((x_1,\bar{x}),y)\big|\lesssim e^{c_0 t}|x_1'-x_1|^{(1+\delta)/2}\mathsf{G}_t^{(c_1,-N-\delta)}(x-y)$$

for any $|m|_{\mathfrak{s}}=N-1$ and

$$\big|\partial_x^m Q_t(x',y)-\partial_x^m Q_t(x,y)\big|\lesssim e^{c_0 t}\|x'-x\|_{\mathfrak{s}}^{\delta}\mathsf{G}_t^{(c_1,-N-\delta)}(x-y)$$

for any $|m|_{\mathfrak{s}}=N$. ▷

A.6 – Decomposition of the fundamental solution. In the rest of this appendix we prove Proposition *27* and Proposition *28*.

Proof of Proposition 27 – Recall from the proof of Theorem *34* that we can decompose $\partial_x^k Q_t^b$ in the form

$$\partial_x^k Q_{t,s}^b(x,y)=\partial_x^k L+\partial_x^k*\Phi=\partial_x^k Z_{t-s}^{b(s,y)}(x-y)+\mathsf{G}_{\alpha,\alpha}^{\alpha-k}. \tag{A.22}$$

By using the analyticity of $\lambda\mapsto Z^\lambda$, we can replace $b(s,y)$ with $b(t,x)$ and obtain

$$\partial_x^k Q_{t,s}^b(x,y)=\partial_x^k Z_{t-s}^{b(t,x)}(x-y)+\mathsf{G}_{\alpha,\alpha}^{\alpha-k}, \tag{A.23}$$

which implies (4.21) for $p=0$.

Next we consider the case $p=1$. By two decompositions (A.22) and (A.23) of $\partial_x^2 Q^b$ and by Lemma *32*, it is sufficient to consider the integral

$$\int_{(s,t)\times\mathbf{R}}(\partial_x^k Z_{t-u})^{b(t,x)}(x-z)(\partial_x^2 Z_{u-s})^{b(s,y)}(z-y)dudz.$$

By the semigroup property of $Z_t^\lambda=e^{-ct}e^{\lambda t\Delta}$, the above integral is equal to

$$\begin{aligned}\int_s^t(\partial_x^{k+2}Z)^{b(t,x)(t-u)+b(s,y)(u-s)}(x-y)\,du&=\frac{\partial_x^k Z_{t-s}^{b(s,y)}-\partial_x^k Z_{t-s}^{b(t,x)}}{b(s,y)-b(t,x)}\\&=\partial_\lambda\partial_x^k Z_{t-s}^\lambda|_{\lambda=b(t,x)}+\mathsf{G}_{\alpha,\alpha}^{\alpha-k}.\end{aligned}$$

The general case $p\geq 2$ is an easy extension. ▷

Proof of Proposition 28 – From (A.22), we have the higher order expansion

$$\partial_x^k Q^b=\partial_x^k L+\partial_x^k L*K+(\mathsf{G}_{1,1}^{2-k}).$$

By definitions of L and K and the analyticity of $\lambda\mapsto Z^\lambda$, we have

$$\begin{aligned}\partial_x^k L_t(x,y)&=\partial_x^k Z_t^{b(y)}(x-y)\\&=\partial_x^k Z_t^{b(x)}(x-y)+b'(x)(y-x)\partial_\lambda\partial_x^k Z_t^\lambda(x-y)|_{\lambda=b(x)}+(\mathsf{G}_{1,1}^{2-k})\end{aligned} \tag{A.24}$$

and

$$\begin{aligned}K_t(x,y)&=(b(x)-b(y))\partial_x^2 L_t(x,y)\\&=\big\{-b'(y)(x-y)+O(|x-y|^2)\big\}\big\{\partial_x^2 Z_t^{b(y)}(x-y)+(\mathsf{G}_{1,1}^{-1})\big\}\\&=-b'(y)(x-y)\partial_x^2 Z_t^{b(y)}(x-y)+(\mathsf{G}_{1,1}^{0}).\end{aligned}$$

By convolution and the analyticity of $\lambda\mapsto Z^\lambda$ again, we have

$$\begin{aligned}&(\partial_x^k L*K)_t(x,y)\\&=-b'(y)\int_{(0,t)\times\mathbf{R}}(\partial_x^k Z_{t-s}^{b(x)})(x-z)(z-y)\partial_x^2 Z_s^{b(y)}(z-y)dsdz+(\mathsf{G}_{1,1}^{2-k})\\&=-b'(x)\int_{(0,t)\times\mathbf{R}}(\partial_x^k Z_{t-s}^{b(x)})(x-z)(z-y)\partial_x^2 Z_s^{b(x)}(z-y)dsdz+(\mathsf{G}_{1,1}^{2-k}).\end{aligned} \tag{A.25}$$

By (A.24) and (A.25), we have the decomposition (4.22) with

$$Y_t^{k,\lambda}(x) = -x\partial_\lambda \partial_x^k Z_t^\lambda(x) - \int_{(0,t)\times\mathbf{R}} (\partial_x^k Z_{t-s}^\lambda)(x-y) y \partial_x^2 Z_s^\lambda(y) ds dy.$$

The general $p \geq 1$ case is obtained in a similar way to the proof of Proposition *27*. ▷

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