Renormalization of a stochastic nonlinear Schrödinger (NLS) model

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Joint work with R. Fukuizumi (Tokyo) and L. Thomann (Nancy)

A first glimpse of the model

1-d quadratic Schrödinger model with additive noise:

$$(\imath\partial_t - \Delta)u = |u|^2 + \dot{B}, \quad u_0 = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{T},$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $\dot{B}$ is a stochastic noise.

Two main objectives:

(i) Identify and treat situations where the equation cannot be interpreted in a space of functions, but only *in a space of general distributions*, using some additional renormalization procedure.

(ii) Go beyond the classical white-noise-in-time situations and handle the case of a space-time fractional noise $\dot{B}$ (= test pathwise approach).

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Outline

At the crossroads of two lines of research

1. NLS equations with random initial condition: for $p, q \in \mathbb{N}$,

$$(\imath\partial_t - \Delta)u = \overline{u}^p u^q, \quad u(0,.) = \Phi, \quad t \in [-T, T],\ x \in \mathbb{R}^d \text{ or } \mathbb{T}^d,$$

where Φ is a random distribution of low regularity on $\mathbb{R}^d$ or $\mathbb{T}^d$.

- Bourgain, Burq, Tzvetkov, Oh, Thomann, Robert, Deng, Nahmod, Yue...

2. Stochastic NLS models: for regular maps b, σ,

$$(\imath\partial_t - \Delta)u = b(u) + \sigma(u)\,\dot{W}, \quad u(0,.) = 0, \quad t \in [-T, T],\ x \in \mathbb{R}^d \text{ or } \mathbb{T}^d,$$

where $\dot{W}$ is a random noise on $[0, T] \times \mathbb{R}^d$ or $[0, T] \times \mathbb{T}^d$.

In the literature: $\dot{W}$ is a white noise in time $\rightarrow$ stochastic *Itô-type controls*

- De Bouard-Debussche, Brzézniak-Millet, Hornung, Cheung-Mosincat,...

$\longrightarrow$ Only possible if the equation can be treated in a space of *functions*

Outline

Fractional noise

$$\boxed{(\imath\partial_t - \Delta)u = |u|^2 + \dot{B}, \quad u_0 = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{T}.} \tag{1}$$

Definition. We call **space-time fractional noise** of indexes $H_0, H_1 \in (\frac{1}{2}, 1)$ the centered Gaussian noise $\dot{B}$ on $\mathbb{R} \times \mathbb{R}$ with

$$\mathbb{E}\big[\langle \dot{B}, \varphi\rangle\langle \dot{B}, \psi\rangle\big] = \int dsdt \int dxdy\, \varphi(s,x)\psi(t,y)|s-t|^{2H_0-2}|x-y|^{2H_1-2}.$$

When $H_0, H_1 \to \frac{1}{2}$, $\dot{B}$ converges to a space-time white noise $\dot{W}$.

Recall that $\dot{W} = \partial_t\partial_x W$, with W a Brownian sheet.

$\longrightarrow$ In the same way, $\boxed{\dot{B} = \partial_t\partial_x B}$ with B a **fractional Brownian sheet** (= 2-parameter version of a *fractional Brownian motion*).

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Definition. For $H \in (0,1)$, we call **fractional Brownian motion** of index H the centered Gaussian process B on $\mathbb{R}_+$ with covariance

$$\mathbb{E}\big[B_s B_t\big] = \frac{1}{2}\big\{|s|^{2H} + |t|^{2H} - |t-s|^{2H}\big\}.$$

Remark. When $H = \frac{1}{2}$, one has $\mathbb{E}\big[B_s B_t\big] = \frac{1}{2}\big\{|s| + |t| - |t-s|\big\} = s \wedge t$.

$\Longrightarrow$ B is a (standard) Brownian motion

Properties.

Self-similar: for every $a \in \mathbb{R}$, $\{B_{at},\ t \geq 0\} \sim \{|a|^H B_t,\ t \geq 0\}$.

Stationary increments: $B_t - B_s \sim B_{t-s}$.

Pathwise regularity: a.s., $|B_t - B_s| \lesssim |t-s|^{H-\varepsilon}$ for all $0 \leq s, t \leq 1$.

Long-range dependence: one has for instance, for $n \geq 1$,

$$\mathbb{E}\big[(B_1 - B_0)(B_{n+1} - B_n)\big] = \tfrac{1}{2}\big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\big).$$

*When $H \neq \frac{1}{2}$, **memory effect**: disjoint increments are not independent.*

When $H \neq \frac{1}{2}$, B is not a martingale process.

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Fractional noise

The fractional noise is a standard noise model in the SDE/SPDE literature

- SDE: classical application of rough-paths theory

- Heat/wave models: either Skorohod or pathwise interpretation

- Schrödinger: ??

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$$\dot{B}(t,x) = \sum_{k\in\mathbb{Z}} \Big(\int_0^{2\pi} dy\, e^{-\imath ky}\dot{B}(t,y) \Big) e^{\imath kx}.$$

Remark. Not to be confused with the regularized white noise

$$\dot{W}^{(-\alpha)}(t,x) = (\langle\nabla\rangle^{(-\alpha)}\dot{W})(t,x) = \sum_{k\in\mathbb{Z}} \frac{1}{\langle k\rangle^\alpha} \dot{\beta}_t^{(k)} e^{\imath kx}.$$

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Outline

Mild formulation

$$(\imath\partial_t - \Delta)u = |u|^2 + \dot{B}, \quad u_0 = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{T}.$$

Let us recast the dynamics in the mild form:

$$u(t) = -\imath \int_0^t ds\, e^{-\imath(t-s)\Delta} \dot{B}(s,.) - \imath \int_0^t ds\, e^{-\imath(t-s)\Delta} |u(s,.)|^2$$

$$= e^{-\imath t\Delta}\left[-\imath \int_0^t ds\, e^{\imath s\Delta} \dot{B}(s,.) - \imath \int_0^t ds\, e^{\imath s\Delta} \big(u(s,.)\overline{u(s,.)}\big) \right].$$

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we obtain

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Low regularity issue

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To study the wellposedness and regularity of Ψ, consider a sequence of smooth approximations $\dot{B}^{(n)}$ of $\dot{B}$ (e.g., $\dot{B}^{(n)} := \rho_n * \dot{B}$) and set

$$\Psi^{(n)}(t,.) := -\imath \int_0^t ds\, e^{\imath s\Delta}\dot{B}^{(n)}(s).$$

Proposition. Let $\dot{B}$ be a fractional noise of index (H_0, H_1), and $T > 0$.

(i) If $2H_0 + H_1 > 2$, then $(\Psi^{(n)})_{n\geq 1}$ converges a.s. in $L^2(\mathbb{R}\times\mathbb{T})$.

(ii) If $2H_0 + H_1 \leq 2$, then a.s. $\|\Psi^{(n)}\|_{L^2([0,T]\times\mathbb{T})} \to \infty$ as $n \to \infty$.

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Low regularity issue

Proposition. Assume that $2H_0 + H_1 \leq 2$.

Then, almost surely and for every $T > 0$, the sequence of functions

$$(t,x) \mapsto \int_0^t ds\, e^{\imath s\Delta}\Big(\big(e^{-\imath s\Delta}\underset{\bullet}{\circ}^{(n)}(s)\big)\overline{\big(e^{-\imath s\Delta}\underset{\bullet}{\circ}^{(n)}(s)\big)}\Big)(x)$$

fails to converge *in the general space of distributions on* $[-T,T]\times\mathbb{T}$.

Low regularity issue $\Longrightarrow$ renormalization

$\Longrightarrow$ When $2H_0 + H_1 \leq 2$, we need to renormalize our model

$$(\imath\partial_t - \Delta)u^{(n)} = |u^{(n)}|^2 + \dot{B}^{(n)}, \quad u_0^{(n)} = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{T}.$$

$=$ Find "reasonable" correction

$$(\imath\partial_t - \Delta)\tilde{u}^{(n)} = |\tilde{u}^{(n)}|^2 - \sigma^{(n)} + \dot{B}^{(n)},$$

so that $\tilde{u}^{(n)}$ converges as $n \to \infty$.

Of course we want to avoid trivial renormalization procedures, such that

$$\sigma^{(n)} = |\tilde{u}^{(n)}|^2 \quad \text{or} \quad \sigma^{(n)} = \dot{B}^{(n)}, \quad \dots$$

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Outline

Three examples from the SPDE literature

First example: **Heat** Φ^4_3 **model**

$$(\partial_t - \Delta)u = u^3 + \dot{B}, \quad u_0 = 0, \quad t \in [0,T],\ x \in \mathbb{T}^3.$$

Theorem (Hairer 12', "Wick" renormalization).
Let $\dot{B}$ *be a space-time white noise on* $\mathbb{R}_+ \times \mathbb{T}^3$*, and* $\dot{B}^{(n)} := \rho_n * \dot{B}$*.*
Then there exists a sequence $(\Lambda^{(n)})_{n \geq 1}$ *such that:*

(i) for every $n \geq 1$, $\Lambda^{(n)} \in \mathbb{R}$ is a deterministic constant,

(ii) $\Lambda^{(n)}$ can be explicitly described in terms of $\dot{B}^{(n)}$

(iii) $\Lambda^{(n)} \to \infty$ as $n \to \infty$,

(iv) the renormalized equation

$$\boxed{(\partial_t - \Delta)u^{(n)} = (u^{(n)})^3 - \Lambda^{(n)} u^{(n)} + \dot{B}^{(n)}},\ u_0^{(n)} = 0,\ t \in [0,T],\ x \in \mathbb{T}^3,$$

converges a.s. in $\mathcal{C}([0,T]; \mathcal{H}^{-\alpha}(\mathbb{T}^3))$*, for some* $\alpha > 0$.

Three examples from the SPDE literature

Second example: **Quadratic wave in 3d**

$$(\partial_t^2 - \Delta)u = u^2 + \dot{B}, \quad u_0 = 0, \quad t \in [0, T],\ x \in \mathbb{T}^3,$$

Theorem (Gubinelli-Koch-Oh 18', "Wick" renormalization).
Let $\dot{B}$ be a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^3$, and $\dot{B}^{(n)} := \rho_n * \dot{B}$. Then there exists a sequence $(\sigma^{(n)})_{n\geq 1}$ such that:

(i) for every $n \geq 1$, $\sigma^{(n)} : \mathbb{R}_+ \to \mathbb{R}$ is a deterministic time function,

(ii) for every $t \geq 0$, $\sigma^{(n)}(t)$ can be explicitly described in terms of $\dot{B}^{(n)}$,

(iii) for every $t > 0$, $\sigma^{(n)}(t) \to \infty$ as $n \to \infty$,

(iv) the renormalized equation

$$\boxed{(\partial_t^2 - \Delta)u^{(n)} = (u^{(n)})^2 - \sigma^{(n)} + \dot{B}^{(n)}},\ u_0^{(n)} = 0,\ t \in [0, T],\ x \in \mathbb{T}^3,$$

converges a.s. in $\mathcal{C}([0, T]; \mathcal{H}^{-\alpha}(\mathbb{T}^3))$, for some $\alpha > 0$.

Three examples from the SPDE literature

Third example: **Cubic NLS with rough random initial condition**

$$(\imath\partial_t - \Delta)u = |u|^2 u, \quad u_0 = \Phi, \quad t \in [-T, T],\ x \in \mathbb{T}.$$

Theorem (Colliander-Oh 10', "Bourgain" renormalization).

Let Φ be in a suitable space of rough random distributions on $\mathbb{T}$, and denote by $(\Phi^{(n)})_{n\geq 1}$ its Fourier approximation. Then setting

$$\Lambda^{(n)}(\omega) := 2 \int_{\mathbb{T}} dx\, |\Phi^{(n)}(\omega, x)|^2,$$

the renormalized equation

$$\boxed{(\imath\partial_t - \Delta)u^{(n)} = |u^{(n)}|^2 u^{(n)} - \Lambda^{(n)} u^{(n)}}, \quad u_0^{(n)} = \Phi^{(n)},\ t \in [-T, T],\ x \in \mathbb{T},$$

converges a.s. in $\mathcal{C}([-T, T]; \mathcal{H}^{-\alpha}(\mathbb{T}))$, for some $\alpha > 0$.

Remark.

(i) $\Lambda^{(n)}$ is explicitly described in terms of $\Phi^{(n)}$.

(ii) $\Lambda^{(n)}$ does not depend on (t, x), but *it still depends on* ω.

"Guidelines" for the renormalization terms:

• Depend explicitly on the noise $\dot{B}$, and not on the solution u.

• Offer some "reduction" in the variables, i.e. not depend simultaneously on t, x and ω

Outline

Theorem (D-F-T 23). Let $H_0, H_1 > \frac{1}{2}$ be such that $\frac{7}{4} < 2H_0 + H_1 < 2$, and set

$$\Lambda^{(n)}(t) := \int_{\mathbb{T}} \overline{\stackrel{\circ}{|}^{(n)}(t,x)}\, dx, \quad \sigma^{(n)}(t) := \sum_{k\neq 0} \mathbb{E}\Big[\Big|\stackrel{\circ}{|}^{(n)}_k(t)\Big|^2\Big].$$

Then:

(i) For every $0 < t \leq 1$, $\sigma^{(n)}(t) \overset{n\to\infty}{\sim} c_H\, t\, 2^{2n(2-(2H_0+H_1))}$.

(ii) There exists a (random) time $T_0 > 0$ such that the sequence

$$\boxed{(\imath\partial_t - \Delta)u^{(n)} = |u^{(n)}|^2 - \big[\Lambda^{(n)} u^{(n)} + \sigma^{(n)}\big] + \dot{B}^{(n)}}$$

converges a.s. to some limit u in $\mathcal{C}([-T_0, T_0]; \mathcal{H}^{-\alpha}(\mathbb{T}))$, for some $\alpha > 0$.

Some details about the strategy. Start from the rescaled model

$$(\imath\partial_t - \Delta)u = |u|^2 - \Lambda \cdot u - \sigma + \dot{B}, \quad u_0 = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{T},$$

for some fixed functions $\Lambda : \mathbb{R} \to \mathbb{C}$ and $\sigma : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$.

Mild form:

$$u(t) = e^{-\imath t\Delta}\Big[-\imath \int_0^t ds\, e^{\imath s\Delta}\dot{B}(s,.) - \imath \int_0^t ds\, e^{\imath s\Delta}\big(u(s,.)\overline{u(s,.)}\big) + \imath \int_0^t ds\, \Lambda(s)e^{\imath s\Delta}u(s,.) + \imath \int_0^t ds\, e^{\imath s\Delta}\sigma(s,.)\Big].$$

Setting

$$\mathring{u}(t) := e^{\imath t\Delta}u(t) \quad \text{and} \quad \overset{\circ}{\vert}(t,.) := -\imath \int_0^t ds\, e^{\imath s\Delta}\dot{B}(s) \quad (\text{“linear solution”})$$

we obtain

$$\mathring{u}(t) = \overset{\circ}{\vert}(t) - \imath \int_0^t ds\, e^{\imath s\Delta}\big((e^{-\imath s\Delta}\mathring{u}(s))\overline{(e^{-\imath s\Delta}\mathring{u}(s))}\big) + \imath \int_0^t ds\, \Lambda(s)\mathring{u}(s,.) + \imath \int_0^t ds\, e^{\imath s\Delta}\sigma(s,.).$$

Setting

$$\mathcal{I}(v)(t) = -\imath \int_0^t v(r)\, dr \quad \text{(time integration)},$$

$$\mathcal{M}(v,w)(s) = e^{\imath s\Delta}\big((e^{-\imath s\Delta} v(s))\overline{(e^{-\imath s\Delta} w(s))}\big) \quad \text{(“Schrödinger” product)},$$

the equation can be written as

$$\boxed{\mathring{u} = \Psi + \mathcal{I}\mathcal{M}(\mathring{u}, \mathring{u}) - \mathcal{I}\big(\Lambda \cdot \mathring{u}\big) - \mathcal{I}\big(e^{\imath . \Delta}\sigma\big).}$$

Da Prato-Debussche trick: define

$$z := \mathring{u} - \Psi,$$

and thus recast the equation into the *remainder* equation

$$z = \mathcal{I}\mathcal{M}(z + \Psi, z + \Psi) - \mathcal{I}\big(\Lambda \cdot (z + \Psi)\big) - \mathcal{I}\big(e^{\imath . \Delta}\sigma\big),$$

or otherwise stated

$$\boxed{\begin{aligned} z = \mathcal{I}\mathcal{M}(z,z) - \mathcal{I}\big(\Lambda \cdot z\big) &+ \Big[\mathcal{I}\mathcal{M}(z, \Psi) + \mathcal{I}\mathcal{M}(\Psi, z)\Big] \\ &+ \Big[\mathcal{I}\mathcal{M}(\Psi, \Psi) - \mathcal{I}\big(\Lambda \cdot \Psi\big) - \mathcal{I}\big(e^{\imath . \Delta}\sigma\big)\Big]. \end{aligned}}$$

Proposition (D-F-T). ("Bourgain-Wick" renormalization)

Assume that $H_0, H_1 > \frac{1}{2}$ satisfy $\frac{7}{4} < 2H_0 + H_1 < 2$, and set

$$\Lambda^{(n)}(t) := \int_{\mathbb{T}} \overline{\Psi^{(n)}(t,x)}\, dx, \qquad \text{("Bourgain")}$$

$$\sigma^{(n)}(t) := \sum_{k \neq 0} \mathbb{E}\Big[\big|\Psi^{(n)}_k(t)\big|^2\Big], \qquad \text{("Wick")}$$

as well as

$$\curlyvee^{(n)} := \mathcal{I}\mathcal{M}\big(\Psi^{(n)}, \Psi^{(n)}\big) - \mathcal{I}\big(\Lambda^{(n)} \cdot \Psi^{(n)}\big) - \mathcal{I}\big(e^{\imath . \Delta} \sigma^{(n)}\big).$$

Then there exist $b \in (\frac{1}{2}, 1)$ and $s \in (0, \frac{1}{2})$ such that a.s.

$$\curlyvee^{(n)} \to \curlyvee \quad \text{in } Z^{s,b}. \quad \Big(\|z\|^2_{Z^{s,b}} := \sum_k \langle k \rangle^{2s} \int d\lambda\, \langle \lambda \rangle^{2b} \big|\mathcal{F}(z_k)(\lambda)\big|^2\Big)$$

Proposition (D-F-T). Given $z \in Z^{s,b}$, we can interpret and control

$$\mathcal{I}\mathcal{M}(z,z),\ \mathcal{I}(\Lambda \cdot z),\ \mathcal{I}\mathcal{M}(z, \Psi),\ \mathcal{I}\mathcal{M}(\Psi, z) \quad \text{in } Z^{s,b}.$$

Proposition (D-F-T). ("Bourgain-Wick" renormalization)

Assume that $H_0, H_1 > \frac{1}{2}$ satisfy $\frac{7}{4} < 2H_0 + H_1 < 2$, and set

$$\Lambda^{(n)}(t) := \int_{\mathbb{T}} \overline{\mathrm{I}^{(n)}(t,x)}\, dx, \qquad (\text{"Bourgain"})$$

$$\sigma^{(n)}(t) := \sum_{k \neq 0} \mathbb{E}\Big[\big|\mathrm{I}_k^{(n)}(t)\big|^2\Big], \qquad (\text{"Wick"})$$

as well as

$$\Upsilon^{(n)} := \mathcal{I}\mathcal{M}(\mathrm{I}^{(n)}, \mathrm{I}^{(n)}) - \mathcal{I}\big(\Lambda^{(n)} \cdot \mathrm{I}^{(n)}\big) - \mathcal{I}\big(e^{\imath . \Delta} \sigma^{(n)}\big).$$

Then there exist $b \in (\frac{1}{2}, 1)$ and $s \in (0, \frac{1}{2})$ such that a.s.

$$\Upsilon^{(n)} \to \Upsilon \quad \text{in } Z^{s,b}. \quad \Big(\|z\|^2_{Z^{s,b}} := \sum_k \langle k \rangle^{2s} \int d\lambda\, \langle \lambda \rangle^{2b} \big|\mathcal{F}(z_k)(\lambda)\big|^2\Big)$$

Proposition (D-F-T). Given $z \in Z^{s,b}$, we can interpret and control

$$\mathcal{I}\mathcal{M}(z,z),\ \mathcal{I}\big(\Lambda \cdot z\big),\ \mathcal{I}\mathcal{M}(z, \mathrm{I}),\ \mathcal{I}\mathcal{M}(\mathrm{I}, z) \quad \text{in } Z^{s,b}.$$

Perspectives

Proposition. Assume that $H_0 = H_1 = \frac{1}{2}$, that is $\dot{B}$ is a space-time white noise on $\mathbb{R} \times \mathbb{T}$. Then for every $b \geq \frac{1}{2}$, it holds that

$\mathbb{E}\Big[\Big\|$  $^{(n)}\Big\|^2_{Z^{s,b}}\Big] \overset{n\to\infty}{\longrightarrow} \infty.$

$\Longrightarrow$ space-time white noise is essentially out of reach.

Questions:

- What happens when $\frac{3}{2} \leq 2H_0 + H_1 < \frac{7}{4}$?
- Need for some higher-order expansion ?
- Need for some a-priori expansion of the solution ?
- $d \geq 2$? Cubic nonlinearity ?

Thank you!