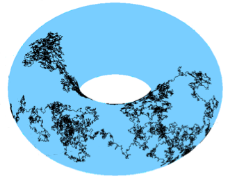
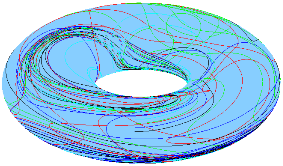
Kinetic Brownian motion in the diffeomorphism group of a closed Riemannian manifold





Joint work with J. Angst and P. Perruchaud (Rennes)

1. Kinetic Brownian motion in $\mathbb{R}^d$

- Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$dx_t = \dot{x}_t \, dt,$$
$$\dot{x}_t = B_{\sigma^2 t},$$

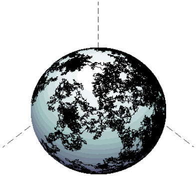
with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.

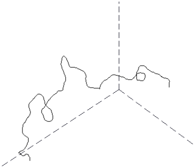
1. Kinetic Brownian motion in $\mathbb{R}^d$

- Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$dx_t = \dot{x}_t \, dt,$$
$$\dot{x}_t = B_{\sigma^2 t},$$

with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.



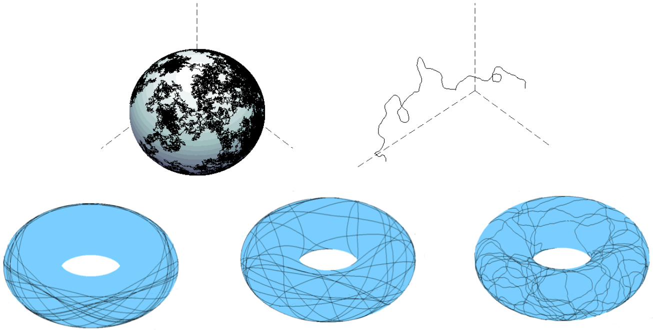


1. Kinetic Brownian motion in $\mathbb{R}^d$

- Definition. **Kinetic Brownian motion $(x_t, \dot{x}_t)$ in $\mathbb{R}^d$** is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$
\begin{aligned}
dx_t &= \dot{x}_t \, dt, \\
\dot{x}_t &= B_{\sigma^2 t},
\end{aligned}
$$

with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.

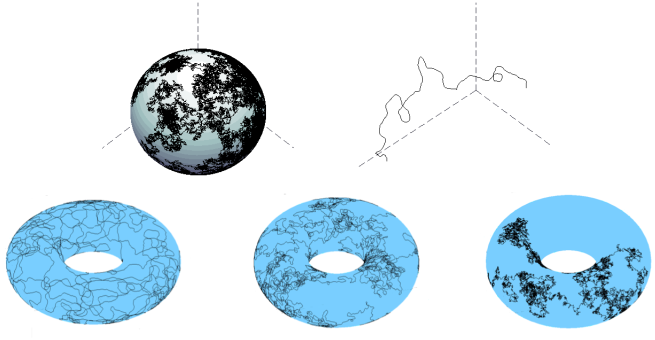


1. Kinetic Brownian motion in $\mathbb{R}^d$

- Definition. **Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$ is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$
\begin{aligned}
dx_t &= \dot{x}_t \, dt, \\
\dot{x}_t &= B_{\sigma^2 t},
\end{aligned}
$$

with B Brownian motion on $\mathbb{S}^{d-1}$, with parameter $\sigma \in [0, \infty)$.



1. Kinetic Brownian motion in $\mathbb{R}^d$

▶ Theorem – **Homogenization.** *The time-rescaled position process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly to a Euclidean Brownian motion with generator* $\frac{4}{d(d-1)}\Delta$, *as* $\sigma \uparrow \infty$.

Idea of proof. The dynamics is given by the SDE

$$
\begin{aligned}
dx_t^i &= \dot{x}_t^i \, dt, \\
d\dot{x}_t^i &= -\sigma^2 \frac{d-1}{2} \dot{x}_t^i \, dt + \sigma \sum_{j=1}^{d} \left(\delta^{ij} - \dot{x}_t^i \dot{x}_t^j\right) dW_t^j.
\end{aligned}
$$

Set $X_t^\sigma := x_{\sigma^2 t}$. Then

$$
X_t^\sigma = x_0 + \frac{2}{d-1} \frac{1}{\sigma^2} \left(\dot{x}_0 - \dot{x}_{\sigma^2 t}\right) + M_t^\sigma,
$$

with

$$
\left\langle M^{\sigma,i}, M^{\sigma,j} \right\rangle_t = \frac{4}{(d-1)^2} \frac{1}{\sigma^2} \int_0^{\sigma^2 t} \left(\delta^{ij} - \dot{x}_s^{\sigma,i} \dot{x}_s^{\sigma,j}\right) ds.
$$

Use **ergodic theorem** and **functional CLT** to conclude. ◁

1. Kinetic Brownian motion in $\mathbb{R}^d$

▶ Theorem – **Homogenization.** *The time-rescaled position process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly to a Euclidean Brownian motion with generator* $\frac{4}{d(d-1)}\Delta$, *as* $\sigma\uparrow\infty$.

Idea of proof. The dynamics is given by the SDE

$$dx_t^i = \dot{x}_t^i\,dt,$$

$$d\dot{x}_t^i = -\sigma^2\frac{d-1}{2}\dot{x}_t^i\,dt + \sigma\sum_{j=1}^{d}\left(\delta^{ij} - \dot{x}_t^i\dot{x}_t^j\right)dW_t^j.$$

Set $X_t^\sigma := x_{\sigma^2 t}$. Then

$$X_t^\sigma = x_0 + \frac{2}{d-1}\,\frac{1}{\sigma^2}\left(\dot{x}_0 - \dot{x}_{\sigma^2 t}\right) + M_t^\sigma,$$

with

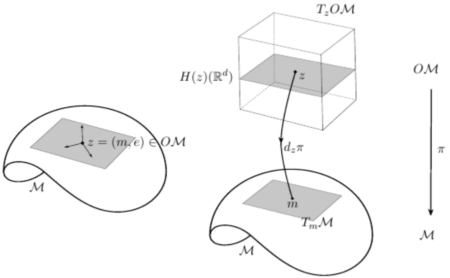
$$\left\langle M^{\sigma,i}, M^{\sigma,j}\right\rangle_t = \frac{4}{(d-1)^2}\,\frac{1}{\sigma^2}\int_0^{\sigma^2 t}\left(\delta^{ij} - \dot{x}_s^{\sigma,i}\dot{x}_s^{\sigma,j}\right)ds.$$

Use **ergodic theorem** and **functional CLT** to conclude. ◁

2. Manifold-valued Kinetic Brownian motion

Let (M, g) be a d-dimensional Riemannian manifold.

- **Cartan development**: a useful way to construct Brownian motion on M. Let $\pi : OM \to M$, stand for the **orthonormal frame bundle** over M; generic point $z = (m, e)$, with e orthonormal basis of $T_m M$. For $z \in OM$, let $H(z) \in L(\mathbb{R}^d, T_z OM)$ stand for the (metric-dependent) horizontal form at z.



$$dz_t = H(z_t) \circ dW_t, \text{ in OM}, \qquad w_t := \pi(z_t), \text{ in M}.$$

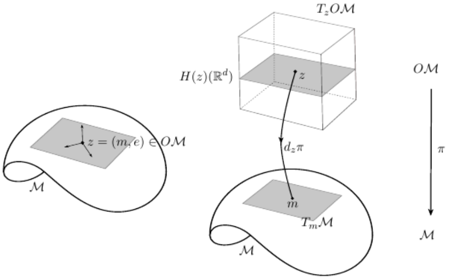
- Definition. **Kinetic Brownian motion** m_t^σ **in** M **via Cartan development.** For X_t^σ time rescaled kinetic Brownian motion in $\mathbb{R}^d$, set

$$dz_t^\sigma = H(z_t^\sigma)\, dX_t^\sigma, \text{ in OM}, \qquad m_t^\sigma := \pi(z_t^\sigma), \text{ in M}.$$

2. Manifold-valued Kinetic Brownian motion

Let (M, g) be a d-dimensional Riemannian manifold.

- **Cartan development**: a useful way to construct Brownian motion on M. Let $\pi : OM \to M$, stand for the **orthonormal frame bundle** over M; generic point $z = (m, e)$, with e orthonormal basis of $T_m M$. For $z \in OM$, let $H(z) \in L(\mathbb{R}^d, T_z OM)$ stand for the (metric-dependent) horizontal form at z.



$$dz_t = H(z_t) \circ dW_t, \text{ in OM}, \qquad w_t := \pi(z_t), \text{ in M}.$$

- Definition. **Kinetic Brownian motion** m_t^σ **in** M **via Cartan development.** For X_t^σ time rescaled kinetic Brownian motion in $\mathbb{R}^d$, set

$$dz_t^\sigma = H(z_t^\sigma)\, \dot{X}_t^\sigma\, dt, \text{ in OM}, \qquad m_t^\sigma := \pi(z_t^\sigma), \text{ in M}.$$

2. Manifold-valued Kinetic Brownian motion

► Theorem – **Homogenization.** *Assume (M, g) is complete and stochastically complete. Then the process $\left(m_t^\sigma\right)_{0\leq t\leq 1}$ converges weakly to a Brownian motion with generator $\frac{4}{d(d-1)}\Delta$, as $\sigma \uparrow \infty$.*

Idea of proof. Back in $\mathbb{R}^d$ with time rescaled kinetic Brownian motion X_t^σ. Prove that the canonical rough path lift $\mathbf{X}^\sigma$ of $(X_t^\sigma)_{0\leq t\leq 1}$ converges weakly in a rough path sense to the Stratonovich Brownian rough path.

- Prove first weak convergence in uniform norm of $\mathbf{X}^\sigma$ to the Stratonovich Brownian rough path, using weak convergence results on stochastic integrals.
- Prove σ-uniform moment bounds on X_{ts}^σ and $\int_s^t X_{us}^\sigma \otimes dX_u^\sigma$, and use Lamperti-type tightness result.

Use the continuity of the Itô-Lyons solution map for the equation

$$dz_t^\sigma = H(z_t^\sigma)\, dX_t^\sigma = H(z_t^\sigma)\, d\mathbf{X}_t^\sigma, \quad z_t^\sigma \in OM,$$

to transport weak convergence of $\mathbf{X}^\sigma$ from the rough paths side to the dynamics on OM and M. ◁

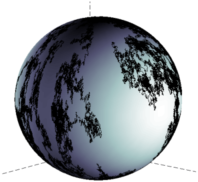
3. Anisotropic Kinetic Brownian motion in $\mathbb{R}^d$

Let Σ be a positive-definite symmetric matrix – no loss in assuming $\Sigma = \mathrm{diag}(\alpha_i^2)$.

► Definition. **Anisotropic Kinetic Brownian motion** $(x_t, \dot{x}_t)$ **in** $\mathbb{R}^d$, with anisotropy Σ, is the hypoelliptic diffusion with state space $\mathbb{R}^d \times \mathbb{S}^{d-1}$

$$dx_t = \dot{x}_t \, dt,$$
$$d\dot{x}_t = \sigma P_{\dot{x}_t} \circ dW_t,$$

where W is an $\mathbb{R}^d$-valued Brownian motion with covariance Σ, and $P_{\dot{x}} : \mathbb{R}^d \to \langle \dot{x} \rangle^\perp$, the orthogonal projection. (Note $\langle \dot{x} \rangle^\perp = T_{\dot{x}} \mathbb{S}^{d-1}$.)



3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

- Theorem – **Homogenization.**
  - *The invariant measure* μ *of the velocity process* $\dot{x}$ *on the sphere is the image by the radial projection on the sphere of the measure on* $\mathbb{R}^d$ *with density* $|x|^{-1}$ *wrt the Gaussian measure with covariance* Σ*.*
  - *The time-rescaled process* $(x_{\sigma^2 t})_{0\leq t\leq 1}$ *converges weakly as* $\sigma \uparrow \infty$ *to a Euclidean Brownian motion with covariance matrix* $\mathrm{diag}(\gamma_i)$*, with*

$$\gamma_i := 2\int_0^\infty \mathbb{E}_\mu\left[\dot{x}_0^i\,\dot{x}_t^i\right]dt, \quad 1\leq i\leq d.$$

  - We have weak convergence of the associated rough path $\mathbf{X}^\sigma$ to the corresponding Stratonovich Brownian rough path, as $\sigma\uparrow\infty$.

Idea of proof. The dynamics of velocity $\dot{x}_t$ is given by the SDE

$$d\dot{x}_t^i = -\frac{\sigma^2}{2}\left(\alpha_i^2 + \sum_{k=1}^d \alpha_k^2 - 2\sum_{\ell=1}^d \alpha_\ell^2\left|\dot{x}_t^\ell\right|^2\right)\dot{x}_t^i\,dt + \sigma\left(\alpha_i dW_t^i - \dot{x}_t^i\sum_{\ell=1}^d \alpha_\ell \dot{x}_t^\ell dW_t^\ell\right)$$

No clear description of $X_t^\sigma = x_{\sigma^2 t}$, when Σ different from a constant multiple of identity.

Give up the analysis of the SDE and **use ergodic properties of** $\dot{x}$.

3. Anisotropic Kinetic Brownian motion on $\mathbb{R}^d$

1. We have for any probability measure λ on $\mathbb{S}^{d-1}$

$$\left\| P_t^* \lambda - \mu \right\|_{\mathrm{TV}} \lesssim e^{-ct},$$

for some positive constant c. This implies σ-uniform moment estimates

$$\sup_{\sigma \geq 0} \left\| X_t^\sigma - X_s^\sigma \right\|_{L^p} \lesssim |t-s|^{p/2},$$
$$\sup_{\sigma \geq 0} \left\| \mathbb{X}_{ts}^\sigma \right\|_{L^p} \lesssim |t-s|^{p},$$

where $\mathbb{X}_{ts}^\sigma := \int_s^t X_{us}^\sigma \otimes dX_u^\sigma$, implying **tightness** for the laws of the canonical rough paths $\mathbf{X}^\sigma$ associated with anisotropic kinetic Brownian motion.

2. We prove that any limit law turns the canonical process on the rough paths space into a continuous **Lévy process**. We identify its generator using the invariance of the invariant measure μ by the **symmetries**

$$(\theta_1, \ldots, \theta_d) \in \mathbb{S}^{d-1} \mapsto (\theta_1, \ldots, \theta_{i-1}, -\theta_i, \theta_{i+1}, \ldots, \theta_d) \in \mathbb{S}^{d-1}. \qquad \triangleleft$$

4. Geometry of the diffeomorphism group

- (M, g) a Riemannian manifold = **domain of the fluid flow**,
  $\mathscr{D} := \{\text{Diffeo of } M\}$ or $H^s(M, M)$: a Fréchet/Hilbert manifold,
  $T_\varphi \mathscr{D} = \{\text{smooth}/H^s \text{ 'vector fields' at } \varphi\} = \{m \in M \to u(m) \in T_{\varphi(m)} M\}$.

(Variant with volume preserving diffeomorphism group and divergence-free vector fields on M.)

- **Weak Riemannian metric** on $\mathscr{D}$

$$\langle u, v \rangle := \int_M g_{\varphi(m)}\big(u(m), v(m)\big) \, \mathrm{VOL}_g(dm).$$

Induced topology on $\mathscr{D}$ weaker than smooth or H^s topology. There may be no good notion of parallel transport... But Ebin-Marsden (69') prove there is one! It is a *smooth map*, and its *exponential map* is *well-defined and smooth* in a neighbourhood of the zero section of $T\mathscr{D}$.

Geodesics on the 'submanifold' of volume preserving diffeomorphisms are solution of **Euler's equation for incompressible fluids**

$$\partial_t u + u \nabla u + \nabla p = 0,$$

for a pressure field $p : M \to \mathbb{R}$. (V.I. Arnol'd, 66')

4. Geometry of the diffeomorphism group

On the 2-dimensional torus, for the group of volume preserving diffeomorphisms.

- Orthonormal basis of $\mathrm{L{\scriptsize IE}}(\mathscr{D})$, $k \in \mathbb{Z}\backslash\{0\}$

$$A_k = |k|^{-1}\big(k_2 \cos(k\cdot\theta)\partial_1 - k_1 \cos(k\cdot\theta)\partial_2\big),$$
$$B_k = |k|^{-1}\big(k_2 \sin(k\cdot\theta)\partial_1 - k_1 \sin(k\cdot\theta)\partial_2\big).$$

- Geodesic equation $u := \partial_t\varphi \circ \varphi^{-1}$

$$\partial_t u + \Gamma(u,u) = 0,$$

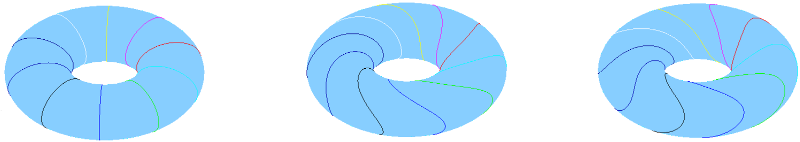
with explicit Christoffel symbols Γ, e.g.

$$\Gamma(A_k, A_\ell) = [k,\ell]\big(\alpha_{k,\ell} B_{k+\ell} + \beta_{k,\ell} B_{k-\ell}\big).$$

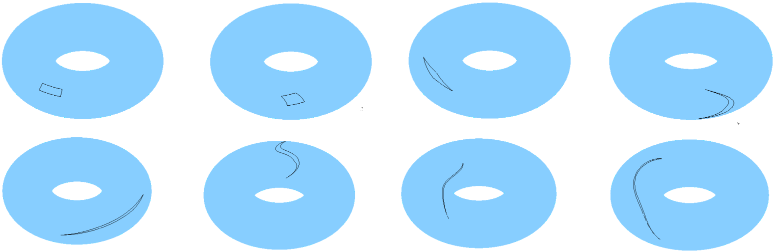
$\Gamma(A_k,\cdot), \Gamma(B_k,\cdot)$ unbounded antisymmetric operators that do not induce nice evolutions on the "orthonormal group" in $\mathrm{L{\scriptsize IE}}(\mathscr{D})$.

4. Geometry of the diffeomorphism group

- **Time 1 flow with $\sigma = 0$, for different initial momentum in volume preserving diffeomorphism group.**



- **Evolution of an area element along geodesic motion in volume preserving diffeomorphism group.**



5. Kinetic Brownian motion in the diffeomorphism group

1. On $\text{Lie}(\mathscr{D}) \simeq H^s(TM)$. Write $\mathbb{S}$ for unit sphere of $H^s(TM)$,

$$du_t = \dot{u}_t\, dt,$$
$$d\dot{u}_t = \sigma P_{\dot{u}_t} \circ dW_t,$$

with W an $H^s(TM)$-valued (anisotropic!) Brownian motion – with trace-class covariance operator Σ.

2. Follow Ebin-Marsden' strategy, showing one can formulate **Cartan's development** operation as solving nice ODE on the infinite-dimensional configuration space (= a substitue for the orthonormal frame bundle above $\mathscr{D}$)

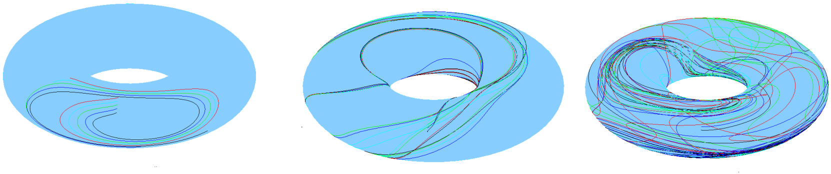
$$TH^s(\mathcal{F}M) \times L(H^s(TM)),$$

driven by a *smooth* vector field. Set $\varphi_t :=$ projection of dynamics on the diffeomorphism space $\mathscr{D}$.

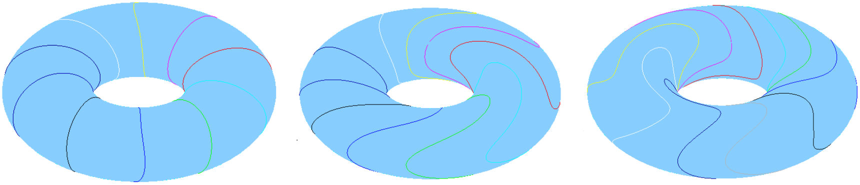
(Variant for volume preserving diffeomorphism group and divergence-free vector fields on M.)

5. Kinetic Brownian motion in the diffeomorphism group

- **Examples of flows with time, for noise parameter $\sigma = 1$.**



- **Time 1 snapshots for increasing noise parameter σ, with same initial momentum.**



5. Kinetic Brownian motion in the diffeomorphism group

Set $U_t^\sigma := u_{\sigma^2 t} \in \text{LIE}(\mathscr{D})$. Wlog $\Sigma = \text{diag}(\alpha_i^2)$, non-increasing eigenvalues α_i.

► Theorem – **Homogenization in LIE($\mathscr{D}$).** Assume $3\,\alpha_1^2 < \text{tr}(\Sigma)$.

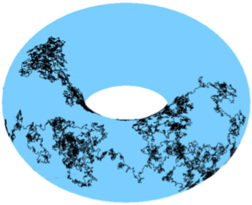
- *The invariant measure μ of the velocity process $\dot{u}$ on the sphere is the image by the radial projection on the sphere of the measure on $\mathbb{H}$ with density $|u|^{-1}$ wrt the Gaussian measure with covariance Σ.*
- *The time-rescaled process $(U_t^\sigma)_{0\le t\le 1}$ converges weakly as $\sigma \uparrow \infty$ to a Brownian motion B in LIE($\mathscr{D}$) with **covariance***

$$\Theta(f) := 2\int_0^\infty \mathbb{E}_\mu\big[f(u_0)\,f(u_t)\big]\,dt, \quad f \in \mathbb{H}'$$

- The rough path lift $\mathbf{U}^\sigma$ of $(U_t^\sigma)_{0\le t\le 1}$ converges to the Stratonovich Brownian rough path associated with B.

Using the above mentioned version of Cartan's development machinery, one can define kinetic Brownian motion in $\mathscr{D}$ in a *small time interval*. (Warning! $\mathscr{D}$ may not be geodesically complete and may have finite diameter.)

► Theorem – **Homogenization in $\mathscr{D}$.** *Kinetic Brownian motion in $\mathscr{D}$ provides an* ***interpolation*** *between the dynamics of a(n incompressible) fluid and the projection on the diffeomorphism group of a Brownian flow on a larger space.*



Thank you!