Regularity structures and paracontrolled calculus

Joint works with M. Hoshino

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1. Pointwise description devices

2. Fourier-type description devices

3. From paracontrolled systems to models and modelled distributions

The multiplication problem in singular PDEs

Singular PDEs = **multiplication problem**, e.g.

$$(\partial_t - \Delta)u = u\,\zeta, \quad \text{in 2-dimensional torus,}$$
$$(\partial_t - \partial_x^2)u = \xi + (\partial_x u)^2, \quad \text{in 1-dimensional torus,}$$

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Leads to **regularity structures, models and modelled distributions**, and **paracontrolled calculus and paracontrolled systems**.

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Regularity structures (**RS**) and paracontrolled calculus (**PC**) have their roots in rough paths theory for ODEs driven by irregular controls

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Our **aim** in a singular PDE setting

microscopic description $\Longleftrightarrow$ **macroscopic** description

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Consistency of repeated re-expansion around different points and requirement that the $g(\tau)$ form a sufficiently rich family to describe an algebra of functions, directly lead to the definition of a concrete regularity structure $\mathscr{T}$ and a model (g, Π) on it.

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    for all $\tau \in T^+$,
  - There is a map $\Pi : T \to \mathcal{S}'(\mathbb{T}^d)$, such that

    $$\Pi_x \tau = (\Pi \otimes g_x^{-1})\Delta\tau$$

    has $C^{|\tau|}$-regularity at x (only).

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$$\mathcal{D}^\gamma(T, g) := \Big\{\mathbf{f} := \big(f^\tau(x)\big)_{\tau \in \mathcal{B},\, x \in \mathbb{T}^d}\,;\, \Big|\mathbf{f}(z) - \widehat{g_{zy}}\big(\mathbf{f}(y)\big), \tau\rangle\Big| \lesssim |z-y|^{\gamma - |\tau|},$$
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Reconstruction theorem (Hairer) – *Given a model* (g,Π) *on a regularity structure* $\mathscr{T}$, *there exists a linear continuous operator*

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$$\Big|\Big\langle \mathbf{Rf} - \sum_\tau f^\tau(x)\Pi_x\tau, \varphi_x^\lambda\Big\rangle\Big| \lesssim \lambda^{|\tau|};$$

this map $\mathbf{R}$ is unique if $\gamma > 0$. It is called the **reconstruction map**.

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Theorem (B.-Hoshino 2018) – *Fix a regularity structure* $\mathscr{T}$ *and a model* $\mathsf{M} = (\mathsf{g}, \Pi)$ *on* $\mathscr{T}$*.*

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By Littlewood-Paley, a distribution $a = \sum a_i$, with a_i smooth and $\mathrm{supp}(\widehat{a_i}) \subset \{\text{annulus of size} \simeq 2^i\}$. Write

$$ab = \sum_{i \ll j} a_i b_j + \sum_{i \sim j} a_i b_j + \sum_{j \ll i} a_i b_j$$
$$= P_a b + \Pi(a,b) + P_b a.$$

The **paraproduct** terms $P_a b, P_b a$ are always well-defined, not the case of the **resonant** term $\Pi(a,b)$. In $P_a b$ one modulates the high frequencies of b by low frequencies of a; one can say that $P_a b$ looks like b in a Fourier sense. Write

$$\Delta\sigma = \sum_{\mu \leq \sigma} \mu \otimes (\sigma/\mu) \in T \otimes T^+, \quad \Delta^+\tau = \sum_{\nu \leq^+ \tau} \nu \otimes (\tau/^+\nu) \in T^+ \otimes T^+.$$

Theorem (B.-Hoshino 2018) – *Fix a regularity structure $\mathscr{T}$ and a model* $\mathrm{M} = (\mathrm{g}, \Pi)$ *on $\mathscr{T}$. One can construct 'reference functions/distributions'* $\big\{[\tau]^{\mathrm{g}} \in C^{|\tau|}(\mathbb{T}^d)\big\}_{\tau \in \mathcal{B}^+}$ *and* $\big\{[\sigma]^{\mathrm{M}} \in C^{|\sigma|}(\mathbb{T}^d)\big\}_{\sigma \in \mathcal{B}}$ *such that*

$$\mathrm{g}(\tau) = \sum_{\mathbf{1} <^+ \nu <^+ \tau} \mathrm{P}_{\mathrm{g}(\tau/^+\nu)}[\nu]^{\mathrm{g}} + [\tau]^{\mathrm{g}},$$

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We talk of **para-remainders** $[\tau]^{\mathsf{g}}, [\sigma]^{\mathsf{M}}$*; they depend continuously on the model M.*

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Theorem (B.-Hoshino 2018) – *Fix a regularity exponent* $\gamma \in \mathbb{R}$, *and a model* $\mathsf{M} = (\mathrm{g}, \Pi)$ *on the regularity structure* $\mathscr{T}$. *One can associate to any modelled distribution*

$$\mathbf{f} = \sum_{\sigma \in \mathcal{B}; |\sigma| < \gamma} f^{\sigma} \sigma \in \mathcal{D}^{\gamma}(T, g),$$

a distribution $[\mathbf{f}]^{\mathsf{M}} \in \mathcal{C}^{\gamma}(\mathbb{T}^{\mathbf{d}})$ *such that one defines a reconstruction* $\mathbf{Rf}$ *of* $\mathbf{f}$ *setting*

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for an $[f^\sigma]^{\mathsf{g}} \in \mathcal{C}^{\gamma - |\sigma|}(\mathbb{T}^d)$. *Moreover, the* **para-remainder map**

$$\mathbf{f} \mapsto \Big([\mathbf{f}]^{\mathsf{M}}, \big([f^\sigma]^{\mathsf{g}}\big)_{\sigma \in \mathcal{B}} \Big)$$

from $\mathcal{D}^\gamma(T, g)$ *to* $\mathcal{C}^\gamma(\mathbb{T}^d) \times \prod_{\tau \in \mathcal{B}} \mathcal{C}^{\gamma - |\tau|}(\mathbb{T}^d)$, *is continuous.*

3. From paracontrolled systems to models and modelled distributions

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Proposition – *If g is given, then for any family $\big([\sigma] \in C^{|\sigma|}(\mathbb{T}^d)\big)_{\sigma\in\mathcal{B}_\bullet,|\sigma|<0}$ there exists a unique model (g, Π) on $\mathscr{T}$ such that*

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One assumes fairly weak assumptions on $\mathscr{T}$, satisfied by all reasonable regularity structures, like the regularity structures used for the study of singular PDEs. Assume in particular $\mathcal{B}^+$ freely generated by $\mathcal{B}^+_\bullet$ and monomials.

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  (1 – Generating set) *There exists a finite subset $\mathcal{G}^+_\bullet$ of $\mathcal{B}^+_\bullet$ such that*

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Theorem (B. Hoshino 2019) – *Under weak assumptions, for any family $\big([\tau]\in C^{|\tau|}(\mathbb{T}^d)\big)_{\tau\in\mathcal{G}^+_\bullet}$ there exists a unique g map on (T^+, Δ^+) such that*

$$g(\tau) = \sum_{\mu<^+\tau,\,\mu\in\mathcal{B}^+} P_{g(\tau/^+\mu)}[\mu]^g + [\tau], \quad \forall\,\tau\in\mathcal{G}^+_\bullet.$$

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Theorem (B. Hoshino 2018) – *For the regularity structures used for singular PDEs, one has a bi-Lipschitz parametrization of the space of admissible models by*

$$\prod_{\sigma\in\mathcal{B}_\bullet,|\sigma|<0} C^{|\sigma|}(\mathbb{T}^d).$$

(Generalizes greatly a result by Tapia and Zambotti (2018) on the parametrization of the set of branched rough paths – they used completely different methods.)

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for $\mathbf{f} = \sum_{\sigma\in\mathcal{B}} f^\sigma\sigma \in \mathcal{D}^\gamma(T, g)$ *– recall* $\mathcal{B}_\bullet = \mathcal{B}\backslash\text{polynomials}$.

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$$\prod_{\sigma\in\mathcal{B}_\bullet} C^{\gamma-|\sigma|}(\mathbb{T}^d),$$

via the paracontrolled representation

$$f^\sigma = \sum_{\sigma<\mu;|\mu|<\gamma} \mathsf{P}_{f^\mu}[\mu/\sigma]^{\mathsf{g}} + [f^\sigma]^{\mathsf{g}}, \tag{1}$$

for $\mathbf{f} = \sum_{\sigma\in\mathcal{B}} f^\sigma \sigma \in \mathcal{D}^\gamma(T, \mathsf{g})$ – recall $\mathcal{B}_\bullet = \mathcal{B}\backslash\text{polynomials}$.

(Proving this statement happens to be equivalent to an extension problem for the map g.)

3. From paracontrolled systems to modelled distributions

Previous assumptions are on the regularity structure. Here is an assumption on a basis of T.

Assumption (H) – *For any $\sigma \in \mathcal{B}_\bullet$, there is no term fo the form $\mu \otimes X^k$ with $k \neq 0$, in the formula for $\Delta\sigma$.*

Proposition (B. Hoshino 2019) – *The regularity structures built be Bruned, Hairer and Zambotti for the study of singular PDEs have a basis that satisfy this assumption. (The natural basis does not satisfy it!)*

Theorem (B. Hoshino 2019) – *Let $\mathscr{T}$ be a reasonable regularity structures satisfying further assumption* **(H)**. *Let (g, Π) be a model on $\mathscr{T}$. The space $\mathcal{D}^\gamma(T, g)$ of modelled distributions is bi-Lipscthiz homeomorphic to the product space*

$$\prod_{\sigma \in \mathcal{B}_\bullet} C^{\gamma - |\sigma|}(\mathbb{T}^d),$$

via the paracontrolled representation

$$f^\sigma = \sum_{\sigma < \mu; |\mu| < \gamma} \mathsf{P}_{f^\mu} [\mu/\sigma]^{\mathsf{g}} + [f^\sigma]^{\mathsf{g}}, \qquad (1)$$

for $\mathbf{f} = \sum_{\sigma \in \mathcal{B}} f^\sigma \sigma \in \mathcal{D}^\gamma(T, g)$ – *recall* $\mathcal{B}_\bullet = \mathcal{B} \backslash \text{polynomials}$.

The use of paracontrolled systems like (1) are the starting point of the paracontrolled approach to singular PDEs.

Thank you for your attention!