Spatial coagulation with bounded coagulation rate

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Abstract. We prove that the spatial coagulation equation with bounded coagulation rate is well posed for all times in a given class of kernels if the convection term of the underlying particle dynamics has divergence bounded below by a positive constant. Multiple coagulations, fragmentation, and scattering are also considered.

1. Introduction

This article is concerned with Smoluchowski's picture of a chemical reaction involving two kinds of dynamics. Particles involved in the reaction are described by their location $x \in \mathbb{R}^n$ and an inner structure, which we describe as an element y of an Abelian semigroup. It can be its mass, in $\mathbb{R}^*$ or $\mathbb{N}$, or a finer structure saying for instance which smaller masses compose that mass, as in [11]. On the one hand, particles move as a result of the action of some convection flow and thermal diffusive effects. On the other hand, close particles may coagulate to form a new particle with a new characteristic. We shall mainly be interested here in binary collisions. Not to overload notations, let us describe a particle by its position $x \in \mathbb{R}^n$ and its mass $y \in \mathbb{R}_+^*$.

The evolution of the system is described by a time-dependent kernel $\mu_t(x\,;dy)$ from $\mathbb{R}^n$ to $\mathbb{R}_+^*$, with $\mu_t(x\,;(\alpha,\beta))$ representing the concentration of particles at location x with a mass in between α and β. Let K be a measurable symmetric function on $\mathbb{R}_+^* \times \mathbb{R}_+^*$. Given a signed measure μ on $\mathbb{R}_+^*$, define a signed measure $K(\mu,\mu)$ on $\mathbb{R}_+^*$ setting

$$K(\mu,\mu) = \frac{1}{2} \int \left\{ \delta_{y+y'} - \delta_y - \delta_{y'} \right\} K(y,y')\, \mu(dy)\mu(dy').$$

Smoluchowski's picture of the microscopic dynamics leads heuristically to the formal evolution equation on kernels

$$\partial_t \mu_t = L_x^* \mu_t + K(\mu_t, \mu_t), \qquad (1.1)$$

where $L_x = \frac{1}{2} a^{ij}(x, y) \partial_{x_i} \partial_{x_j} + b^i(x, y) \partial_{x_i}$ is a convection-diffusion differential operator. This heuristic result was proved in an important special case by Norris in [12] and later in [4] and [14] by Hammond, Rezakhanlou, and Yaghouti. Well-posedness for equation (1.1) has been investigated in a relatively small number of works. Amann's article [1] is the only work where the problem is tackled in this generality. Supposing all the coefficients in the dynamics bounded, he obtains the existence of a unique maximal solution depending continuously on the initial condition using highly nontrivial functional analytic tools. Under some smoothness and moment hypothesis on the initial condition, he is able to prove well-posedness for all times in space dimension 1 or if the diffusivity coefficients do not depend on y, and $b = 0$. This work was partly extended in [2], in the Brownian case where $b = 0$ and

$$\frac{1}{2} a^{ij}(x, y) \partial_{x_i} \partial_{x_j} = d(y)\triangle_x, \qquad (1.2)$$

with $d(y)$ bounded away from 0 and ∞, and well-posedness for all times was proved under some moment and size conditions on the initial condition. They allow linear growth for the coagulation rate. The works [3,8–10], identify different settings in which one can prove the existence of *a* weak solution to equation (1.1) (in different appropriate senses), defined for all times and for unbounded coagulation kernels controlled in different ways. They all investigate equation (1.1) without convection and with a diffusion of the form (1.2). Hammond and Rezakhanlou are able to give in [5] a well-posedness result for all times for a discrete mass space, a bounded decreasing diffusivity $d(y)$, and some moment conditions. (See also [13].)

All these works investigate the differential problem (1.1) in different functional settings. Norris [12] and Laurencot-Mischler [9] consider a mild integral formulation of equation (1.1) rather than its differential counterpart. They work with no convection term and a diffusivity of the form (1.2). Relying crucially on Gaussian estimates, Norris is able to prove a well-posedness result for all times under some weak control on the coagulation rate and some moment condition on the initial condition. His method does not apply to position-dependent diffusivity dynamics. Consult also [13] for a recent existence and uniqueness result under some moment assumptions.

We refer the reader to the above works for further and older references on the spatial coagulation equation (1.1). Note yet that the methods developed in [2] and [8,9], should be sufficiently robust to deal with more general dynamics than (1.2) on bounded smooth domains as they essentially rely on some compactness property of the operator.

Following Amann's view [1], we consider in this work a bounded coagulation kernel and a general convection-diffusion spatial dynamics. It is quite likely that under some reasonably general conditions on the convection-diffusion coefficients, equation (1.1) is well-posed for all times in a good functional framework. Despite all the subtle works done, this remains yet an open problem. We identify in this work a previously unnoticed case where this problem can be solved. Roughly speaking, a condition on the divergence of the convection term prevents the particles in the underlying microscopic

dynamics from concentrating too much; this enables us to obtain some uniform bound in the spatial coordinate. Although far from opening a road to solving the general problem, the simplicity of our framework and argument contrasts with the sophistications developed in the above references.

We suppose that the coefficients a^{ij} and b^i are measurable functions of $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+^*$, of class C^∞ as functions of $x \in \mathbb{R}^n$, and with all their derivatives bounded, uniformly in $y \in \mathbb{R}_+^*$; this allows linear growth of the drift. To give an integral form of (1.1), let us suppose here that one can write the matrix a as $\sigma\sigma^*$ for some bounded matrix-valued measurable function σ on $\mathbb{R}^n \times \mathbb{R}_+^*$, of class C^∞ as a function of x. Denote by $\phi_t(x, y)$ the well-defined random flow on $\mathbb{R}^n \times \mathbb{R}_+^*$ of the stochastic differential equation

$$
\begin{aligned}
dx_t &= \sigma(x_t, y_t)\, dB_t + b(x_t, y_t)dt, \\
dy_t &= 0.
\end{aligned}
\tag{1.3}
$$

For a kernel v on $\mathbb{R}^n \times \mathbb{R}_+^*$, and $t \geqslant 0$, set

$$
\left(\widehat{P}_t v\right)(x\,;\,dy) = \mathbb{E}\left[e^{-\int_0^t (\operatorname{div} b)(\phi_s(x,y),y)ds}\, v\left(\phi_t(x, y)\,;\,dy\right)\right],
$$

where div b is the divergence of the vector field $b(\cdot, y)$ on $\mathbb{R}^n$, for fixed y. The integral form of equation (1.1) is

$$
\mu_t = \widehat{P}_t \mu_0 + \int_0^t \widehat{P}_{t-s}\left(K(\mu_s, \mu_s)\right)\, ds.
\tag{1.4}
$$

We now introduce the functional setting in which we are going to solve this equation. Let v_0 be a nonnegative finite measure on $\mathbb{R}_+^*$ such that

$$
v_0 \star v_0 \leqslant C\, v_0
\tag{1.5}
$$

for some positive constant C. Define $\mathcal{N}$ as the set of kernels $v(x\,;\,dy)$ from $\mathbb{R}^d$ to $\mathbb{R}_+^*$ of the form $\{f(x; y)\, v_0(dy)\}_{x\in\mathbb{R}^d}$, where $f(x\,;\,y)$ is a bounded measurable function *uniformly equicontinuous in x*:

$$
\forall \epsilon > 0,\, \exists\, \delta > 0,\, \forall\, y \in \mathbb{R}_+^*,\, |x - x'| \leqslant \delta \Rightarrow \left|f(x\,;\,y) - f(x'\,;\,y)\right| \leqslant \epsilon.
$$

Setting

$$
\|v\| = \sup_{x\in\mathbb{R}^d}\, v_0\text{-ess}\sup_{y\in\mathbb{R}_+^*}\, |f(x\,;\,y)|
$$

defines a norm on $\mathcal{N}$.

THEOREM 1. *Suppose the following two conditions hold.*

(i) *There exists some positive constant α, β such that $\alpha\, \mathrm{Id} \leqslant a(x, y) \leqslant \beta\, \mathrm{Id}$, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+^*$.*

(ii)   *There exists a positive constant ε such that the vector $b(\cdot, y)$ has a divergence bounded below by ε, for all $y \in \mathbb{R}_+^*$.*

   Let $M > 0$ be an upper bound for K. Then, for any $\mu_0 \in \mathcal{N}$, equation (1.4) has a unique maximal solution in $\mathcal{N}$, continuous with respect to t, and defined for all times provided $\|\mu_0\|$ is small enough; it is nonnegative if μ_0 is nonnegative.

REMARKS.   (1) Use Laplace transform to construct a positive continuous and LEB-integrable function g on $\mathbb{R}_+^*$ such that the finite measure $\nu_0(dy) := g(y)\, dy$ satisfies (1.5) for some positive constant C. Suppose $\mu_0(x\,;dy) = h_0(x\,;y)\, dy$, for a bounded measurable function h_0 with support in $\mathbb{R}^n \times (0, m)$, for some $m < \infty$. It follows from theorem 1 that μ_t is well-defined for all times provided $\|h_0\|_\infty$ is small enough, and each $\mu_t(x\,;dy)$ is absolutely continuous with respect to Lebesgue measure.

(2) It will be clear from the proof of theorem 1 that the conclusion remains unchanged if the coagulation rate $K(y, y')$ is allowed to depend on x also, as an equicontinuous function with a modulus of continuity independent of $y, y' \in \mathbb{R}_+^*$.

(3) Although all our statements are given here and below with $y \in \mathbb{R}_+^*$, all the results hold true with y taking values in a general Abelian semigroup describing the inner structure of particles, at the price of introducing the formalism used in [11]. All results hold in particular for the discrete masses spatial coagulation equation.

(4) Note that since we are working with a bounded coagulation kernel, we do not require any moment assumption on the initial condition.

(5) A similar result holds when $\mathbb{R}^n$ is replaced by a bounded domain of $\mathbb{R}^n$, with smooth boundary, and Dirichlet or Neumann conditions are imposed on the boundary. Condition (ii) makes unnecessary the use of spectral properties of the diffusion/convection operators.

(6) Our method of proof is similar in spirit to the method used in [2], where the quadratic coagulation term is balanced by a linear term.[1] Our functional setting is more elementary and the above result new.

(7) The introduction of the map $\widehat{P}_t$ is better understood by considering first a pure convection dynamics, where a (and so σ) is null. In that case, the equation $\dot{\mu}_t = L_x^* \mu_t$ on kernels reads

$$\dot{\mu}_t = -b^i \partial_{x_i} \mu_t - (div\, b)\mu_t. \tag{1.6}$$

   Introducing the flow φ_t of the ordinary differential equation $\dot{x} = b(x)$ on $\mathbb{R}^n$, equation (1.6) has solution

$$\mu_t(x\,;dy) = \mu_0\left(\varphi_t(x)\,;dy\right)\, e^{-\int_0^t (div\, b)(\varphi_s(x), y)ds}$$

---

[1] The main ingredients of [2] are mass conservation of solutions and the exponential decay of a semigroup on functions with zero spatial mean. We do not need mass conservation but use crucially the exponential decay of $\widehat{P}_t$.

In the case of a general second-order differential operator L_x as above, the associated dynamics is no longer deterministic, but given by the stochastic differential equation (1.3). Equation (1.4) is the integral form of (1.1) obtained by varying the constant.

2. Proof of theorem 1

2.1. Well-posedness

The proof of the following lemma is straightforward and left to the reader.

LEMMA 2. *The space* $(\mathcal{N}, \|\cdot\|)$ *is a Banach space.*

The proof of theorem 1 is an application of the elementary fixed point theorem for contractions of a Banach space. Fix $T > 0$. Given an $\mathcal{N}$-valued path $\nu = (\nu_s)_{0 \leqslant s \leqslant T}$, define for each $t \in [0, T]$ the kernel

$$F(\nu)_t = \widehat{P}_t \nu_0 + \int_0^t \widehat{P}_{t-s}(K(\nu_s, \nu_s)) \, ds.$$

PROPOSITION 3. *The map* F *is a locally Lipschitz map from* $\mathcal{C}([0, T], \mathcal{N})$ *into itself.*

It follows classically from proposition 3 that one can associate to any $\mu_0 \in \mathcal{N}$ a time T depending only on $\|\mu_0\|$, in an increasing way, such that F has a unique fixed point in $\mathcal{C}([0, T], \mathcal{N})$ with initial value μ_0. The following proposition proves then that $T = \infty$.

PROPOSITION 4. *We have* $\|\mu_t\| \leqslant \|\mu_0\|$, *for all* $t \in [0, T]$, *if* $\|\mu_0\|$ *is small enough.*

We now proceed to proving the two propositions.

Proof of proposition 3. We first deal with the operator $\widehat{P}_t$, for $t > 0$.

LEMMA 5.   *a) The linear map* $\widehat{P}_t$ *sends* $\mathcal{N}$ *into itself and has norm no greater than* $e^{-\varepsilon t}$.
    b) Fix $\nu \in \mathcal{N}$. *The map* $t \in \mathbb{R}_+ \mapsto \widehat{P}_t \nu \in (\mathcal{N}, \|\cdot\|)$ *is continuous.*

Proof.   **a)** For $\mu(x; dy) = f(x; y)\nu_0(dy)$ in $\mathcal{N}$,

$$\left(\widehat{P}_t \mu\right)(x; dy) = \mathbb{E}\left[e^{-\int_0^t (\operatorname{div} b)(\phi_s(x,y),y) \, ds} f(\phi_t(x, y); y)\right] \nu_0(dy),$$

so we need to see that the above expectation defines a (y-)uniformly equicontinuous function of x; denote it by $\left(P_t^{\operatorname{div} b} f\right)(x, y)$. Rewrite equation (1.3) as a stochastic differential equation driven by vector fields V_i

$$dx_t = V_i(x_t, y) \, dB_t^i + b(x_t) \, dt,$$

with $V_i(x, y)$ being the ith column of the matrix $\sigma(x, y)$. Kusuoka and Stroock have proved in [6] that we have[2] for each $i \in \{1, \ldots, n\}$

$$\sup_{x \in \mathbb{R}^n} \left| V_i P_t^{\mathrm{div} b} f \right| (x, y) \leqslant c \, t^{-\frac{1}{2}} \sup_{x \in \mathbb{R}^n} |f(x \, ; \, y)|, \tag{2.1}$$

for each $y \in \mathbb{R}_+^*$, where the constant c depends only on an upper bound for the uniform norm of the V_i's and all their derivatives. It can be taken to be independent of $y \in \mathbb{R}_+^*$ from our assumptions. Also, from the uniform ellipticity of the matrices $a(x, y)$, inequality (2.1) implies[3] the (y-)uniform equicontinuity of $P_t^{\mathrm{div} b} f$. This proves that $\widehat{P}_t$ sends $\mathcal{N}$ into itself; the statement on its norm is straightforward from hypothesis (ii) of theorem 1.

b) It suffices to prove the continuity at $t = 0$. For $\nu(x, dy) = f(x \, ; \, y)\nu_0(dy)$ in $\mathcal{N}$, we have

$$\|\widehat{P}_t \nu - \nu\| \leqslant \sup_{x, y} \mathbb{E}\left[\left| e^{-\int_0^t (\mathrm{div}\, b)(\phi_s(x, y), y)\, ds} f(\phi_t(x, y) \, ; \, y) - f(x \, ; \, y) \right| \right]$$

$$\leqslant \sup_{x, y} \mathbb{E}\left[\left| f(\phi_t(x, y) \, ; \, y) - f(x \, ; \, y) \right| \right]$$

$$+ \|f\|_\infty \sup_{x, y} \mathbb{E}\left[\left| e^{-\int_0^t (\mathrm{div}\, b)(\phi_s(x, y) \, ; \, y)\, ds} - 1 \right| \right]$$

$$\leqslant \sup_{x, y} \mathbb{E}\left[|f(\phi_t(x, y) \, ; \, y) - f(x \, ; \, y)| \right] + c \, \|f\|_\infty t. \tag{2.2}$$

for some positive constant c. By the uniform equicontinuity of $f(\cdot \, ; \, y)$, one can associate with any $\alpha > 0$ a $\delta > 0$ such that

$$\forall \, y \in \mathbb{R}_+^*, \ \forall x, x' \in \mathbb{R}^n, \ \ |x' - x| \leqslant \delta \Rightarrow \left| f(x, \, ; \, y) - f(x' \, ; \, y) \right| \leqslant \alpha.$$

Then, we have for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+^*$

$$\mathbb{E}\left[\left| f(\phi_t(x, y) \, ; \, y) - f(x \, ; \, y) \right| \right] \leqslant \alpha \, \mathbb{P}\left(|\phi_t(x, y) - x| \leqslant \delta \right)$$

$$+ 2\|f\|_\infty \, \mathbb{P}\left(|\phi_t(x, y) - x| > \delta \right)$$

$$\leqslant \alpha + 2\|f\|_\infty \frac{\mathbb{E}\left[|\phi_t(x, y) - x|^2 \right]}{\delta^2}.$$

---

[2]They prove their results for σ and b and all their derivatives bounded; their proof of the following result extends in a straightforward way to our elliptic setting.

[3]Due to the uniform ellipticity of the matrices $a(x, y)$, inequality (2.1) is equivalent to the existence of a constant c', independent of $i \in \{1, \ldots, n\}$, such that $\sup_{x \in \mathbb{R}^n} \left| \partial_{x_i} P_t^{\mathrm{div} b} f \right| \leqslant c' t^{-1/2} \sup_{x \in \mathbb{R}^n} |f(x, y)|$. The y-uniform equicontinuity of $P_t^{\mathrm{div} b} f$ follows from the boundedness of f as $t > 0$ is fixed.

As the isometry property of stochastic integration with respect to Brownian motion and Jensen's inequality give

$$\mathbb{E}\left[|\phi_t(x, y) - x|^2\right] \leqslant 2\mathbb{E}\left[\left(\int_0^t a(\phi_s(x, y), y)dB_s\right)^2\right]$$

$$+2\mathbb{E}\left[\left(\int_0^t b(\phi_s(x, y), y)ds\right)^2\right]$$

$$\leqslant 2\,\mathbb{E}\left[\int_0^t |a(\phi_s(x, y), y)|^2\,ds\right]$$

$$+2t\,\mathbb{E}\left[\int_0^t |b(\phi_s(x, y), y)|^2\,ds\right] \leqslant Ct$$

for some positive constant C depending only on $\|a\|_\infty$ and $\|b\|_\infty$, it follows that

$$\sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^*_+} \mathbb{E}\left[|f(\phi_t(x, y)\,;\, y) - f(x\,;\, y)|\right] \leqslant \alpha + \frac{2C\|f\|_\infty}{\delta^2}\,t.$$

Together with (2.2), this inequality implies that $\|\widehat{P}_t v - v\| \to 0$ as t goes to 0. □

It follows from lemma 5 that proposition 3 will be proved if we can prove that the kernel $K(\mu, \mu)$ belongs $\mathcal{N}$ if μ does (**c**) and is a locally Lipschitz function of μ (**d**). Recall we write M for an upper bound of K.

c) Given $\mu(x, dy) = f(x\,;\, y)\,v_0(dy)$ in $\mathcal{N}$, set, for any $x \in \mathbb{R}^n$,

$$K^+(\mu, \mu)(x\,;\, dz) = \frac{1}{2}\int \delta_{y+y'}(dz)\,K(y, y')\,f(x\,;\, y')\,v_0(dy')\,f(x\,;\, y)v_0(dy),$$

and

$$K^-(\mu, \mu)(x\,;\, dz) = \left(\int K(y, z)f(x\,;\, y)\,v_0(dy)\right)f(x; z)\,v_0(dz),$$

so that $K(\mu, \mu) = K^+(\mu, \mu) - K^-(\mu, \mu)$. As v_0 satisfies hypothesis (1.5) we have[4]

$$\left|K^+(\mu, \mu)(x\,;\, \cdot)\right| \leqslant M\|f\|_\infty^2\,v_0 \star v_0 \leqslant CM\|f\|_\infty^2\,v_0;$$

we can thus write

$$K^+(\mu, \mu)(x\,;\, dy) = k^+(x, y)\,v_0(dy)$$

for some bounded measurable function $k^+(x, y)$. Taking any bounded function g, we see by dominated convergence that the integral

$$\int g(y)\,K^+(\mu, \mu)(x\,;\, dy)$$

---

[4]We write $|\rho|$ for the absolute value of a signed measure ρ.

is a uniformly continuous function of x, with a modulus of continuity independent of g. It follows that one can construct a version of $k^+(x, y)$ which is (y-)uniformly equicontinuous in x. So K^+ sends $\mathcal{N}$ into itself; it is straightforward to see that K^- does the same.

d) Denote by $|\cdot|_{TV}$ the total variation norm on finite signed measures on $\mathbb{R}_+^*$. The elementary inequality

$$\left| \left(K(\mu, \mu) - K(\mu', \mu') \right) (x\,;\cdot) \right|_{TV}$$
$$\leqslant M \left(|\mu(x\,;\cdot)|_{TV} + |\mu'(x\,;\cdot)|_{TV} \right) \left| \mu(x\,;\cdot) - \mu'(x\,;\cdot) \right|_{TV}, \quad x \in \mathbb{R}^d,$$

shows that K is locally Lipschitz in $(\mathcal{N}, \|\cdot\|)$.

$\square$

Proof of proposition 4. Set $m = v_0(\mathbb{R}_+^*)$. As equation (1.4) gives for all $t \in [0, T]$

$$\|\mu_t\| \leqslant e^{-\varepsilon t} \|\mu_0\| + M \left(\frac{C}{2} + m \right) \int_0^t e^{-\varepsilon(t-s)} \|\mu_s\|^2 \, ds,$$

the real-valued function $\|\mu_t\|$ is bounded above by the solution of the equation

$$\dot{z}_t = -\varepsilon z_t + M \left(\frac{C}{2} + m \right) z_t^2, \quad z_0 = \|\mu_0\|, \quad 0 \leqslant t \leqslant T.$$

Its solution

$$z_t = \frac{\varepsilon}{M \left(\frac{C}{2} + m \right) + \frac{\varepsilon - M\left(\frac{C}{2} + m \right) \|\mu_0\|}{\|\mu_0\|} e^{\varepsilon t}}$$

is defined for all times and less than or equal to z_0 for $\|\mu_0\| < \dfrac{\varepsilon}{M\left(\frac{C}{2} + m \right)}$. $\square$

2.2. Positivity

We do not suppose in this section that $\mu_0 \in \mathcal{N}$ is small, so we work on a time interval $[0, T]$ where equation (1.4) is known to have a unique solution in $\mathcal{N}$ continuous with respect to t.

Suppose $\mu_0 \geqslant 0$. We prove the positivity of μ_t by the usual method; see for instance theorem 10 in [2]. We recall it here for the reader's convenience. Let α be a positive constant such that

$$\int_{\mathbb{R}_+^*} \mu_s(x\,;dy) \leqslant \|\mu_s\| \, v_0(\mathbb{R}_+^*) \leqslant \frac{\alpha}{M} \tag{2.3}$$

for all $x \in \mathbb{R}_+^*$ and $s \in [0, T]$. Define a map G from $\mathcal{C}\left([0, T], \mathcal{N}\right)$ into itself setting

$$G\left((v_s)_{0 \leqslant s \leqslant T} \right)_t = \widehat{P}_t \mu_0 + \int_0^t \widehat{P}_{t-s} \left(K(\mu_s, v_s) + \alpha v_s \right) ds - \alpha \int_0^t \widehat{P}_{t-s} v_s \, ds.$$

Given two measures ρ and ρ' on $\mathbb{R}_+^*$, set

$$K(\rho, \rho')(dz) = \int \left\{ \delta_{y+y'}(dz) - \delta_y(dz) - \delta_{y'}(dz) \right\} K(y, y')\, \rho(dy)\rho'(dy').$$

As the following inequality between kernels holds pointwise in $x \in \mathbb{R}^n$

$$K(\mu_s, \nu_s)(dz) \geqslant -\left(\int_{\mathbb{R}_+^*} K(y, z)\, \mu_s(dy) \right) \nu_s(dz) + \alpha\, \nu_s(dz)$$

$$\geqslant \left(\alpha - \int_{\mathbb{R}_+^*} K(y, z)\, \mu_s(dy) \right) \nu_s(dz) \overset{(2.3)}{\geqslant} 0,$$

the first integral in the definition of G is nonnegative. The path $(\mu_s)_{0 \leqslant s \leqslant T}$ solves by construction the equation on kernels

$$\nu_t + \alpha \int_0^t \widehat{P}_{t-s} \nu_s\, ds = G\left((\nu_s)_{0 \leqslant s \leqslant T} \right)_t \geqslant 0, \quad t \in [0, T].$$

Taking T smaller if necessary, one can solve this equation by a fixed point method; see e.g. theorem 10 of [2]. As each iteration step of the method preserves nonnegative kernels, its unique fixed point $(\mu_s)_{0 \leqslant s \leqslant T}$ is nonnegative if μ_0 is nonnegative.

REMARK ON THE PROOF OF THEOREM 1. The pattern of proof used to prove theorem 1 works equally well for some sub-elliptic convection-diffusion operators L_x of Hörmander form $\frac{1}{2} \sum_{i=1}^p V_i^2 + B$ satisfying the following assumptions.

- The vector fields $V_i(\,\cdot\,; y)$ and $B(\,\cdot\,; y)$ on $\mathbb{R}^n$ are measurable functions of $(x, y) \in \mathbb{R}^n \times \mathbb{R}_+^*$, of class $\mathcal{C}^\infty$ as functions of $x \in \mathbb{R}^n$, with all their derivatives bounded uniformly in $y \in \mathbb{R}_+^*$,
- The vector fields $B(\cdot, y)$ satisfy hypothesis (ii) of theorem 1,
- Set $\mathcal{A} = \{\emptyset\} \cup \bigcup_{k=1}^\infty \{1, \ldots, p\}^k$, set $V_{[\emptyset]} = 0$ and $V_{[i]} = V_i$, for $i = 1 \ldots, p$, and define inductively the vector fields $V_{[\alpha]}$, for $\alpha \in \mathcal{A}$, setting

$$V_{[\alpha i]} = \left[V_{[\alpha]}, V_i \right], \quad \text{for } i = 1, \ldots, p.$$

Given a positive integer ℓ, write $\mathcal{A}_\ell$ for $\{\emptyset\} \cup \bigcup_{k=1}^\ell \{1, \ldots, p\}^k$. We suppose there exists an ℓ such that

$$\inf_{\xi \in \mathbb{S}^{n-1}} \left\{ \sum_{\alpha \in \mathcal{A}_\ell} \left(V_{[\alpha](x,y)}, \xi \right)^2 ; \ (x, y) \in \mathbb{R}^n \times \mathbb{R}_+^* \right\} > 0. \tag{2.4}$$

Indeed, theorem 2 of [7] provides a uniform in $\alpha \in \mathcal{A}_\ell$ estimate of

$$\sup_{(x,y) \in \mathbb{R}^n \times \mathbb{R}_+^*} \left| V_{[\alpha]} P_t^{\mathrm{div}\, b} f \right| (x, y)$$

in terms of $\| f \|_\infty$, at a fixed time $t > 0$, similar to (2.1). The y-uniform equicontinuity of $P_t^{\mathrm{div}\, b} f$ follows from condition (2.4) as above. The remainder of the proof of theorem 1 does not need any change.

This kind of improvement of theorem 1 seems difficult to get by the analytical methods developed in the works mentioned in the introduction.

3. Multiple coagulations, fragmentation, and scattering

3.1. Multiple coagulations

It is clear from the proof of theorem 1 that it can be extended to a multiple coagulation framework. Write K_2 for the above binary coagulation kernel K, and let K_n be for all $n \geqslant 3$ a symmetric nonnegative valued function on $\left(\mathbb{R}_+^*\right)^n$. Set

$$
\begin{aligned}
&K_n(\mu, \dots, \mu) \\
&= \frac{1}{n!} \int \left\{ \delta_{y_1 + \cdots + y_n} - \left(\delta_{y_1} + \cdots + \delta_{y_n} \right) \right\} K_n(y_1, \dots, y_n)\, \mu(dy_1) \dots \mu(dy_n)
\end{aligned}
$$

The spatial coagulation equation takes in that general framework the form

$$
\partial_t \mu_t = L_x^* \mu_t + \sum_{n \geqslant 2} K_n(\mu_t, \dots, \mu_t). \tag{3.1}
$$

THEOREM 6. *Suppose the coagulation kernels K_n are uniformly bounded above by a constant independent of n. Suppose hypotheses* (i) *and* (ii) *of theorem 1 hold. Then equation* (3.1) *has a unique maximal solution in $\mathcal{N}$, defined for all times if $\|\mu_0\|$ is small enough; it is non-negative if μ_0 is non-negative.*

Proof. The proof of this statement is similar to the proof of theorem 1, noticing that we have $\nu_0^{*n} \leqslant C^n \nu_0$ under condition (1.5), and introducing, for any measure μ on $\mathbb{R}_+^*$, the measures $K_n^+(\mu, \dots, \mu)$ and $K_n^-(\mu, \dots, \mu)$ defined by the formulas

$$
K_n^+(\mu, \dots, \mu)(dz) = \frac{1}{n!} \int \delta_{y_1 + \cdots + y_n}(dz) K_n(y_1, \cdots, y_n)\, \mu(dy_1) \dots \mu(dy_n),
$$

$$
K_n^-(\mu, \dots, \mu)(dz) = \frac{1}{(n-1)!} \left(\int K(y_1, \dots, y_{n-1}, z)\, \mu(dy_1) \dots \mu(dy_{n-1}) \right) \mu(dz).
$$

Denote by M an upper bound for all the coagulation kernels, and write m for $\nu_0(\mathbb{R}_+^*)$. To prove that the unique maximal solution is defined for all times for a small enough initial condition, remark that equation (3.1) implies the inequality

$$
\|\mu_t\| \leqslant e^{-\varepsilon t} \|\mu_0\| + \sum_{n \geqslant 2} \left(\frac{MC^n}{n!} + \frac{m^{n-1}}{(n-1)!} \right) \int_0^t e^{-\varepsilon(t-s)} \|\mu_s\|^n \, ds,
$$

so $\|\mu_t\|$ is bounded above by the solution of the differential equation

$$
\dot{z}_t = -\varepsilon z_t + M \left(e^{C z_t} - 1 - C z_t \right) + z_t \left(e^{m z_t} - 1 \right), \quad z_0 = \|\mu_0\|.
$$

Its solution is decreasing and defined for all times provided z_0 is small enough. $\square$

3.2. Coagulation, fragmentation, and scattering

More realistic models of chemical reactions have fragmentation and scattering terms. The fragmentation term accounts for the spontaneous breakage of particles

into smaller particles; it is defined by the following formula, where ρ, here and below, is used as a generic measure on $\mathbb{R}_+^*$,

$$\mathfrak{B}(\rho)(dz) := \int \left\{ F(y\,;dz) - \delta_y(dz) \right\} B(y)\,\rho(dy).$$

A particle of mass y splits into smaller particles at rate $B(y)$, the result of the splitting being given by the nonnegative measure $F(y\,;dz)$ with support in $(0,\,y)$ and total mass $\int z\,F(y\,;dz) = y$. The scattering term accounts for the fact that real particles have a mass no bigger than some maximum mass y_0, in a given situation, so if coagulation forms a particle with mass bigger than y_0, this particle breaks instantaneously into smaller particles. The scattering term $\mathfrak{S}$ is determined by a scattering kernel S from $(y_0, 2y_0)$ to the set of nonnegative measures on $\mathbb{R}_+^*$, and the formula

$$\mathfrak{S}(\rho, \rho)(dz) :$$
$$= \int_{(0,y_0)^2} \left\{ S(y + y'; dz) - \delta_y(dz) - \delta_{y'}(dz) \right\} \mathbf{1}_{y+y'>y_0} K(y, y')\,\rho(dy)\rho(dy').$$

This term was first introduced in [15] and studied further in [16] and [17]. The coagulation term is changed in this model into

$$K(\rho, \rho) = \int \left\{ \delta_{y+y'} - \delta_y - \delta_{y'} \right\} \mathbf{1}_{y+y'\leqslant y_0}\,K(y, y')\,\rho(dy)\,\rho(dy').$$

The proof of the following theorem is analog to the proof of theorem 1.

THEOREM 7. *Let $y_0 > 0$ be an upper bound for the size of the particles in the system. Suppose the following conditions hold.*

(i) *There exists some positive constant α, β such that $\alpha Id \leqslant a(x, y) \leqslant \beta Id$, for all $(x, y) \in \mathbb{R}^n \times (0, y_0)$.*
(ii) *There exists a positive constant ε such that the vector field $b(\cdot, y)$ has divergence bounded below by ε, for all $y \in (0, y_0)$.*
(iii) *The coagulation and breakage rates K, B are bounded, and the fragmentation kernel $F(y\,;dz)$ has the form $\mathfrak{f}(y\,;z)\,v_0(dz)$, with $\|B\|_\infty\,(1 + \|\mathfrak{f}\|_\infty) < \varepsilon$.*
(iv) *We have*

$$\int S(y + y'; dz)\,v_0(dy)v_0(dy') \leqslant C\,v_0(dz) \tag{3.2}$$

  for some positive constant C.

  Given $\mu_0 \in \mathcal{N}$, the equation

$$\mu_t = \widehat{P}_t\mu_0 + \int_0^t \widehat{P}_{t-s}\,(K(\mu_s, \mu_s) + \mathfrak{B}(\mu_s) + \mathfrak{S}(\mu_s, \mu_s))\,ds, \quad t \in [0, T]$$

has a unique solution in $\mathcal{N}$ continuous with respect to t, and defined for all times provided $\|\mu_0\|$ is small enough; it is non-negative if μ_0 is non-negative.

Condition (iv) will for instance hold if $v_0 = \mathrm{LEB}_{|(0, y_0)}$ and the measures $S(a\,;dz)$ have a uniformly bounded density with respect to Lebesgue measure on $(0, y_0)$, for $a \in (y_0, 2y_0)$.

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