Small time fluctuations for bridges of sub-Riemannian diffusions

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From diffusion operators to sub-Riemannian geometry

$\mathbb{M}$: compact manifold

$\mathcal{L}$: second order differential operator on $\mathbb{M}$, with $\mathcal{L}\mathbf{1} = 0$, and principal symbol $\sigma : T^*\mathbb{M} \to T\mathbb{M}$, characterized by

$$(df)\big(\sigma(dg)\big) = \frac{1}{2}\big(\mathcal{L}(fg) - f\mathcal{L}(g) - g\mathcal{L}(f)\big).$$

- **Horizontal paths**: $\dot{\gamma}_t \in \sigma\big(T^*_{\gamma_t}\mathbb{M}\big)$.
- We have a well-defined positive-definite **scalar product on** $\sigma(T^*\mathbb{M})$:

$$(p, p')_{\mathcal{L}} := e\big(\sigma(e')\big),$$

for any $p = \sigma(e), p' = \sigma(e')$, with $e, e' \in T^*_m\mathbb{M}$, $m \in \mathbb{M}$.

Assume any two points of $\mathbb{M}$ *are the end-points of a horizontal path*, so Hörmander's conditions hold.

- **Bridge law** $\mathbf{P}_\epsilon^{x,y}$ on

$$\Big\{\omega \in \mathcal{C}\big([0,1],\mathbb{M}\big); \omega_0 = x, \omega_1 = y\Big\},$$

diffusion process associated with $\epsilon\mathcal{L}$, conditionned on going from x to y in a unit time.

- **Aim.** *Describe the* **asymptotic behaviour of** $\mathbf{P}_\epsilon^{x,y}$ *as* ϵ *goes to* 0, *in terms of the geometry of horizontal paths.*

Prototype results for Brownian motion

- Theorem [Hsu 90'] – **LDP and first order asymptotics**. *Set*

$$J(\omega) = \int_0^1 \left|\dot{\omega}_s\right|^2 ds - d(x,y)^2.$$

The family $\left(\mathbf{P}_\epsilon^{x,y}\right)_{0<\epsilon\leq 1}$ *satisfies a* **LDP with good rate function** J. *If further* x *and* y *are joined by a* unique minimizing geodesic γ, *then* $\mathbf{P}_\epsilon^{x,y}$ *converges weakly to a* **Dirac mass on** γ.

Prototype results for Brownian motion

▶ Theorem [Hsu 90'] – **LDP and first order asymptotics**.

Use coordinates $\mathfrak{f}$ on a neighbourhood of γ, and set

$$Z_t^\epsilon = \epsilon^{-\frac{1}{2}} \left(D_{\gamma_t}\mathfrak{f}\right)^{-1}\big(\mathfrak{f}(X_t) - \mathfrak{f}(\gamma_t)\big) \quad \in T_{\gamma_t}\mathbb{M}.$$

Define

$$\mathfrak{S}_\gamma^0 := \big\{\text{sections of } T\mathbb{M} \text{ over } \gamma \text{ with null ends}\big\},$$

$$\mathfrak{I}_\gamma^0 := \left\{p_\bullet \in \mathfrak{S}_\gamma^0\,;\, \int_0^1 \mathbf{g}(p_s, p_s)\, ds < \infty\right\},$$

and the **second variation of the energy functional**

$$\mathfrak{Q}(p_\bullet) = \int_0^1 \left(\left|\nabla_{\gamma_s} p_s\right|^2 - \big(R(p_s, \dot\gamma_s)\dot\gamma_s, p_s\big)\right) ds$$

▶ Theorem [Molchanov 75'] – **Small time fluctuations for Brownian bridges.** *If x and y are non-conjugate along γ then the finite dimensional laws of $Z_\bullet^\epsilon$ converge weakly to a Gaussian measure on $\mathfrak{S}_\gamma^0$, with* **Cameron-Martin space** $\left(\mathfrak{I}_\gamma^0, \mathfrak{Q}\right)$.

Examples

1. When σ is **definite-positive**, $(\cdot,\cdot)_{\mathcal{L}}$ defines a **Riemannian metric**, and
$$\mathcal{L} = \triangle + V.$$

2. **Carnot groups.** Lie groups G whose (finite dim.) Lie algebra $\mathfrak{g}$ has a stratification
$$\mathfrak{g} = \oplus_{i=1}^{\ell}\mathfrak{g}_i, \qquad \mathfrak{g}_i = \left[\mathfrak{g}_1, \mathfrak{g}_{i-1}\right], \ \left[\mathfrak{g}_1, \mathfrak{g}_\ell\right] = 0.$$
Choose an adapted basis in $\mathfrak{g}$, and define $\sigma(\mathbf{g}e^*) = \mathbf{g}e_1$, for $\mathbf{g} \in G$ and $e^* = \sum e_i^* \in \mathfrak{g}$.

3. **Intrinsic sub-Riemannian Laplacian.** E sub-bundle of $T\mathbb{M}$ with constant rank, $g_E(\cdot,\cdot)$ Riemannian metric on E, and $E^i := E^{i-1} \oplus \left[E, E^{i-1}\right]$, with $E^{\ell-1} \neq E^\ell = T\mathbb{M}$. Set
$$\mathcal{L}f = \mathrm{div}_E\left(\nabla_E f\right),$$
with horizontal gradient $g_E\left(\nabla_E f, q\right) = (Df)(q)$, for all $q \in \Gamma(E)$, $f \in \mathcal{C}^\infty(\mathbb{M})$.

Sub-Riemannian pecularities

Riemannian geometry: minimizing paths are projections of bicharacteristics in $T^*\mathbb{M}$, with Hamiltonian

$$H(m,p) = p\big(\sigma_m(p)\big) = |p|_m^2.$$

Sub-Riemannian geometry: situations where minimizing paths are *not* projections of bicharacteristics (Martinet-type distributions).

Assumption A. The two end-points are joined by a unique minimizing path γ, which is the projection of a bicharacteristic.

Sub-Riemannian pecularities

In a neigh. of γ, write $\mathcal{L} = \frac{1}{2}\sum V_i^2 + V$, and for a control $g \in H^1$, set

$$\dot\gamma_t^g = V_i(\gamma_t^g)\,\dot g_t^i, \quad \gamma_0^g = x.$$

Riemannian geometry: With $y = \gamma_1^h$, we always have

$$\frac{d\gamma_1^g}{d\,g}_{\big|g=h}(H^1) = T_y\mathbb{M}. \qquad (1)$$

Sub-Riemannian geometry: no longer true. However, if $\gamma^h = \gamma$ with h of minimal H^1-norm the space $\frac{d\gamma_1^g}{d\,g}_{\big|g=h}(H^1)$ **depends only on** σ.

Assumption B. Identity (1) holds.

Equiv. to invertibility of some deterministic Malliavin covariance matrix.

Sub-Riemannian pecularities

Non-constant rank of σ may cause troubles. Work in $\mathbb{M} = (-1, 1)$, with $\sigma(m, p) = m^2 p$. For the **horizontal path** $m_t = \frac{t^2}{2}$, the relation $\dot{m}_t = \sigma(m_t, p_t)$ imposes $p_t = \frac{4}{t}$, so $\int_0^1 p_s\big(\sigma(p_s)\big) = 4$, while $\int_0^1 p_s^2\, ds = \infty$, so $\int_0^1 |p_s|^2_{m_s}\, ds = \infty$, *for any continuous Riemannian metric on* $\mathbb{M}$.

- Definition. *A* **horizontal path** $(m_t)_{0 \leq t \leq 1}$ **with finite energy** *is said to be* **regular** *if there exists a section* $(p_t)_{0 \leq t \leq 1}$ *of* $T^*\mathbb{M}$ *s.t.* $\dot{m}_t = \sigma(p_t)$, *and* $\int_0^1 |p_s|^2_{m_s}\, ds < \infty$, *for some (hence all) Riemannian metric on* $\mathbb{M}$.

Assumption C. The path γ is regular. (Always holds if σ has constant rank.)

Non-conjugacy. Under assumptions A,B,C, γ is the projection of a *unique* bicharacteristic; let $p_0 \in T_x\mathbb{M}$, $p_1 \in T_y\mathbb{M}$ be its **initial and final momenta**. Let $(\psi_t)_{0\le t\le 1}$ stand for the **Hamiltonian flow** in $T^*\mathbb{M}$. Define, for $0 \le t \le 1$, the **Jacobi operators**

$$J_t : T_x^*\mathbb{M} \to T_{\gamma_t}\mathbb{M}, \quad K_{1-t} : T_y^*\mathbb{M} \to T_{\gamma_t}\mathbb{M},$$

setting

$$J_t(p) = \frac{d}{da}\Big|_{a=0}(\pi \circ \psi_t)(x, p_0 + ap), \quad p \in T_x^*\mathbb{M},$$

$$K_{1-t}(p') = \frac{d}{da}\Big|_{a=0}(\pi \circ \psi_{-(1-t)})(x, p_1 + ap'), \quad p' \in T_y^*\mathbb{M}.$$

▶ Definition. *The two end-points x and y of γ are said to be* **non-conjugate along** γ *if J_1 is invertible.*

Assumption D. The two points x and y are non-conjugate along γ.

Main results

- Theorem – **LDP and first order asymptotics.**
  - *Assume* $\mathcal{L} = \frac{1}{2}\sum V_i^2 + V$, *and set*

    $$J(\omega) = \frac{1}{2}\Big(\inf\left\{\|g\|^2 \,;\, \gamma^g = \gamma\right\} - d(x,y)^2\Big).$$

    The family $\left(\mathbf{P}_\epsilon^{x,y}\right)_{0<\epsilon\leq 1}$ *satisfies a* **LDP with good rate function** J*.*
  - *If either* $\mathcal{L} = \frac{1}{2}\sum V_i^2 + V$*, or* γ *satisfies assumptions A,B,C,D, then the measures* $\mathbf{P}_\epsilon^{x,y}$ *converges weakly to a* **Dirac mass on** γ*.*

Main results

Recall

$$Z_t^\epsilon = \epsilon^{-\frac{1}{2}} \left(D_{\gamma_t}\mathfrak{f}\right)^{-1}\left(\mathfrak{f}(X_t) - \mathfrak{f}(\gamma_t)\right) \quad \in T_{\gamma_t}\mathbb{M}.$$

► Theorem – **Small time fluctuations for bridges of degenerate diffusions.** *Under assumptions A,B,C,D, then*

1. *the map*

$$0 \leq s \leq t \leq 1 : (s,t) \mapsto J_s J_1^{-1} K_{1-t}^* \in L\left(T_{\gamma_t}^*\mathbb{M}, T_{\gamma_s}^*\mathbb{M}\right)$$

is the covariance function of a unique zero-mean Gaussian measure $\mathbf{Q}^{x,y}$ *on* $\mathfrak{S}_\gamma^0$, *with an explicit Cameron-Martin space;*

2. *the distribution of* $Z_\bullet^\epsilon$ *converges weakly to* $\mathbf{Q}^{x,y}$.

Tools for the proofs

- **LDP.** Follow Hsu's proof based on heat kernel estimates available in our setting, after Ben-Arous, Léandre works.
- **First order deterministic asymptotics.** Follows from LDP when $\mathcal{L} = \frac{1}{2}\sum V_i^2 + V$, and from the proof of the small time fluctuations theorem under assumptions A,B,C,D.
- **Small time fluctuations.** **Main piece of work**: **constructing** the Gaussian measure $\mathbf{Q}^{x,y}$ and characterizing its **Cameron-Martin space** in terms of an analogue of the quadratic form in the **second variation of the energy functional**. Difficulty: get expressions of some geometric and probabilistic quantities in terms of σ only.

  On the *probabilistic side*: use **Malliavin calculus** and the **stationnary phase method** to get the Gaussian fluctuations.