Lifetime of relativistic diffusions

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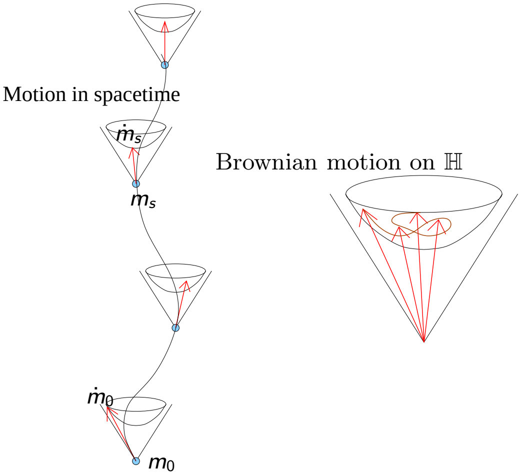
Layout of the talk

- Relativistic diffusions
- Lifetime of relativistic diffusions
  - Non-explosion criteria
  - Explosion criteria
- Time function on a Lorentzian manifold

• $m_s = m_0 + \int_0^s \dot{m}_r \, dr$:
$X_s = (m_s, \dot{m}_s) \in \mathbb{R}^{1,3} \times \mathbb{H}$
• Law of $\{X_s\}_{s \geqslant 0}$ frame-independent, *i.e.* intrinsic.

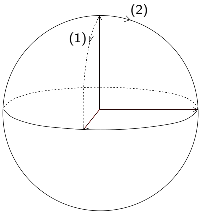
Theorem (Dudley, 66')

If X is continuous then $\dot{m}_s$ is a Brownian motion on $\mathbb{H}$.

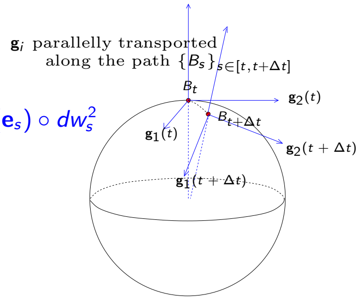


Brownian motion on $\mathbb{S}^2$. Infinitesimal Euclidean rotations:

$E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; $\mathbf{e} \in \mathbb{O}_3 : V_i(\mathbf{e}) = \mathbf{e}E_i$: invariant vector fields on $O(3)$.



$$\circ d\mathbf{e}_s = V_1(\mathbf{e}_s) \circ dw_s^1 + V_2(\mathbf{e}_s) \circ dw_s^2$$



Brownian motion on $\mathbb{H}$. Idem : use infinitesimal hyperbolic rotations:

$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, etc.

Brownian motion on $\mathbb{S}^2$. Infinitesimal Euclidean rotations:
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Brownian motion on $\mathbb{H}$. Infinitesimal hyperbolic rotations:
$E_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$;
$\mathbf{e} \in SO(1,3) : V_i(\mathbf{e}) = \mathbf{e}E_i$: invariant vector fields on $SO(1,3)$.

Dudley's diffusion on $\mathbb{R}^{1,3} \times SO(1,3)$.
$(m_s, \mathbf{e}_s) = \big(m_s, (\mathbf{e}_0(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s))\big)$ solves the sde

$$dm_s = \mathbf{e}_0(s)\, ds,$$
$$d\mathbf{e}_s = V_i(\mathbf{e}_s) \circ dw_s^i,$$

has generator

$$\mathcal{G} = H_0 + \frac{1}{2}\sum_{i=1}^{3} V_i^2.$$

- $(\mathbb{M}, g)$: Lorentzian manifold (oriented, time-oriented) of dimension $1+3$.
- $T^1\mathbb{M}$: future unit bundle ; $(m, \dot{m}) \in T^1\mathbb{M}$.
- $\mathbb{OM}$: orthonormal frame bundle, $\Phi = \big(m, (\mathbf{e}_0, \cdots, \mathbf{e}_3)\big) \in \mathbb{OM}$, $\mathbf{e}_0$ timelike and future.
- H_0: generates the geodesic flow, $(V_i)_{1\leqslant i\leqslant 3}$ canonical vertical vector fields.

- Θ-diffusion: $\Theta : T^1\mathbb{M} \to \mathbb{R}_+$ (function of the curvature, of the stress-energy tensor...)

$$d\Phi_s = H_0(\Phi_s)\, ds + \frac{1}{4}\big(V_i\Theta\big)\, V_i + \sqrt{\Theta(\Phi_s)}\, V_i \circ dw_s^i$$

$$\mathcal{G} = H_0 + \frac{1}{2}\sum_{i=1}^{3} V_i(\Theta V_i)$$

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Motivations from physics. Undesirable phenomena: exploding curvature invariants, inextendible incomplete geodesics...

Theorem (Hawking-Penrose)

The following three conditions cannot hold at the same time on a Lorentzian manifold

- *(Causality condition) The chronology condition holds.*
- *(Energy condition) Any complete causal geodesic has a pair of conjugated points.*
- *(Initial/boundary condition) The space has a "trapped set".*

Non-explosion

Lyapounov functions • *If there exists a function f and a positive constant C such that $\mathcal{G}f \leqslant Cf$, and f diverges to $+\infty$ along any timelike path living any compact set, then the relativistic diffusion has as an infinite lifetime.*

- U vector field on $\mathbb{M}$: $f(\Phi) = g(U, \mathbf{e}_0)$, for $\Phi = \big(m, (\mathbf{e}_0, ..., \mathbf{e}_d)\big) \in \mathbb{OM}$
- Generalised warped product: $\mathbb{R} \times S$, $ds^2 = a(m)^2\, dt^2 - h_{ij}(m)\, dx^i dx^j$

THEOREM (B.-F. '10)

Take $\Theta(\Phi) = \Theta(m)$. Then the Θ-diffusion has as an infinite lifetime if the function $(m, \dot{m}) \in T^1\mathbb{M} \to \nabla_{\dot{m}}(\log a)$ is bounded below $\Longleftrightarrow$ ∇a is everywhere non-spacelike and future.

A "volume-growth" condition

Riemannian case. X a symmetric conservative diffusion, with generator L, $X_0 \sim$ invariant measure

$$f(X_s) = f(X_0) + M_s + \int_0^s Lf(X_r)\,dr$$

$$f(X_s) = f(X_{T-(T-s)}) = f(X_T) + \widetilde{M}_s + \int_0^s Lf(X_{T-(T-r)})\,dr$$

so $df(X_s) = \frac{1}{2}(dM_s + \widetilde{M}_s)$: Gaussian control of the increments of $f(X)$ if $\langle M\rangle, \langle \widetilde{M}\rangle$ controlled.

X: reflected Brownian motion on the boundary of a (large) ball.

Theorem (Grigor'yan '86 / Hsu-Qin '10)

On a complete Riemannian manifold, Brownian motion is conservative if $\int_1^\infty \frac{r}{\ln|B(r)|}\,dr = \infty$.

Lorentzian case. *Difficulties:* no balls on $\mathbb{M}$; what could play the role of the conservative, symmetric diffusion?

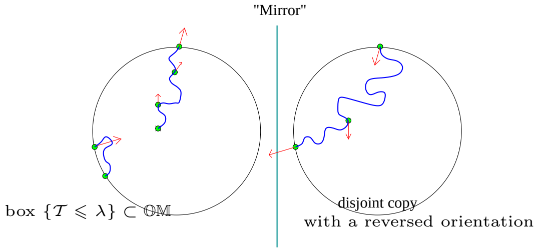
• *A Riemannian metric on $\mathbb{OM}$?* Parallelisable manifold: set $H_0, \ldots, H_d, (V_{ij})_{0\leqslant i<j\leqslant d}$ to be an orthonormal basis everywhere.

Proposition (B.-F. '10)

$\Theta(\Phi) = \Theta(m)$, *bounded. The Θ-diffusion is conservative if $\mathbb{OM}$ is complete.*

• *A weaker geometric control.* Controlled dynamics: $\dot{\Psi}_s = H_0 u_s^0 + V_i u_s^i$
Reference point Φ_{ref}. Define $\mathcal{T}(\Phi) =$ minimal traveling time from Φ_{ref} to Φ, with controls $|u^i|_\infty \leqslant 1$.

$$|H_0 \mathcal{T}| + \sum_{i=1}^{d} |V_i \mathcal{T}| \leqslant 1.$$



$\Theta_r := \max \Theta(\cdot)$ on the set $B_r := \{\mathcal{T} \leqslant r\}$.

Theorem (B.-F. '10)

The Θ-diffusion is conservative if

$$\int^{\infty} \frac{r\,dr}{\Theta_r \log\big(\Theta_r \mathrm{Vol}(B_r)\big)} = \infty.$$

If $\mathbb{M} \sim \mathbb{R}^{1,3}$, Θ bounded, and g, g^{-1} and their first derivative are bounded, then non-explosion.

Explosion

Write ζ for the lifetime of the diffusion.

PROPOSITION

If $\mathbb{P}_\Phi(\zeta < \infty) > 0$, then $\mathbb{P}_\Phi(\zeta < \epsilon) > 0$, for all $\epsilon > 0$.

A simple explosion criterion. On some manifold $\mathcal{M}$.

LEMMA

Suppose there exists two smooth functions $f \leqslant h$ and two constants $0 \leqslant c' < c$ such that

$$\mathcal{G}f \geqslant c\, f \quad \textit{and} \quad \mathcal{G}h \leqslant c'\, h.$$

Let $x_0 \in \mathcal{M}$ be such that $f(x_0) > 0$. Then the diffusion with generator $\mathcal{G}$ started from x_0 explodes with positive probability.

In our case:

$$\mathcal{G} = H_0 + \frac{1}{2} \sum_{i=1..3} V_i^2 .$$

Given $\Phi = (m, \mathbf{e}) = \big(m, (\mathbf{e}_0, \cdots, \mathbf{e}_3)\big) \in \mathbb{OM}$, set

$$U(\Phi) = \frac{3}{2} \int_{T^1_m \mathbb{M}} G(\mathbf{e}_0, y) Ric_m(y, y)\, dy,$$

where G is the Green function of Laplacian on each fiber of $T^1\mathbb{M}$. One can apply the preceding explosion lemma to $f = Ric_{|T^1\mathbb{M}}$ and $h := Ric_{|T^1\mathbb{M}} + U$, under some conditions. Write R for the scalar curvature.

Theorem (B. 10')

Let $(\mathbb{M}, g)$ be a Lorentzian manifold satisfying the following conditions.

(1') *Static energy condition.* $Ric \geqslant 0$ *non-constant,* $R \leqslant 0$.

(2') *Regularity condition.* $\exists\ 0 < \alpha < 1,\ 0 \leqslant c' < c$ *and* $c < 2 < \frac{c'}{\alpha}$ *such that* $c - c' < \ 2\,(1-\alpha)$ *and*

$$\frac{1-\alpha}{\alpha}\, Ric_{|\,T^1\mathbb{M}} \leqslant U.$$

(3') *Dynamic energy condition.* (i) $H_0\, h \leqslant (c' - 2\alpha)\, h$,
(ii) $H_0\, Ric \geqslant (c-2)\, Ric$.

Let $\Phi_0 \in \mathbb{OM}$ be such that $Ric(\Phi_0) > 0$. Then the relativistic diffusion started from Φ_0 explodes with positive probability.

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Globally hyperbolic spacetimes. A function T on a Lorentzian manifold is said to be a (global) time function if *(1)* ∇T is everywhere timelike, *(2)* each level hypersurface $\{T = t\}$ is a (connected) spacelike submanifold, *(3)* each integral curve of ∇T meets each hypersurface $\{T = t\}$ at precisely one point.

THEOREM (Geroch '70)

A Lorentzian manifold has a global time function iff it is globally hyperbolic.

On $\mathbb{OM}$. *A weaker definition:* A (smooth) function $T : \mathbb{OM} \rightarrow \mathbb{R}$ is a time function if it increases along any (lifting to $\mathbb{OM}$ of a) timelike geodesic.

THEOREM (B. '10)

In any strongly causal Lorentzian manifold, the orthonormal frame bundle $\mathbb{OM}$ *has a time function.*

Some questions. (With an eye towards Penrose, Hawking incompleteness theorems...)

• Find a pathwise version of the global explosion criterion used.

• What happens if the diffusion enters a region satisfying the conditions of a geometric incompleteness theorem?

• Can a relativistic diffusion miss a naked singularity? (Like in the fast rotatoing Kerr black-holes.)