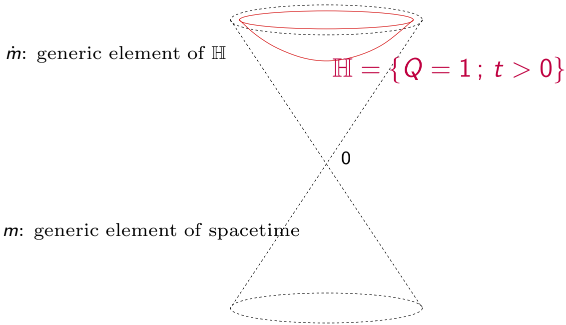
A pathwise approach to relativistic diffusions

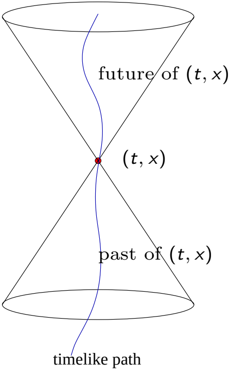
Ismaël Bailleul,
(Statistical Laboratory, Cambridge)

Lvov, June 2009.

- Randomly moving in Minkowski spacetime: from Langevin equation to $(V, \mathfrak{z})$-processes
- One-particle distribution function
- Dynamics on a Lorentzian manifold

- $Q(m) = t^2 - |x|^2$ defines the geometry of Minkowski spacetime $\mathbb{R}^{1,3}$.





In an *empty environment*

- $m_s = m_0 + \int_0^s \dot{m}_r \, dr$: process $X_s = \big(m_s, \dot{m}_s\big) \in \mathbb{R}^{1,3} \times \mathbb{H}$
- Law of $\{X_s\}_{s \geqslant 0}$ independent of the rest frame, *i.e.* defined intrinsically.

Theorem (Dudley, 66')

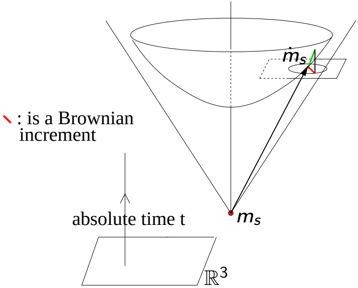
If X is continuous, then $\dot{m}_s$ is a Brownian motion on $\mathbb{H}$.

In a *non-empty environment*: Relativistic Langevin process (Debbasch, Mallick, Rivet, 97')

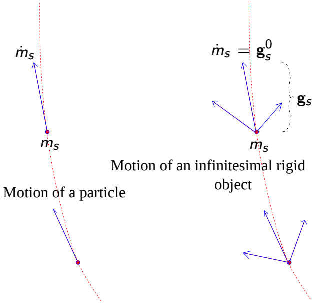
q: $\mathbb{R}^3$-part of $\dot{m}$; $\gamma(q) = \sqrt{1 + |q|^2}$; W an $\mathbb{R}^3$-Brownian motion

$$dx_t = \frac{q_t}{\gamma(q_t)} dt$$
$$dq_t = -2\alpha \frac{q_t}{\gamma(q_t)} dt + \circ dW_t$$

Jüttner distribution is invariant.



Phase space: orthonormal frame bundle $\mathbb{OR}^{1,3}$ of $\mathbb{R}^{1,3}$, $(m, \mathbf{g})$ a generic element, $\mathbf{g} = (\mathbf{g}^0, \mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3) \in \mathbb{O}(1,3)$
$(m_s, \mathbf{g}_s) \in \mathbb{OR}^{1,3}$



$(V, \mathfrak{z})$-processes: there exists at each (proper) time (of the moving particle) a (local) rest frame where the random part of the acceleration of the particle is Brownian in any spacelike direction of the frame, when computed using the time of the rest frame.

An exemple: Dudley's process in $\mathbb{R}^{1,2}$

No deterministic acceleration. The local frame in which $\mathbf{e}_s = (m_s, \mathbf{g}_s)$ has a Brownian acceleration is $\mathbf{g}_s$ itself.
The green Brownian increment of $\mathbf{g}_s^0$ in $\mathbb{H}$ is indeed Brownian in any spacelike direction of the plane $\langle \mathbf{g}_s^1, \mathbf{g}_s^2 \rangle$.

Infinitesimal hyperbolic rotations:

$$(1) : \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = E_1,$$

$$(2) : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = E_2.$$

$\mathbf{g} \in \mathbb{O}_3 : V_i(\mathbf{g}) = \mathbf{g}E_i$: left invariant vector fields on $SO(1,2)$.

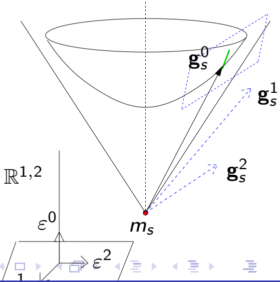
H_0: generates geodesic flow

Generator: $H_0 + \frac{1}{2}\left(V_1^2 + V_2^2\right)$

Dynamics:

$$\circ dm_s = \mathbf{g}_s^0 ds$$

$$\circ d\mathbf{g}_s = \sum_{i=1..2} V_i(\mathbf{g}_s) \circ dW_s^i$$



An exemple: $(0, Id)$*-process*:

For $\circ d\widehat{\mathbf{g}}_t^0 = \sum_{j=1..3} \widehat{\mathbf{g}}_t^j \circ d\widehat{\beta}_t^j$ to have an $\mathbb{R}^3$-part $= \sum_{i=1..3} \varepsilon^i \circ d\widehat{W}_t^i$, we need

$$\circ d\widehat{W}_t^i = -\sum_{j=1..3} q(\varepsilon^i, \widehat{\mathbf{g}}_t^j) \circ d\widehat{\beta}_t^j$$

If $A(\mathbf{g})_{ij} = q\left(\varepsilon^i, \mathbf{g}^j\right)$, then

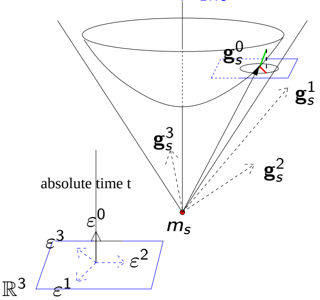
$$\circ d\widehat{\beta}_t = -A(\widehat{\mathbf{g}}_t)^{-1} \circ d\widehat{W}_t$$

Back in proper time s, $\circ d\mathbf{g}_s^0 = \sum_{j=1..3} \mathbf{g}_s^j \circ d\beta_s^j$, with

$$\circ d\beta_s = -q(\varepsilon^0, \mathbf{g}_s^0)^{\frac{1}{2}} A(\mathbf{g}_s)^{-1} \circ dW_t$$

Dynamics: $dm_s = \mathbf{g}_s^0\, ds$

$$d\mathbf{g}_s = \sum_{i=1..3} V_i(\mathbf{e}_s) \circ d\beta_s^i$$



$(V, \mathfrak{z})$-processes: $\{\mathbf{e}_s\}_{s\geqslant 0} = \{(m_s, \mathbf{g}_s)\}_{s\geqslant 0}$

Setting:

- V vector field on $\mathbb{O}_3$:

$$\circ dm_s = \mathbf{g}_s^0\, ds, \quad \circ d\mathbf{g}_s = V(\mathbf{e}_s)ds,$$

unperturbed (or mean) dynamics.

- Random previsible $\mathbb{OR}^{1,3}$-valued functionnal $\{\mathfrak{z}_s(\mathbf{e}_.)\}_{s\geqslant 0}$ such that $\mathfrak{z}_s = (m_s, f_s) = \left(m_s, (f_s^0, ..., f_s^3)\right)$ belongs to $\mathbb{O}_{m_s}\mathbb{R}^{1,3}$.

$(*)$ A time-dependent random matrix:
$A_{ij}(s) = q(f_s^i, \mathbf{g}_s^j)$

$(*)$ $\circ d\beta_s = q(f_s^0, \mathbf{g}_s^0)^{\frac{1}{2}}\, A_s^{-1} \circ dW_s$,
$\mathbb{R}^3$-valued random noise.

Dynamics:

$$\circ dm_s = \mathbf{g}_s^0\, ds,$$
$$\circ d\mathbf{g}_s = V(\mathbf{e}_s)ds + \sum_{i=1..3} V_i(\mathbf{g}_s)\circ d\beta_s^i$$

Generator:

$$B = B(\mathbf{g}) = \left(A(\mathbf{g})^{-1}\right)^* A(\mathbf{g})^{-1}.$$

$$L = H_0 + V + \frac{1}{2}\sum_{i=1..3} V_i B^{ij} V_j$$

- Randomly moving in Minkowski spacetime: from Langevin equation to $(V, \mathfrak{z})$-processes
- One-particle distribution function
- Dynamics on a Lorentzian manifold

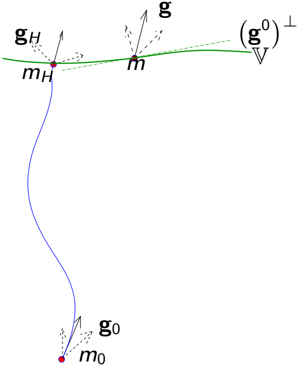
Definition. Given $\mathbf{e} = (m, \mathbf{g}) \in \mathbb{OR}^{1,3}$, set $\mathcal{V}_{\mathbf{e}} = \{\mathbb{V}\,;\ \text{spacelike hypersurfaces such that } m \in \mathbb{V} \text{ and } T_m\mathbb{V} = (\mathbf{g}^0)^{\perp}\}$. For $\mathbb{V} \in \mathcal{V}_{\mathbf{e}}$, set $H = \inf\{s \geqslant 0\,;\ m_s \in \mathbb{V}\}$.

$\sigma_{\mathbb{V}}$: volume element on $\mathbb{V}$
$\mathbb{O}\mathbb{V} = \mathbb{V} \times \mathbb{O}(1,3)$
$d\mathrm{VOL}_{\mathbb{O}\mathbb{V}} = d\widehat{\mathbf{g}} \otimes d\sigma_{\mathbb{V}}$

PROPOSITION/DEFINITION

1. *The random variable* $\mathbf{e}_H \mathbf{1}_{H<\infty}$ *has a smooth density* $f_{\mathbb{V}}(\mathbf{e}_0\,;\,\widehat{\mathbf{e}})$ *with respect to the measure* $\mathrm{VOL}_{\mathbb{O}\mathbb{V}}(d\widehat{\mathbf{e}})$ *on* $\mathbb{O}\mathbb{V}$.
2. *We have* $f_{\mathbb{V}'}(\mathbf{e}_0\,;\,\mathbf{e}) = f_{\mathbb{V}}(\mathbf{e}_0\,;\,\mathbf{e})$ *for any other* $\mathbb{V}'$ *in* $\mathcal{V}_{\mathbf{e}}$.

$f_{\mathbb{V}}(\mathbf{e}_0\,;\,\mathbf{e})$ *is independent of* $\mathbb{V} \in \mathcal{V}_{\mathbf{e}}$*; call it the value at point* $\mathbf{e}$ *of the* **one-particle distribution function of the** $(V, \mathfrak{z})$**-diffusion started from** $\mathbf{e}_0$. *We shall denote it by* $f(\mathbf{e}_0\,;\,\mathbf{e})$*; it is defined for* $\mathbf{e} \neq \mathbf{e}_0$.



Recall: $L = H_0 + V + \frac{1}{2} \sum_{i=1..3} V_i B^{ij} V_j$, generator of the random motion.

THEOREM

We have $L^* f(\mathbf{e}_0\,;\,\cdot) = 0$ *in* $\mathbb{OM} \backslash \{\mathbf{e}_0\}$.

Let f, g such that $L^* f = L^* g = 0$. Set

$$X(m) = -\int_{\mathbb{O}(1,3)} \mathbf{g}^0\, f(m,\mathbf{g}) \ln \frac{f(m,\mathbf{g})}{g(m,\mathbf{g})}\, d\mathbf{g}.$$

THEOREM (H-theorem)

$divX \geqslant 0$.

- Randomly moving in Minkowski spacetime: from Langevin equation to $(V, \mathfrak{z})$-processes

- One-particle distribution function

- Dynamics on a Lorentzian manifold

• Geometrical framework. $(\mathbb{M}, q)$ Lorentzian manifold of dimension $1 + d$, $\mathbb{OM}$ orthonormal frame bundle over $\mathbb{M}$; generic element $\mathbf{e} = (m, \mathbf{g}) = \big(m, (\mathbf{g}^0, \mathbf{g}^1, \dots, \mathbf{g}^d)\big)$. Geodesic flow on $\mathbb{OM}$ induced by vector field H_0. Vertical vector fields $V_i, i = 1..d$: generate hyperbolic rotations in each fiber over $\mathbb{M}$.

V: vertical vector field on $\mathbb{OM}$. *Unperturbed dynamics*:
$d\mathbf{e}_s = H_0(\mathbf{e}_s)ds + V(\mathbf{e}_s)ds$

• Step-by-step description of the random dynamics (for $V = 0$).
$\mathfrak{z}(\mathbf{e}_{.}) = \big(f^0(\mathbf{e}_{.}), f^1(\mathbf{e}_{.}), \dots, f^d(\mathbf{e}_{.})\big)$: previsible 'action-process'.

1. $m_{s+\delta s} = m_s + \mathbf{g}^0_s\, \delta s$,
2. $\mathbf{g}^0_{s+\delta s} = \mathbf{g}^0_s + \delta_{geod}\mathbf{g}^0_s + \delta_{rand}\mathbf{g}^0_s$. Increment $\delta_{rand}\mathbf{g}^0_s$: only vector of $T_{\mathbf{g}^0_s}(\mathbb{H}_m\mathbb{M})$ s.t. its projection in $\mathrm{span}\big(f^1(\mathbf{e}_s), \dots, f^d(\mathbf{e}_s)\big) \,/\,/\, f^0(\mathbf{e}_s)$ = scaled Brownian increment
$$q\big(f^0(\mathbf{e}_s), \mathbf{g}^0_s\big)^{\frac{1}{2}} \sum_{i=1}^{d} f^i(\mathbf{e}_s) \circ dW^i_s.$$
3. Transport parallelly $\{\mathbf{g}^1_s, \dots, \mathbf{g}^d_s\}$ along the increment $\delta\mathbf{g}^0_s$ of $\mathbf{g}^0_s$.

• Example: Langevin equation in spacially flat Robertson-Walker spacetime.
$Q = dt^2 - a(t)^2|dx|^2$. Co-ordinates: $(x, q) \in \mathbb{R}^3 \times \mathbb{R}^3$. Set $\gamma_t(q) = \sqrt{1 + a(t)^2|q|^2}$.

$$dx_t = \frac{q_t}{\gamma_t(q_t)}dt,$$
$$dq_t = -2\alpha\, a(t)^2\frac{q_t}{\gamma_t(q_t)}dt + \frac{1}{a(t)} \circ dw_t.$$

• *H-theorem holds for general $(V, \mathfrak{z})$-diffusions.*

- Asymptotic behaviour of $(0, \mathbf{e}_.)$-process $\{(m_s, \mathbf{g}_s)\}_{s\geqslant 0}$

THEOREM

In Minkowski spacetime, $\{m_s\}_{s\geqslant 0}$ almost-surely converges towards a random point of the causal boundary. Any open set of the causal boundary can be hit.

CONJECTURE

In strongly causal spacetimes, the $\mathbb{M}$-part of the $(0, \mathbf{e}_.)$-process almost-surely converges towards a random point of the causal boundary.

- Life-time of the process.

OPEN PROBLEM

Is null geodesic incompleteness equivalent to explosion of the $(0, \mathbf{e}_.)$-process with positive probability?