Rough flows

Ismael Bailleul,
Univ. Rennes 1, France

ASC - IMS 2014 – Sydney

It is easy to construct flows!

$$\text{approximate flow} \quad \xrightarrow[\textit{continuous}]{\textit{simple}} \quad \text{flow}$$

Deal with

- **classical RDEs** with **infinite dimensional** state space/**signal**
- **stochastic mean field RDEs**,
- a 'rough analogue' of **stochastic flows**

Layout of the talk

1. **Flows and approximate flows**
2. **An illustration: From controlled ODEs to RDEs**
3. **Rough flows**

1. 'Approximate flows' and flows

- Definition. *A* $\mathcal{C}^1$-**approximate flow** *is a family of* $\mathcal{C}^2$ *maps* $\mu_{ts} : E \to E$, *continuous wrt* (s,t) *for uniform convergence, with* $\|\mu_{ts} - I\|_{\mathcal{C}^2} \leq o_{t-s}(1)$, *and*

$$\|\mu_{tu} \circ \mu_{us} - \mu_{ts}\|_{\mathcal{C}^1} \leq c_1 \, |t-s|^a$$

for some positive constants c_1 *and* $a > 1$ *and all* $0 \leq s \leq u \leq t \leq T$.

For a partition $\pi_{ts} = \{s < t_1 < \cdots < t_n < t\}$ of $[s,t]$, set

$$\mu_{\pi_{ts}} := \bigcirc_{i=0}^{n-1} \mu_{t_{i+1}t_i}.$$

- Theorem [B, 12']. **A** $\mathcal{C}^1$**-approximate flow** μ **defines a unique flow** φ *st.* $\|\varphi_{ts} - \mu_{ts}\|_\infty \lesssim |t-s|^a$; *moreover*

$$\|\varphi_{ts} - \mu_{\pi_{ts}}\|_\infty \lesssim c_1^2 \, |\pi_{ts}|^{a-1}.$$

- Remark. *Elementary* and *short* proof.

  Choice of μ_{ts} guided by local considerations on " *Taylor expansions*".

Given $h \in \mathcal{C}^\alpha, \alpha > \frac{1}{2}$ and $\mathrm{F} = (V_1, \dots, V_\ell)$ vector fields on E, of class $\mathcal{C}_b^2$

$$dz_t = \mathrm{F}(z_t)\, dh_t. \tag{1}$$

▶ Definition. *A* **solution flow** *to equation* (1) *is a flow* φ *with a* "uniform Taylor expansion", *at any time* s *and any point* x, *of the form*

$$f\big(\varphi_{ts}(x)\big) = f\,(x) + h^i_{ts}(V_i f)(x) + O\big(|t-s|^{>1}\big), \tag{2}$$

for all f *regular enough.*

▶ **Method for constructing the solution flow to equation** (1)

1. Candidate for a map μ_{ts} with **good Taylor expansion**

   $$\mu_{ts}(x) = x + h^i_{ts} V_i(x).$$

   It satisfies (2) but is *not a flow*.
2. μ is a $\mathcal{C}^1$-**approximate flow**: $\big\|\mu_{tu} \circ \mu_{us} - \mu_{ts}\big\|_{\mathcal{C}^1} \leq c_1\, |t-s|^{2\alpha}$.
3. Its associated flow satisfies (2) since $\big\|\varphi_{ts} - \mu_{ts}\big\|_\infty \lesssim |t-s|^{2\alpha}$.

2. Flows generated by classical RDEs

$\mathrm{F} = (V_1, \dots, V_\ell) : \mathrm{Lip}_3$ vector fields on E, $\mathbf{X}$ a weak geometric Hölder p-rough path over $\mathbb{R}^\ell$,

$$dz_t = \mathrm{F}(z_t)\,\mathbf{X}(dt). \tag{3}$$

▶ Definition. *A* **solution flow** *to equation* (3) *is a flow* φ *with* "uniform Taylor expansion", *at any time s and any point x, of the form*

$$f\big(\varphi_{ts}(x)\big) = f(x) + \sum_{1 \le |I| \le [p]} X^I_{ts}(V_I f)(x) + O\big(|t-s|^{>1}\big),$$

with V_i *identified with a first order diff. operator and*

$$V_I f = V_{i_1} \cdots V_{i_k} f, \;\; \textit{if } I = (i_1, \dots, i_k).$$

Set $V_{[I]} = \Big[V_{i_1}, \big[\dots, [V_{i_{k-1}}, V_{i_k}], \dots\big]\Big]$ and $\boldsymbol{\Lambda}_{ts} := \log \mathbf{X}_{ts}$. Define μ_{ts} as the **time 1 map of the ODE**

$$\dot{y}_u = \big(\Lambda^I_{ts} V_{[I]}\big)(y_u), \quad 0 \le u \le 1.$$

▶ Proposition [B, 12']. *We have*

$$\left\| f \circ \mu_{ts} - \left(f + \sum_{1 \leq |I| \leq [p]} X^I_{ts} (V_I f) \right) \right\|_\infty \leq c_f(\mathbf{X}) \, |t-s|^{>1},$$

and μ *is a* $\mathcal{C}^1$-**approximate flow**.

▶ Theorem [B, 12']. *The RDE* $dz_t = \mathrm{F}(z_t)\,\mathbf{X}(dt)$ *has a* **unique solution flow** φ. *It satisfies*

$$\left\| \varphi_{ts} - \mu_{\pi_{ts}} \right\|_\infty \lesssim c_1^2 \left| \pi_{ts} \right|^{a-1},$$

where c_1 *is polynomial in the norm of* $\mathbf{X}$.

▶ Remarks [B, 12'-13']. The approach can deal with '**Banach space-valued' rough paths** and **unbounded vector fields with linear growth**, giving well-posedness results.

3. From stochastic flows to rough flows

- **Ito** setting $\circ dz_t = V_i(z_t) \circ dB_t^i$ one can **separate space** (V_i) and **noise** (B)
- **stochastic flow** setting $dy_t = \mathrm{F}(y_t, \circ dB_t)$ one **cannot separate space** from **noise**

  Fundamental object **vector field-valued** Brownian motion, or **semimartingale**.

- **Rough path** setting **lift** B into a rough path $\mathbf{B}$ RDE $dz_t = \mathrm{F}(z_t)\mathbf{B}(dt)$
- **Rough flow** setting **lift** $\mathrm{F}(y_t, \circ dB_t)$ into **?** **?!**

3.1 Rough vector fields

Let $2 < p < 3$ be given, and $V(\cdot, t)$ be a time-dependent **velocity field** on E. Set $V_{ts}(\cdot) = V(\cdot, t) - V(\cdot, s)$.

▶ Definition. A **(weak geometric Hölder) p-rough vector field** is a family $\left(\mathbf{V}_{ts}\right)_{0 \leq s \leq t \leq T}$, where $\mathbf{V}_{ts} = \left(V_{ts}, \mathbb{V}_{ts}\right)$, and $\mathbb{V}_{ts}$ is a second order differential operator s.t.

(i) the **vector fields** V_{ts} are $\mathcal{C}^3_b$, with $\sup_{0 \leq s \leq t \leq T} \frac{\left\| V_{ts} \right\|_{\mathcal{C}^3}}{|t-s|^{\frac{1}{p}}} < \infty$,

(ii) the second order differential operators $W_{ts} := \mathbb{V}_{ts} - \frac{1}{2} V_{ts}^2$, are actually **vector fields**, and

$$\sup_{0 \leq s \leq t \leq T} \frac{\left\| W_{ts} \right\|_{\mathcal{C}^2}}{|t-s|^{\frac{2}{p}}} < \infty,$$

(iii) we have $\mathbb{V}_{ts} = \mathbb{V}_{tu} + \mathbb{V}_{us} + V_{us} V_{tu}$, for all $0 \leq s \leq u \leq t \leq T$.

3.2 Rough flows

Let μ_{ts} be the **time 1 map of the ODE**

$$\dot{y}_u = \big(V_{ts} + W_{ts}\big)(y_u), \quad 0 \leq u \leq 1.$$

- Theorem [BR, 14']. *We have*

$$\big\| f \circ \mu_{ts} - \big(f + V_{ts} f + \mathbb{V}_{ts} f\big) \big\|_\infty \leq c_f(\mathbf{V})\, |t-s|^{>1}$$

*and μ is a $\mathcal{C}^1$-**approximate flow** which depends continuously on $\mathbf{V}$. The unique flow associated to μ is said to solve the **RDE on flows***

$$d\varphi_s = \mathbf{V}(\varphi, \circ dt),$$

*and called a **rough flow**; it is a continuous function of $\mathbf{V}$.*

- Remarks. **Continuous semimartingale vector fields** have rough lifts; their associated **rough flows** are the awaited **stochastic flows**. One can also lift **Gaussian vector fields** to rough vector fields, and study their *associated flows*: **stable/unstable manifold theorems** for such dynamics.

References

[B, 12'] **Flows driven by rough paths**

(Submitted)

[B, 13'] **Flows driven by Banach- space-valued rough paths**

(To appear in *Séminaires de Probabilités*, 2014)

[BR, 14'] **Rough flows,** with S. Riedel (T.U. Berlin),

(To be submitted)