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P Baird

Harmonic maps with symmetry, harmonic morphisms and deformations of metrics

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Preface

We consider the basic problem of harmonic maps, that of finding a harmonic representative for a given homotopy class of maps between two Riemannian manifolds. Given that we are unable to solve this problem for fixed metrics, we can ask whether there exist metrics with respect to which there exists a harmonic representative - this is known as the 'rendering problem'. In particular, we study these two problems for maps between spheres. We do this by considering maps where 'reduction occurs', that is, because the map possesses certain symmetries (equivariant), the problem of harmonicity reduces to solving a certain second-order nonlinear ordinary differential equation (the reduction equation). The origin of this method is the Thesis of R.T. Smith (1972), in which certain harmonic maps between Euclidean spheres are constructed.

The symmetry which we make use of is that the map should preserve families of parallel hypersurfaces with constant mean curvature. It should be noted that this seems to be a more natural symmetry in the context of harmonic maps than the more commonly exploited symmetry of 'equivariance with respect to group actions'.

We prove a reduction theorem for harmonic maps between space forms, and provide many examples of maps satisfying the conditions of the theorem. In some instances the reduction equation is easily soluble, on the other hand we find examples where we have little idea about appropriate solutions to the corresponding reduction equation.

Using the stress-energy tensor associated to harmonic maps we study harmonic morphisms, proving a theorem characterizing those harmonic morphisms with minimal fibres. We then consider harmonic morphisms defined by homogeneous polynomials, relating such maps to the construction of maps between spheres where reduction occurs.

By allowing deformations of the metrics, first for harmonic morphisms and then for maps where reduction occurs, we are able to solve the rendering problem for all classes of the homotopy groups $\pi_n(S^n) = \mathbb{Z}$ for all n.

Recently it has become apparent that harmonic maps have a significant role to play in certain problems of theoretical physics. A particular example is the description of solitons in terms of harmonic maps from S^2 into the complex projective space CP^n. It is to be hoped that this work will lead to a greater understanding of solutions with symmetry of some of the variational problems of physics.

I would like to express my most sincere thanks to Jim Eells who has provided continued support and encouragement during the preparation of this work.

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Paul Baird

Contents

0.	INTRODUCTION	1
1.	FIRST CONSTRUCTIONS	7
1.1	The Laplacian on the sphere	7
1.2	Harmonic maps into spheres	9
1.3	Joins of spheres and Smith's construction	11
1.4	Outline of the solution of Smith's equation	16
1.5	Hyperbolic space	19
1.6	Polar coordinates on hyperbolic space and an analogous construction to Smith's	22
1.7	Solving the equation for hyperbolic spaces	24
2.	ISOPARAMETRIC FUNCTIONS	27
2.1	Definition of isoparametric function	27
2.2	Properties of isoparametric functions and Münzner's classification theorem	29
2.3	Examples of isoparametric functions	34
2.4	Generalizing the notion of isoparametric families of hypersurfaces	39
3.	THE STRESS-ENERGY TENSOR	42
3.1	Derivation of the stress-energy tensor	42
3.2		46
3.3	The eigenvalue decomposition of the stress-energy tensor	47
4.	EQUIVARIANT THEORY	49
4.1	Maps which are equivariant with respect to isoparametric functions	49

4.2 Generalized equivariant maps between Riemannian manifolds 65

5. EXAMPLES OF EQUIVARIANT MAPS 69

5.1 Maps from Euclidean space to the sphere 69

5.2 Maps from hyperbolic space to the sphere 76

5.3 Maps from sphere to sphere 80

6. ON CERTAIN ORDINARY DIFFERENTIAL EQUATIONS OF THE PENDULUM TYPE 94

6.1 Existence of solutions 94

6.2 Asymptotic estimates 111

6.3 Smoothness of certain equivariant harmonic maps 116

7. THE GENERAL THEORY OF HARMONIC MORPHISMS 122

7.1 General theory 122

7.2 Examples and non-examples of harmonic morphisms 125

7.3 Maps $\phi : (M,g) \longrightarrow (N,h)$ where $\phi^* h$ has two distinct non-zero eigenvalues 133

8. HARMONIC MORPHISMS DEFINED BY HOMOGENEOUS POLYNOMIALS 135

8.1 Properties of harmonic polynomial morphisms 135

8.2 Some examples 139

8.3 Harmonic morphisms defined by homogeneous polynomials of degree bigger than two 142

8.4 Harmonic polynomial morphisms and equivariant maps between spheres 146

9. DEFORMATIONS OF METRICS 153

9.1 Deformations of the metric for harmonic morphisms 153

9.2 Examples 160

9.3 Deformations of metrics for equivariant maps 162

9.4 Examples 172

REFERENCES 177

INDEX OF DEFINITIONS 180

Introduction

Let (M, g) denote the Riemannian manifold M together with its metric g. Let ϕ be a map between Riemannian manifolds:

$$\phi : (M, g) \longrightarrow (N, h)$$

(all manifolds, metrics and maps will be assumed smooth unless otherwise stated). Then the derivative of ϕ, $d\phi$, is a section of the bundle of 1-forms on M with values in the pull-back bundle ϕ^{-1} TN:

$$d\phi \in \mathscr{C}(T^*M \otimes \phi^{-1} TN).$$

The bundle $T^*M \otimes \phi^{-1} TN$ is a Riemannian vector bundle and has a Levi-Civita connection ∇ acting on sections:

$$\nabla : \mathscr{C}(T^*M \otimes \phi^{-1} TN) \longrightarrow \mathscr{C}(\otimes^2 T^*M \otimes \phi^{-1} TN).$$

Call $\nabla d\phi \in \mathscr{C}(\odot^2 T^*M \otimes \phi^{-1} TN)$ the 2^{nd} fundamental form of the map ϕ. Then we say that ϕ is harmonic if $\operatorname{trace}_g \nabla d\phi = \sum_i \nabla d\phi(X_i, X_i)$ is zero at each point of M, where $(X_i)_{1 \leq i \leq \dim M}$ is a local orthonormal frame field for M.

Given a map ϕ as above; there are two basic problems of harmonic maps:

(i) letting $[\phi]$ denote the homotopy class of ϕ, then does there exist a harmonic representative $\tilde{\phi} \in [\phi]$ such that $\tilde{\phi} : (M, g) \longrightarrow (N, h)$ is harmonic?

(ii) Do there exist metrics $\tilde{g}$ and $\tilde{h}$ on M and N respectively, such that there exists a representative $\tilde{\phi} \in [\phi]$ with $\tilde{\phi} : (M, \tilde{g}) \longrightarrow (N, \tilde{h})$ harmonic?

The first of these problems has been considered throughout the history of elliptic analysis and differential geometry. The first existence theorem was proved by Eells and Sampson in [12]. The second problem is known as the rendering problem. This has been considered by Lemaire [28] in the case when the domain is a surface. One of the main objectives of this work is to study these two problems for maps $\phi : S^m \longrightarrow S^n$ between spheres, where the homotopy classes are represented by the groups $\pi_m(S^n)$.

Up until about 1970, the only harmonic maps $\emptyset : S^m \longrightarrow S^n$, $m \geq n$, that were known were a few homogeneous polynomial maps (defined by eigenfunctions of the Laplacian on S^m):

(a) the identity maps $I^n : S^n \longrightarrow S^n$ for all $n \geq 0$;

(b) the Hopf maps $H^n : S^{2n-1} \longrightarrow S^n$, for $n = 2, 4$ and 8;

(c) the maps $G^k : S^1 \longrightarrow S^1 : z \longrightarrow z^k$, where $z \in \mathbb{C}$, $|z|^2 = 1$ and k is any integer.

In his Thesis in 1972, Smith [33] gave a method whereby, given two such homogeneous polynomial maps, and provided certain "damping conditions" are satisfied, one can construct another harmonic map between spheres.

Smith's method was to construct a 1-parameter family of maps all homotopic to the join of the two polynomial maps. This family is parametrized by a function α from the interval to itself. Smith then exploited the symmetry of the so constructed map $\emptyset_\alpha$ to reduce the problem of whether $\emptyset_\alpha$ is harmonic or not to solving a 2^{nd} order non-linear ordinary differential equation in α. This procedure of reducing the problem of harmonicity of a certain map to solving an ordinary differential equation we shall simply call reduction. The corresponding differential equation we shall call the reduction equation. Smith's equation has a simple physical interpretation - that of a pendulum moving under the influence of a variable gravity force with variable damping. The above damping conditions are sufficient conditions for this equation to have a solution.

We place Smith's method into a more general setting by viewing his maps as sending "wavefronts to wavefronts" in the sense of geometric optics. That is, we have a commutative diagram of the form:

$$\begin{array}{ccc} M & \xrightarrow{\ \emptyset_\alpha\ } & N \\ f \downarrow & & \downarrow g \\ I_f & \xrightarrow[\ \alpha\ ]{} & I_g \end{array} ,$$

where f, g are real valued functions on M, N respectively, with values in intervals I_f, I_g respectively., and $\alpha : I_f \longrightarrow I_g$ is a smooth function. The appropriate class of functions to which f and g must belong in order to provide a good theory turns out to be the class of isoparametric functions as defined by Cartan in 1938 [5]. These objects provide a rich and beautiful geometry on spheres, and it is indeed pleasing to find that they should be related to constructing harmonic maps between spheres.

After Cartan's impressive study of isoparametric functions in papers dated 1938 - 1940 [5,6,7,8], the subject lay dormant until Nomizu's paper of about 1970 [32], surveying Cartan's work and giving some open problems. That revived interest in the subject, and Münzner proved an important classification theorem [30]. Also many new examples of isoparametric functions on spheres were found, see [32,34,40,33,17] In particular, examples of isoparametric functions on spheres whose level surfaces are non-homogeneous were found by Ozeki and Takeuchi [33] and Ferus, Karcher and Münzner [17]. These examples are very important from our point of view, since we make essential use of them in Theorem 5.3.8, Example 8.2.2 and in Section 8.4. This demonstrates that isoparametric functions are more natural for our purposes than families of homogeneous submanifolds of space forms (from which point of view one could conceivably derive our theory).

R. Wood was the first person to make a connection between isoparametric functions and harmonic maps [43]. One of the essential features he observed was that such functions satisfy an eikonal equation of the form

$$|df(x)|^2 = c\,|x|^{2p-2} ,$$

for some constant c, where p is an integer, $p \geq 2$. This remarkable fact makes the connection with geometric optics even more striking.

Isoparametric functions as studied by Cartan are defined on space forms - that is, complete connected manifolds of constant curvature. In order to consider the construction of harmonic maps between more general Riemannian manifolds, we make what we consider to be a suitable definition of an isoparametric function on an arbitrary Riemannian manifold in Section 2.4.

In Chapter 4 we develop the general framework, and define the necessary conditions on ϕ in order that reduction occurs. Maps ϕ satisfying these conditions we call equivariant. We prove our main theorem (Theorem 4.1.8) which is a reduction theorem for harmonic maps between space forms.

As a consequence of the above generalization, we can construct explicit harmonic maps from Euclidean spaces to spheres and from hyperbolic spaces to spheres - this is done in Sections 5.1 and 5.2. In Section 5.3 we find a very large number of classes $\pi_m(S^n)$ where reduction occurs. However, many of the corresponding reduction equations have a qualitatively different nature to those of Smith, and we are unable at present to solve many of them. One important class of equations does have similar properties to those of Smith, and the solutions yield some new and interesting harmonic maps between Euclidean spheres of the same dimension. This is proved in Theorem 5.3.8.

Two of the most significant examples in Smith's Thesis are

(i) the construction of harmonic representatives for all classes of $\pi_n(S^n) = \mathbb{Z}$, for all $n \leq 7$, and

(ii) the consideration of the classes of $\pi_3(S^2) = \mathbb{Z}$, which are parametrized by their Hopf invariant (see [26]).

The important point in example (i) is that the condition $n \leq 7$ is a consequence of the damping conditions. Thus Smith's method fails to give harmonic representatives for the classes of $\pi_n(S^n)$ with $n > 7$. In example (ii) the classes of $\pi_3(S^2)$ are parametrised by their Hopf invariant $d \in \mathbb{Z}$. When d is the square of an integer, $d = k^2$ for $k \in \mathbb{Z}$; Smith gives a harmonic representative of this class. However, when $d = kl$, $k, l \in \mathbb{Z}$, $k, l \neq 0$ and $k \neq 1$; Smith provides a representative of the class where reduction occurs, but demonstrates that the corresponding reduction equation does not have a solution.

With the above two examples in mind we consider deformations of the metrics on space forms (Chapter 9) - this fits quite naturally into our general framework. We thus come to one of the main consequences of the reduction theorem. This is the solution of the rendering problem for all classes of $\pi_n(S^n)$ for all n (Theorem 9.4.5). The deformed spheres are familiar ellipsoids whose eccentricities depend only on n and the degree. Even allowing various deformations of metrics, we are still at present unable to solve the rendering problem for $\pi_3(S^2)$.

In Chapter 7 we undertake a general study of harmonic morphisms - maps between Riemannian manifolds which pull back germs of harmonic functions to germs of harmonic functions. We prove a theorem characterizing those harmonic morphisms for which the fibres are minimal submanifolds. This generalizes a result of Eells and Sampson [12] stating that every Riemannian submersion which is harmonic has minimal fibres.

Harmonic morphisms were first considered in detail by Fuglede [19], and we make use of his ideas in Chapter 8 when we consider harmonic morphisms defined by homogeneous polynomials. Here we find an interesting connection with isoparametric functions, and we prove a theorem which associates to a certain class of harmonic polynomial morphisms an interesting harmonic Riemannian submersion. By using this theorem we can construct more maps between spheres where reduction occurs.

One of the important tools used in Chapters 7, 8 and 9 is the stress-energy tensor - a divergence free symmetric 2-tensor field on M. Such objects are well-known and important in relativity theory, where they in some sense model the matter distribution in a space-time model. For if $(S_{ij})_{1 \leq i, j \leq 3}$ are the space components of a symmetric 2-tensor, then S_{ij} is the i-component of a force associated to the j-vector (the "stress" acting on a unit area orthogonal to j). Since force is a time rate of change of momentum, this represents the rate of flow of the i-component of momentum through a unit area orthogonal to j. We introduce the time components of S, to obtain the 4-tensor $(S_{ij})_{0 < i, j \leq 3}$, where now S_{i0} represents energy flow, and S_{00} energy density. That div S = 0 and S be symmetric corresponds to various conservation laws. For if X is a Killing vector field, then

$$\operatorname{div}(S(X)) = 0 ,$$

which corresponds to conservation of momentum if X is spacelike, and conservation of energy if X is timelike (using the terminology of relativity theory, see [14]). See [18] for more details.

Hilbert was the first person to derive the stress-energy tensor from a variational principle [22], and it was following a suggestion of Taub (1963) that the stress-energy tensor was used to study harmonic maps in [2].

In chapter 3 we briefly derive and study basic properties of the stress-energy tensor for harmonic maps. For a much more detailed account see [2].

The emphasis with this work is on examples of harmonic maps. However, in chapter 4 we attempt to provide a general framework. This framework is rather unwieldy because of (a) the large number of conditions on the map $\emptyset$, and (b) the necessary consideration of special cases. This detracts from the underlying simplicity, and often it is better to consider each example individually.

1 First constructions

1.1 THE LAPLACIAN ON THE SPHERE

Let M and N be smooth Riemannian manifolds of dimension m and n respectively. Henceforth we shall write (M,g) to denote M together with its metric g.

Let $\phi : M \to N$ be a smooth map, then the derivative $d\phi$ is a section of the bundle of ϕ^{-1} TN-valued 1-forms on M:

$$d\phi \in \mathscr{C}(T^*M \otimes \phi^{-1} TN).$$

Let ∇^M denote the Levi-Civita connection on the bundle $TM \to M$, then ∇^M extends to a connection on the bundle of tensors $\otimes^p TM \otimes \otimes^q T^*M \to M$. The bundle $\phi^{-1} TN \to M$ has a Riemannian structure induced from that of $TN \to N$, allowing us to define the Levi-Civita connection ∇ on the bundle $T^*M \otimes \phi^{-1}TN \to M$. The section $\nabla(d\phi) \in \mathscr{C}(\odot^2 T^*M \otimes \phi^{-1} TN)$ is called the second fundamental form of the map ϕ.

Lemma 1.1.1 If $X, Y \in \mathscr{C}(TM)$, then

$$\nabla d\phi(X,Y) = -d\phi(\nabla^M_X Y) + \nabla^N_{d\phi(X)} d\phi(Y). \tag{1.1.1}$$

Remark 1.1.2 This lemma is saying that contraction and covariant differentiation commute.

We define the Laplacian $\Delta\phi$ of ϕ, to be the trace with respect to g of the second fundamental form $\nabla d\phi$:

$$\Delta\phi(x) = \sum_a \nabla d\phi_x(X_a(x), X_a(x)),$$

at each point x of M, where $X_a \in \mathscr{C}(TM)$, a = 1,2, ... , m, form a local orthonormal frame field about x. Then ϕ is harmonic if $\Delta\phi = 0$.

Example 1.1.3 If $M = \mathbb{R}^m$ with its standard Euclidean metric, and standard coordinates $x_1, x_2, \dots, x_m$, and $\phi : \mathbb{R}^m \to \mathbb{R}$ is a smooth function, then

$$\Delta \emptyset = \sum_i \frac{\partial^2 \emptyset}{\partial x_i^2} .$$

If $\emptyset : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\emptyset = (\emptyset_1, \ldots, \emptyset_n)$, $\emptyset_r : \mathbb{R}^m \longrightarrow \mathbb{R}$, $r = 1, \ldots, n$, then $\Delta\emptyset = (\Delta\emptyset_1, \ldots, \Delta\emptyset_n)$.

Remark 1.1.4 We define the energy density of the map $\emptyset$ to be the function $e(\emptyset) : M \longrightarrow \mathbb{R}$ given by

$$e(\emptyset)(x) = \frac{1}{2} |d\emptyset(x)|^2$$

$$= \frac{1}{2} \sum_a h(d\emptyset_x(X_a), d\emptyset_x(X_a)) ,$$

for each $x \in M$, where (X_a) is an orthonormal basis for T_xM and h is the metric of N. Suppose that M is compact; using the canonical measure associated with g, we define the energy of $\emptyset$ to be the number

$$E(\emptyset) = \int_M e(\emptyset)(x)\, dx .$$

Then a C^2 map $\emptyset : M \longrightarrow N$ is harmonic if and only if it is an extremal of the energy integral E [12].

Suppose $\emptyset : M \longrightarrow N$ and $\psi : N \longrightarrow P$ are two maps, then the second fundamental form of the composition is given by

$$\nabla d(\psi \circ \emptyset) = d\psi \circ \nabla d\emptyset + \nabla d\psi(d\emptyset, d\emptyset) . \qquad (1.1.2)$$

By taking traces we obtain the formula [12]:

$$\Delta(\psi \circ \emptyset) = d\psi \circ \Delta \emptyset + \operatorname{trace} \nabla d\psi (d\emptyset, d\emptyset). \qquad (1.1.3)$$

Let S^{m-1} denote the standard unit sphere in $\mathbb{R}^m$, and let $i: S^{m-1} \longrightarrow \mathbb{R}^m$ be the inclusion map.

Lemma 1.1.5 If $f: \mathbb{R}^m \longrightarrow \mathbb{R}$ is a smooth function then

$$\Delta^{S^{m-1}}(f \circ i) = \left(\Delta^{\mathbb{R}^m} f - \frac{\partial^2 f}{\partial r^2} - (m-1) \frac{\partial f}{\partial r} \right) \circ i , \qquad (1.1.4$$

where $r^2(x) = |x|^2$, for all $x \in \mathbb{R}^m$.

Proof Using the formula (1.1.3) we have

$$\Delta^{S^{m-1}}(f \circ i) = df(\Delta i) + \operatorname{trace} \nabla df(di, di).$$

Also

$$\Delta^{\mathbb{R}^m}(f) \circ i = \operatorname{trace} \nabla df(di, di) + \frac{\partial^2 f}{\partial r^2} \circ i$$

$$= \Delta^{S^{m-1}}(f \circ i) + \frac{\partial^2 f}{\partial r^2} \circ i - df(\Delta^{S^{m-1}} i).$$

Let x be a point of S^{m-1}, then at x

$$\Delta^{S^{m-1}} i = \operatorname{trace} \nabla di(x)$$

$$= \sum_a \nabla^{\mathbb{R}^m}_{di_x(X_a)} di_x(X_a),$$

where $(X_a)_{1 \le a \le m}$ forms an orthonormal basis for $T_x S^{m-1}$, and $X_a = \gamma_a'(0)$,

where $\gamma_a(t)$ are geodesics in S^{m-1} with $\gamma_a(0) = x$. Thus

$$\Delta^{S^{m-1}} i = \sum_a \gamma_a''(0),$$

but $\gamma_a''(0) = -\gamma_a(0) = -\partial/\partial r$. Thus

$$\Delta^{S^{m-1}} i = -(m-1)\frac{\partial}{\partial r}. \qquad \square$$

Corollary 1.1.6 If $f: \mathbb{R}^m \to \mathbb{R}$ is a harmonic homogeneous polynomial of degree p, then

$$\Delta^{S^{m-1}}(f \circ i) = -p(p+m-2) f \circ i,$$

so $f \circ i$ is an eigenfunction of $\Delta^{S^{m-1}}$.

Remark 1.1.7 All eigenfunctions of $\Delta^{S^{m-1}}$ arise from harmonic homogeneous polynomials on $\mathbb{R}^m$ in this way.

1.2 Harmonic maps into spheres

Let S^{n-1} denote the standard sphere in Euclidean n-space, and $j: S^{n-1} \to \mathbb{R}^n$ the inclusion map. If $\phi: (M,g) \to S^{n-1}$ is a map; let Φ denote the composition

$j \circ \phi : M \to \mathbb{R}^n$.

Lemma 1.2.1 The map ϕ is harmonic if and only if

$$\Delta \Phi = - 2e(\Phi) \Phi . \qquad (1.2.1)$$

Proof From equation (1.1.3)

$$\Delta (j \circ \phi) = dj(\Delta \phi) + \text{trace } \nabla dj(d\phi, d\phi).$$

Since j is an isometric immersion $\nabla dj(X,Y)$ is perpendicular to $dj(Z)$, for all $X,Y,Z \in \mathscr{C}(TS^{n-1})$ (so $\Delta \phi$ is the projection of $\Delta(\Phi)$ onto TS^{n-1}); in particular, ϕ is harmonic if and only if $\Delta(j \circ \phi)$ is proportional to Φ .

Now, writing $< , >$ for the Euclidean metric on $\mathbb{R}^n$

$$< d(j \circ \phi), j \circ \phi > = 0 .$$

We differentiate this formula at $x \in M$. Let $(X_a)_{1 \le a \le m}$ be an orthonormal basis for $T_x M$, with $\nabla^M_{X_a} X_a = 0$ at x. Then, using Lemma 1.1.1 and summing over repeated indices

$$0 = \nabla^M_{X_a} < d(j \circ \phi)(X_a), j \circ \phi >$$

$$= < \nabla^{\mathbb{R}^n}_{d(j \circ \phi)(X_a)} d(j \circ \phi)(X_a), j \circ \phi > + < d(j \circ \phi)(X_a), d(j \circ \phi)(X_a) >$$

$$= < \Delta (j \circ \phi), \Phi > + 2e(\Phi) . \qquad \square$$

Corollary 1.2.2 If $\phi : S^{m-1} \to S^{n-1}$ is defined by harmonic homogeneous polynomials of common degree k, then ϕ is harmonic and has constant energy density $k(k + m - 2)/2$.

Proof Corollary 1.1.6 and Lemma 1.2.1 give the result. $\square$

Example 1.2.3 The map from $\mathbb{C}$ to $\mathbb{C}$, $z \to z^k$, $z \in \mathbb{C}$, $k = 1,2, \ldots$ is defined by harmonic homogeneous polynomials of degree k, and restricts to a map G^k : $S^1 \to S^1$ harmonic of degree k.

Example 1.2.4 Let $f: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ be an orthogonal multiplication; i.e. f is bilinear and $|f(x,y)| = |x| |y|$, for all $x \in \mathbb{R}^p$ and all $y \in \mathbb{R}^q$. Then the restriction of f produces a map $\psi : S^{p-1} \times S^{q-1} \to S^{n-1}$ which is a totally geodesic embedding in each variable separately, and hence is harmonic of constant energy density $(p+q-2)/2$.

If now $p = q = n$; the Hopf map $H^n : S^{2n-1} \longrightarrow S^n$:

$$H^n(x,y) = (|x|^2 - |y|^2, 2f(x,y)),$$

is a harmonic polynomial map with constant energy density 2n. In particular, if f is multiplication of complex, quaternionic or Cayley numbers, then we obtain the Hopf fibrations $S^3 \longrightarrow S^2$, $S^7 \longrightarrow S^4$ or $S^{15} \longrightarrow S^8$ respectively. For other examples see [33].

1.3 Joins of spheres and Smith's construction

The join $S^{p-1} * S^{q-1}$ of the two Euclidean spheres S^{p-1} and S^{q-1} is the sphere S^{m-1}, $m = p+q$, obtained by writing each point $z \in S^{m-1}$ as $z = (\cos s\, x, \sin s\, y)$, where $x \in S^{p-1}$, $y \in S^{q-1}$ and $s \in [0, \pi/2]$. We can think of the join as introducing "polar coordinates" on the sphere S^{m-1}.

If $g_1 : S^{p-1} \longrightarrow S^{r-1}$ and $g_2 : S^{q-1} \longrightarrow S^{s-1}$ are two maps; we can form the join $g_1 * g_2 : S^{p+q-1} \longrightarrow S^{r+s-1}$ of g_1 and g_2:

$$g_1 * g_2(\cos s\, x, \sin s\, y) = (\cos s\, g_1(x), \sin s\, g_2(y)). \tag{1.3.1}$$

Consider the case when g_1 and g_2 are defined by harmonic polynomials with $2e(g_1) = a_1$ and $2e(g_2) = a_2$ say, where a_1 and a_2 are constants. Smith's idea was to allow a "reparametrization" of s in equation (1.3.1), by defining $\phi : S^{p+q-1} \longrightarrow S^{r+s-1}$ as

$$\phi(\cos s\, x, \sin s\, y) = (\cos \alpha(s)\, g_1(x), \sin \alpha(s)\, g_2(y)), \tag{1.3.2}$$

for some $\alpha : [0, \pi/2] \longrightarrow [0, \pi/2]$ with $\alpha(0) = 0$ and $\alpha(\pi/2) = \pi/2$. We will derive this map again later in Example 5.3.1. The problem of ϕ being harmonic then reduces to solving a second order non-linear ordinary differential equation in α. To see this we calculate $\Delta \phi$, but in order to simplify later calculations we proceed differently, and by a slightly longer route to Smith [36], and for this we must introduce briefly the notion of harmonic morphism (much more will be said about these maps later in Chapter 7).

A map $\phi : M \longrightarrow N$ is a harmonic morphism if it pulls back germs of harmonic functions to germs of harmonic functions, i.e. if $f: V \to \mathbb{R}$ is harmonic, V a domain in N, then $f \circ \phi$ is harmonic on $\phi^{-1}(V)$ in M. Consequently, if ϕ is a non-constant harmonic morphism, then $\dim M \geq \dim N$, and if $\psi: N \to P$ is a harmonic map, then so is $\psi \circ \phi: M \to P$.

For $x \in M$, let $\mathcal{V}_x = \ker d\phi_x \subset T_x M$, and let $\mathcal{H}_x$ be the perpendicular complement of $\mathcal{V}_x$ in $T_x M$ with respect to g; say that ϕ is <u>horizontally conformal</u> if

$$\lambda(x)^2 \, g(X,Y) = \phi^* h(X,Y) ,$$

for all $X, Y \in \mathcal{H}_x$, where $\lambda: M \to \mathbb{R}$ is some function (not necessarily smooth). Call λ the <u>dilation of</u> ϕ.

<u>Theorem 1.3.1</u> [19,27]: <u>A map</u> $\phi: M \to N$ <u>is a harmonic morphism if and only if</u> ϕ <u>is harmonic and horizontally conformal.</u>

<u>Corollary 1.3.2</u> <u>If</u> $\phi: M \to N$ <u>is a harmonic morphism and</u> $\psi: N \to P$ <u>a harmonic map with energy density</u> $e(\psi)$, <u>then</u> $\psi \circ \phi$ <u>is harmonic and has energy density given by</u>

$$e(\psi \circ \phi)(x) = \lambda(x)^2 \, e(\psi)(\phi(x)) , \qquad (1.3.3)$$

<u>for all</u> $x \in M$.

<u>Proof</u> The result follows from Theorem 1.3.1 and the equation (1.1.3). □

Consider the join $S^{m-1} = S^{p-1} * S^{q-1}$, $p + q = m$. Regard s as a function $s: S^{m-1} \to \mathbb{R}$, by letting $s(\cos s_0 x, \sin s_0 y)) = s_0$, for all $s_0 \in [0, \pi/2]$. Let V_1 be the subset of S^{m-1} given by $V_1 = s^{-1}(0)$, then V_1 is isometric to S^{p-1}. Similarly, let $V_2 = s^{-1}(\pi/2)$; V_2 is isometric to S^{q-1}. Define $\pi_1: S^{m-1} \backslash V_2 \to V_1$ by

$$\pi_1((\cos s x, \sin s y)) = x ,$$

and $\pi_2: S^{m-1} \backslash V_1 \to V_2$ by

$$\pi_2((\cos s x, \sin s y)) = y .$$

<u>Lemma 1.3.3</u> <u>The maps</u> π_1, π_2 <u>are harmonic morphisms, with dilations</u> λ_1, λ_2 <u>respectively given by</u> $\lambda_1(z)^2 = 1/\cos^2 s$, $\lambda_2(z)^2 = 1/\sin^2 s$ <u>for all</u> $z = (\cos s\, x, \sin s\, y)$.

Proof For each $x_0 \in V_1$, the horizontal space with respect to π_1 through $(\cos s_0 x_0, \sin s_0 y_0)$, $s_0 \in [0,\pi/2)$, $y_0 \in S^{q-1}$, consists of the tangents to all curves $\Gamma(u) = (\cos s_0\, \gamma(u), \sin s_0 y_0)$, where $\gamma(u)$ is a curve in S^{p-1} with $\gamma(0) = x_0$. Then $\pi_1(\cos s_0\, \gamma(u), \sin s_0 y_0) = \gamma(u)$, so that, if $\tilde{\Gamma}(u) = (\cos s_0\, \tilde{\gamma}(u), \sin s_0 y_0)$ is another such horizontal curve, then

$$\begin{aligned} \pi_1^* h(\Gamma'(0), \tilde{\Gamma}'(0)) &= h(\gamma'(0), \tilde{\gamma}'(0)) \\ &= g(\Gamma'(0), \tilde{\Gamma}'(0))/\cos^2 s_0 , \end{aligned}$$

where h is the metric on S^{p-1} and g the metric on S^{m-1}. Hence π_1 is horizontally conformal with $\lambda_1^2 = 1/\cos^2 s$.

To see that π_1 is harmonic, we compute trace $\nabla d\pi_1$ with respect to a particular orthonormal basis. Choose $z_0 = (\cos s_0 x_0, \sin s_0 y_0) \in S^{m-1} \setminus V_2$ and curves through z_0:

$$\begin{aligned} \Gamma_i(u) &= (\cos s_0\, \gamma_i(u), \sin s_0 y_0) , \\ \Sigma_r(u) &= (\cos s_0 x_0, \sin s_0\, \sigma_r(u)) , \end{aligned}$$

where $\gamma_i(u)$ is geodesic in S^{p-1} with $\gamma_i(0) = x_0$, and $\sigma_r(u)$ is geodesic in S^{q-1} with $\sigma_r(0) = y_0$. Let $X_i = \Gamma_i'(0)$ and $Y_r = \Sigma_r'(0)$. Define $\nu(s) = (\cos s\, x_0, \sin s\, y_0)$, so that $\xi = \nu'(s_0) \in \mathscr{V}_{z_0}$. Then

$$\begin{aligned} \Delta \pi_1 &= \text{trace } \nabla d\pi_1 \\ &= \sum_i \nabla d\pi_1(X_i, X_i) + \sum_r \nabla d\pi_1(Y_r, Y_r) + \nabla d\pi_1(\xi, \xi). \end{aligned}$$

Since $\gamma_i(u)$ and $\sigma_r(u)$ are both geodesics in S^{p-1}, S^{q-1} respectively; both $\nabla^{S^{m-1}}_{X_i} X_i$ and $\nabla^{S^{m-1}}_{Y_r} Y_r$ are proportional to ξ at z_0. Thus

$$\begin{aligned} \nabla d\pi_1(X_i, X_i) &= - \nabla^{S^{p-1}}_{d\pi_1(X_i)} d\pi_1(X_i) \\ &= - (1/\cos^2 s_0) \nabla^{S^{p-1}}_{X_i} X_i \\ &= 0 . \end{aligned}$$

Also $\nabla d\pi_1(Y_r, Y_r) = 0$ since $d\pi_1(Y_r) = 0$, and $\nabla d\pi_1(\xi, \xi) = 0$ since $\nabla_\xi \xi = 0$ and $d\pi_1(\xi) = 0$. Hence π_1 is harmonic. □

We now rewrite Smith's map $\emptyset : S^{m-1} \to S^{n-1}$ of equation (1.3.2) as

$$\emptyset(z) = (\cos \alpha(s(z))\, g_1 \circ \pi_1(z),\ \sin \alpha(s(z)) g_2 \circ \pi_2(z)) \qquad (1.3.4)$$

for all $z \in S^{m-1} \setminus (V_1 \cup V_2)$; if $z \in V_1$ we define $\emptyset(z) = (g_1(z), 0)$, and if $z \in V_2$ we define $\emptyset(z) = (0, g_2(z))$.

Let $t : S^{n-1} \to \mathbb{R}$ be the function given by $t((\cos t_0 v, \sin t_0 w)) = t_0$, where $v \in S^{r-1}$, $w \in S^{s-1}$ and $t_0 \in [0, \pi/2]$. Writing $\Phi = i \circ \emptyset$, where $i : S^{n-1} \to \mathbb{R}^n$ is the inclusion, it is straightforward to compute $\Delta \Phi$, and one obtains, using Lemma 1.3.3 and Lemma 1.2.1 on the maps $g_1 \circ \pi_1$ and $g_2 \circ \pi_2$,

$$\begin{aligned} \Delta \Phi(z) &= -\alpha'(s)^2 |ds|^2 \Phi(z) + (\alpha''(s)|ds|^2 + \alpha'(s)\Delta s)\nabla t_{\emptyset(z)} \\ &\quad + \cos \alpha(s) \Delta (g_1 \circ \pi_1)(z) + \sin \alpha(s) \Delta (g_2 \circ \pi_2)(z) \\ &= -\alpha'(s)^2 |ds|^2 \Phi(z) + (\alpha''(s)|ds|^2 + \alpha'(s)\Delta s)\nabla t_{\emptyset(z)} \\ &\quad - \cos \alpha(s) \frac{a_1 (g_1 \circ \pi_1)(z)}{\cos^2 s} - \sin \alpha(s) \frac{a_2 (g_2 \circ \pi_2)(z)}{\sin^2 s} \end{aligned}$$

This lies in the plane spanned by $\nabla t_{\emptyset(z)}$ and $\Phi(z)$; hence $\emptyset$ is harmonic if and only if $< \Delta \Phi(z), \nabla t_{\emptyset(z)} > = 0$, for all $z \in S^{m-1}$. Now $\nabla t_{\emptyset(z)} =$ $(-\sin \alpha(s(z))\, g_1 \circ \pi_1(z), \cos \alpha(s(z)) g_2 \circ \pi_2(z))$, hence $\emptyset$ is harmonic if and only if

$$\alpha''(s) |ds|^2 + \alpha'(s)\Delta s + \sin \alpha(s) \cos \alpha(s) \left(\frac{a_1}{\cos^2 s} - \frac{a_2}{\sin^2 s} \right) = 0, \quad (1.3.5)$$

with $\alpha(0) = 0$ and $\alpha(\pi/2) = \pi/2$.

<u>Lemma 1.3.4</u>: <u>The Laplacian Δs of s is given by</u>

$$\Delta s = (q-1) \cot s - (p-1) \tan s.$$

<u>Proof</u> Let $s_0 \in (0, \pi/2)$, then $M_{s_0} = s^{-1}(s_0)$ is a hypersurface of S^{m-1}. Define $i_{s_0} : M_{s_0} \to S^{m-1}$ to be the inclusion map. Then

$$0 = \Delta(s \circ i_{s_0}) = ds(\Delta i_{s_0}) + \text{trace}\, \nabla ds(di_{s_0}, di_{s_0}).$$

But $\xi = \partial/\partial s$ is affine geodesic, hence $\nabla_\xi \xi = 0$, so that $\nabla ds(\xi,\xi) = ds(\nabla_\xi \xi) + \nabla^{\mathbb{R}}_{ds(\xi)} ds(\xi) = 0$ ($ds(\xi)$ is the unit tangent vector along $\mathbb{R}$). That is trace $\nabla ds(di_{s_0}, di_{s_0}) = \Delta s$. Thus

$$0 = ds(\Delta i_{s_0}) + \Delta s .$$

But since Δi_{s_0} is proportional to ξ, we conclude that

$$\Delta i_{s_0} = -\Delta s\, \xi .$$

Thus Δs is $-(m-2) \times$ (mean curvature of M_{s_0}).

Choose $z \in M_{s_0}$; $z = (\cos s_0 x, \sin s_0 y)$, $x \in S^{p-1}$, $y \in S^{q-1}$. Let $\Gamma(u) = (\cos s_0 \gamma(u), \sin s_0 y)$ be a curve in M_{s_0}, $\Gamma(0) = z$, then $X = \Gamma'(0) = (\cos s_0 \gamma'(0), 0)$.

Now $\xi_{\Gamma(u)} = (-\sin s_0 \gamma(u), \cos s_0 y)$. So

$$\begin{aligned} \nabla_X \xi &= \frac{d}{du} \xi_{\Gamma(u)} \Big|_{u=0} \\ &= (-\sin s_0 \gamma'(0), 0) \\ &= -\tan s_0 X . \end{aligned}$$

Thus X is a principal curvature vector of M_{s_0} in S^{m-1}, with principal curvature $\tan s_0$. Similarly, vectors tangent to curves $\Sigma(u) = (\cos s_0 x, \sin s_0 \sigma(u))$ are principal curvature vectors with principal curvature $-\cot s_0$. The result follows by summing over the principal curvatures. □

Equation (1.3.5) now becomes

$$\alpha''(s)|ds|^2 + \alpha'(s)((q-1)\cot s - (p-1)\tan s) + \sin\alpha(s)\cos\alpha(s)\left(\frac{a_1}{\cos^2 s} - \frac{a_2}{\sin^2 s}\right) = 0 . \tag{1.3.6}$$

The equation has singularities at $s = 0, \pi/2$ (that is the coefficients of $\alpha'(s)$ and $\sin\alpha(s)\cos\alpha(s)$ become infinite). We can remove these by reparametrizing (1.3.6) using the parameter u defined by $e^u = \tan s$. Then we obtain

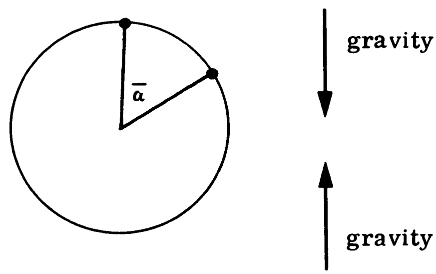
$$\alpha''(u) + \frac{1}{e^u + e^{-u}} ((q-2)e^{-u} - (p-2)e^u)\,\alpha'(u)$$

$$+ \frac{1}{e^u + e^{-u}} \sin\alpha(u)\cos\alpha(u)(a_1 e^u - a_2 e^{-u}) = 0 , \qquad (1.3.7)$$

with $\alpha(-\infty) = 0$ and $\alpha(\infty) = \pi/2$. This is the equation obtained by Smith in [36].

1.4 Outline of the solution of Smith's equation

Equation (1.3.7) is the equation of a pendulum with variable damping acted upon by a force of variable gravity. The position of the pendulum is described by the angle $\bar{\alpha} = 2\alpha$ which is measured from the upward vertical. The idea is to find an exceptional trajectory of the pendulum, so it stands vertically upwards at time $u = -\infty$, and hangs straight down at time $u = +\infty$.



Smith shows that such a solution exists if either

(i) $p = q$ and $a_1 = a_2$, or

(ii) $(p-2)^2 < 4a_1$ and $(q-2)^2 < 4a_2$. (1.4.1)

We call (ii) the damping conditions. An outline of his construction is as follows.

Fix u_0 to be the time when gravity vanishes and manipulate the initial conditions $\alpha_0 = \alpha(u_0)$ and $\alpha_0' = \alpha'(u_0)$. For a given $\alpha_0 \in (0, \pi/2)$, throw the pendulum just hard enough ($\alpha_0' = \alpha_0^-(\alpha_0)$) so that $\alpha(u) \to 0$ as $u \to -\infty$. Similarly, choose $\alpha_0^+(\alpha_0)$ to get $\alpha(u) \to \pi/2$ as $u \to +\infty$. Then, since the coefficients in equation (1.3.7) are smooth; α_0^- and α_0^+ are continuous in α_0 [9]. Further, $\alpha_0^- \to 0$ as $\alpha_0 \to 0$ and $\alpha_0^+ \to 0$ as $\alpha_0 \to \pi/2$. The idea is to find an α_0 such that α_0^+ and α_0^- match. At opposite ends one requires α_0^+ be bounded away from 0 for α_0 near 0, and α_0^- be bounded away from 0 for α_0 near $\pi/2$. This requires the use of a comparison theorem for second order equations, together with the inequalities (ii) above. Smith gives a more sophisticated method of solving equation (1.3.7) in [36]. We give this method in Chapter 6.

In [36] Smith gives many examples of harmonic maps constructed by the above methods. We give a few of these.

Example 1.4.1 Let g_1 be the identity map $I^{p-1}: S^{p-1} \to S^{p-1}$, and $g_2: S^1 \to S^1$ be the map G^k of Example 1.2.3 of degree k. Then for $p < 7$ the damping conditions are satisfied, and so by Smith's construction one can construct a harmonic map $\emptyset: S^{p-1} \to S^{p-1}$, $p < 7$, of degree k, k = 1, 2, ... (in general $\deg(g_1 * g_2) = \deg(g_1)\deg(g_2)$, $g_1: S^{p-1} \to S^{r-1}$, $g_2: S^{q-1} \to S^{s-1}$). By composing with an isometry of degree -1 we obtain harmonic maps of negative degrees. That is, we can represent $\pi_n(S^n) = \mathbb{Z}$ harmonically for $n \leq 7$.

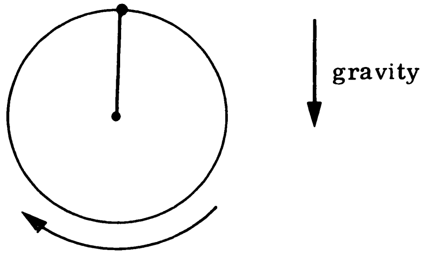
Example 1.4.2 Let $g_1: S^3 \to S^2$ be the Hopf map H^2 of Example 1.2.4. Let g_2 be the identity map $I^{p-1}: S^{p-1} \to S^{p-1}$. Then for $p \leq 5$ the damping conditions are satisfied, and we can represent the non-trivial class of $\pi_{n+1}(S^n) = \mathbb{Z}_2$ harmonically for n = 3,4, ... , 8.

Example 1.4.3 Let $g_1: S^7 \to S^4$ be the Hopf map H^4 of Example 1.2.4. Let $g_2: S^{p-1} \to S^{p-1}$ be the identity map I^{p-1}. Then $g_1 * g_2: S^{p+7} \to S^{p+4}$ represents the generator of $\pi_{p+7}(S^{p+4}) = \mathbb{Z}_{24}$. For p = 1, ..., 6 the damping conditions are satisfied, and we can represent that generator harmonically.

Example 1.4.4 Given an orthogonal multiplication $\psi: S^{p-1} \times S^{q-1} \to S^{n-1}$ as in Example 1.2.4; we can define a map $\emptyset: S^{p+q-1} \to S^n$ by

$$\emptyset(\cos s\, x, \sin s\, y) = (\cos \alpha(s), \sin \alpha(s)\, \psi(x,y)),$$

where $x \in S^{p-1}$, $y \in S^{q-1}$, $s \in [0, \pi/2]$, $\alpha(0) = 0$ and $\alpha(\pi/2) = \pi$. As in Section 1.3 we obtain an equation as a condition of harmonicity similar to equation 1.3.7, but with the gravity always having the same sign. The equation is of a pendulum whose position is described by $\bar{\alpha} = 2\alpha$, and one looks for a trajectory with the pendulum just completing a single rotation.



Similar considerations apply when $\psi: S^1 \times S^1 \longrightarrow S^1$ is given by $\psi(u,v) = u^k v^1$, $u, v \in \mathbb{C}$, $|u|^2 = |v|^2 = 1$. Then the relevant equation is (after reparametrization)

$$\alpha''(u) = \frac{(k^2 e^u + 1^2 e^{-u})}{e^u + e^{-u}} \sin \alpha(u) \cos \alpha(u) , \qquad (1.4.2)$$

with $\alpha(-\infty) = 0$ and $\alpha(\infty) = \pi$. This is the equation of a pendulum with no damping and variable gravity. If $k = 1$, then, due to symmetry considerations one can show that equation 1.4.2 has a solution. However, if $k \neq 1$, then the gravity is either increasing or decreasing. If the gravity is increasing the pendulum will be unable to make a complete rotation. If the gravity is decreasing, the pendulum will go past the upward vertical after performing a complete rotation. Thus there is no solution of equation 1.4.2 with $\alpha(-\infty) = 0$ and $\alpha(\infty) = \pi$ (this can be proved precisely using energy estimates for the pendulum).

Now $\pi_3(S^2) = \mathbb{Z}$ can be parametrized by the integer $k1$, which is the Hopf linking number - the Hopf invariant of each class.

Thus we can represent harmonically the classes of $\pi_3(S^2)$ with Hopf invariant k^2, but we cannot represent harmonically by the methods of Sections 1.3 and 1.4 the classes of $\pi_3(S^2)$ with Hopf invariant $k1$, $k \neq 1$. In particular it is unknown whether the class of $\pi_3(S^2)$ with Hopf invariant 2 has a harmonic representative.

An alternative method of finding harmonic representatives for the classes of $\pi_3(S^2)$ is as follows. The Hopf map $H^2 : S^3 \longrightarrow S^2$ is a harmonic morphism, and hence by Corollary 1.3.2, if $f: S^2 \longrightarrow S^2$ is harmonic; so is $f \circ H^2 : S^3 \longrightarrow S^2$. In particular, if f has degree k, then $f \circ H^2$ has Hopf invariant k^2 [26]. Since we have harmonic maps $f: S^2 \longrightarrow S^2$ of all degrees; we can find harmonic maps with Hopf invariant k^2 for all k.

Similarly there are harmonic maps $f: S^4 \to S^4$ of degree k, for all k (from Example 1.4.1); by composing with the Hopf map $H^4: S^7 \to S^4$, we can represent harmonically the classes of $\pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}$ which have Hopf invariant k^2 (the Hopf invariant parametrizes the $\mathbb{Z}$ factor of $\mathbb{Z} \oplus \mathbb{Z}_{12}$).

1.5 Hyperbolic space

Let M^m denote $\mathbb{R}^m$ equipped with the metric $<,>_M$ given by

$$<u,v>_M = -u_1v_1 + u_2v_2 + \dots + u_mv_m ,$$

for all $u = (u_1, \dots, u_m)$, $v = (v_1, \dots, v_m) \in \mathbb{R}^m$. The space M^m is called m-dimensional Minkowski space. Define (m-1)-dimensional hyperbolic space, or the (m-1)-dimensional pseudo-sphere, to be the space

$$H^{m-1} = \{u \in M^m;\ <u,u>_M = -1\} ,$$

together with the induced metric.

The pseudo-sphere is a "spacelike" hypersurface (if v is tangent to H^{m-1}, then $<v,v>_M > 0$), whose unique normal vector field η is "timelike" ($<\eta, \eta>_M < 0$) in the terminology of relativity theory. There are many analogies between the Euclidean sphere S^{m-1} and the pseudo-sphere H^{m-1}, especially concerning harmonic maps. In particular, the usual stereographic projection sending the sphere less a point to Euclidean space, can be modified to give the well-known isometry between H^{m-1} and $(B^{m-1}, <,>)$, where B^{m-1} is the open ball of radius 2 in $\mathbb{R}^{m-1}$, and the metric $<,> = \sum_{i=1}^{m-1} dx_i \otimes dx_i/(1 - \frac{1}{4}\sum_j x_j^2)^2$, where (x_i) are the standard coordinates on Euclidean space.

Let $i: H^{m-1} \to M^m$ be the inclusion map.

Lemma 1.5.1 If $f: M^m \to \mathbb{R}$ is a smooth function, then

$$\Delta^{H^{m-1}}(f \circ i) = \left(\Delta^{M^m} f + \frac{\partial^2 f}{\partial \eta^2} + (m-1)\frac{\partial f}{\partial \eta}\right) \circ i , \tag{1.5.1}$$

where $\partial/\partial\eta$ denotes differentiation in the normal direction to H^{m-1} and

$$\Delta^{M^m} = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_m^2}$$ is the indefinite Laplacian.

Remark 1.5.2 The unit timelike normal to the pseudo-sphere at x is x itself. Thus,

$$\frac{\partial}{\partial \eta}(x) = x_1 \frac{\partial}{\partial x_1}(x) + x_2 \frac{\partial}{\partial x_2}(x) + \dots + x_m \frac{\partial}{\partial x_m}(x),$$

where $(x_1, \dots, x_m)$ are standard coordinates on $\mathbb{R}^m$.

Proof (of Lemma 1.5.1): Recall equation 1.2.1 for the 2nd fundamental form of the composition of two maps:

$$\nabla d(f \circ i) = df(\nabla di) + \nabla df(di, di),$$

this makes sense with M^m having an indefinite metric (∇ is the Levi-Civita connection with respect to the indefinite metric on the bundle $T^* H^{m-1} \quad i^{-1} T M^m \longrightarrow H^{m-1}$). By taking the trace of this formula with respect to the Riemannian metric on H^{m-1}, we get

$$\Delta^{H^{m-1}}(f \circ i) = df(\Delta^{H^{m-1}} i) + \text{trace}\, \nabla df(di, di). \tag{1.5.2}$$

But,

$$(\Delta^{M^m} f) \circ i = \text{trace}\, \nabla df(di, di) - \frac{\partial^2 f}{\partial \eta^2} \circ i, \tag{1.5.3}$$

where $\Delta^{M^m} = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_m^2}$ is the indefinite Laplacian on M^m. Equations (1.5.2) and (1.5.3) give

$$(\Delta^{M^m} f) \circ i = \Delta^{H^{m-1}}(f \circ i) - \frac{\partial^2 f}{\partial \eta^2} \circ i - df(\Delta^{H^m} i). \tag{1.5.4}$$

Now

$$\begin{aligned} \Delta^{H^{m-1}} i &= \text{trace}\, \nabla di \\ &= \sum_{k=1}^{m-1} \nabla^{M^m}_{di(X_k)} di(X_k), \end{aligned}$$

where we evaluate at a point $x \in H^{m-1}$ with $X_k = \gamma_k'(0)$, $\gamma_k(0) = x$, $\gamma_k(u)$ geodesic in H^{m-1}, $k = 1, \ldots, m-1$. Thus $\Delta^{H^{m-1}} i = \sum_k \gamma_k''(0)$. But $\gamma_k''(0) = \gamma_k(0) = \partial/\partial\eta$. Thus $\Delta^{H^{m-1}} i = (m-1)\,\partial/\partial\eta$. □

Corollary 1.5.3 *If f is a pseudo-harmonic polynomial (i.e. $\Delta^{M^m} f = 0$) of degree k on $\mathbb{R}^m$, then*

$$\Delta^{H^{m-1}} (f \circ i) = k(k+m-2)\, f \circ i ;$$

so $f \circ i$ is an eigenfunction of $\Delta^{H^{m-1}}$.

Let (M,g) be a Riemannian manifold and $\phi: M \to H^{m-1}$ a map. Let $i: H^{m-1} \to M^m$ be the inclusion map, and write $\Phi = i \circ \phi$.

Lemma 1.5.4 *The map ϕ is harmonic if and only if*

$$\Delta \Phi = 2\tilde{e}(\Phi)\Phi ,$$

where $\tilde{e}(\Phi) = \sum_k \langle d\Phi(X_k), d\Phi(X_k)\rangle_M$, (X_k) is an orthonormal basis for M.

Proof From equation 1.1.3

$$\Delta(i \circ \phi) = di(\Delta\phi) + \operatorname{trace} \nabla di(d\phi, d\phi).$$

As in Lemma 1.2.1, ∇di is perpendicular to di with respect to the Minkowski metric. Thus ϕ is harmonic if and only if $\Delta(\Phi)$ is proportional to Φ. That $\langle \Delta\Phi, \Phi\rangle_M = -2\tilde{e}(\Phi)$ is as in Lemma 1.2.1. □

Example 1.5.5 Express each $w = (x,y) \in M^2$ as $w = x + jy$. Multiply $w = x + jy$ and $w' = x' + jy'$ using the rule $w.w' = xx' + yy' + j(yx' + xy')$. Then the map: $M^2 \to M^2$: $w \to w^k$ is pseudo-harmonic, and induces a harmonic map ϕ from H^1 to H^1 with constant energy density $\frac{1}{2}k^2$. If we express H^1 as

$$H^1 = \{(\cosh s, \sinh s);\ s \in [0,\infty)\} \subset M^2 ,$$

then ϕ is simply $\phi((\cosh s, \sinh s)) = (\cosh ks, \sinh ks)$.

Example 1.5.6 Introduce polar coordinates on M^2 by expressing each $(x,y) \in M^2$ as $(x,y) = (\rho \cosh s, \rho \sinh s)$. The Laplacian $\Delta^{M^2} = -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ goes over into

$\frac{1}{\rho^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho}$ (illustrating Lemma 1.5.1). Thus Δ^{H^1} is simply $\frac{\partial^2}{\partial t^2}$. This means that mappings of the form $\psi : H^1 \to S^1$, $\psi(\cosh s, \sinh s) = (\cos\alpha(s), \sin\alpha(s))$ are harmonic if and only if α is linear; $\alpha(s) = ks$; such a map has constant energy density $k^2/2$.

1.6 Polar coordinates on hyperbolic space and an analogous construction to Smith's

Introduce "polar coordinates" on H^{m-1} by expressing each point $z \in H^{m-1}$ in the form $z = (\cosh sx, \sinh sy)$, where $x \in H^{p-1}$, $y \in S^{q-1}$, $p+q = m$ and $s \in [0,\infty)$.

Define a map $\emptyset : H^{p+q-1} \to H^{r+s-1}$ by

$$\emptyset((\cosh sx, \sinh sy)) = (\cosh\alpha(s) g_1(x), \sinh\alpha(s) g_2(y)), \ \alpha(0) = 0,$$

where $g_1 : H^{p-1} \to H^{r-1}$ is harmonic with $|dg_1|^2 = a_1$ a constant, and $g_2 : S^{q-1} \to S^{s-1}$ is harmonic with $|dg_2|^2 = a_2$ a constant. By viewing s as a function, $s : H^{m-1} \to \mathbb{R}$, $s((\cosh s_0 x, \sinh s_0 y)) = s_0$, we can express $\emptyset$ in the form

$$\emptyset(z) = (\cosh\alpha(s(z)) g_1 \circ \pi_1(z), \sinh\alpha(s(z)) g_2 \circ \pi_2(z)), \alpha(0) = 0,$$

for all $z \in H^{m-1}$, where $\pi_1 : H^{m-1} \to H^{p-1}$ is the harmonic morphism: $(\cosh sx, \sinh sy) \to x$, and $\pi_2 : H^{m-1} \setminus H^{p-1} \to S^{q-1}$ is the harmonic morphism: $(\cosh sx, \sinh sy) \to y$, $s \neq 0$. The dilation λ_1 of π_1 is given by $\lambda_1^2 = 1/\cosh^2 s$, and the dilation λ_2 of π_2 is given by $\lambda_2^2 = 1/\sinh^2 s$ (the proof of these facts is similar to the proof of Lemma 1.3.3).

We compute the Laplacians:

$$\Delta(\cosh\alpha(s) g_1 \circ \pi_1) = g_1 \circ \pi_1 (\cosh\alpha(s)\alpha'(s)^2 |ds|^2 + \sinh\alpha(s)\Delta(\alpha(s)) + \cosh\alpha(s)(a_1/\cosh^2 s)),$$

$$\Delta(\sinh\alpha(s) g_2 \circ \pi_2 = g_2 \circ \pi_2(\sinh\alpha(s)\alpha'(s)^2 |ds|^2 + \cosh\alpha(s)\Delta(\alpha(s)) - \sinh\alpha(s)(a_2/\sinh^2 s)).$$

Let $t : H^{n-1} \to \mathbb{R}$, $n = r+s$, be the function given by $t((\cosh s_0 v, \sinh s_0 w) = s_0$, $v \in H^{r-1}$, $w \in S^{s-1}$. Then, writing $\Phi = i \circ \emptyset$ where $i : H^{n-1} \to M^n$ is the inclusion, we see that $\Delta\Phi(z)$ lies in the plane spanned by $\Phi(z)$ and $\nabla t_{\Phi(z)}$, for all $z \in H^{m-1}$. By Lemma 1.5.4, $\emptyset$ is harmonic if and only if $<\Delta\Phi(z), \nabla t_{\Phi(z)}>_M = 0$. Now $\nabla t_{\emptyset(z)} = (\sinh\alpha(s)\, g_1 \circ \pi_1(z), \cosh\alpha(s)\, g_2 \circ \pi_2(z))$, and remembering that $|g_1 \circ \pi_1|^2 = -1$, $|g_2 \circ \pi_2|^2 = 1$; the condition for harmonicity of $\emptyset$ becomes

$$\Delta(\alpha(s)) - \sinh\alpha(s)\cosh\alpha(s)\left(\frac{a_2}{\sinh^2 s} + \frac{a_1}{\cosh^2 s}\right) = 0$$

or,

$$\alpha''(s)|ds|^2 + \alpha'(s)\Delta s - \sinh\alpha(s)\cosh\alpha(s)\left(\frac{a_2}{\sinh^2 s} + \frac{a_1}{\cosh^2 s}\right) = 0, \text{ with } \alpha(0) = 0 \qquad (1.6.1)$$

Lemma 1.6.1 *The Laplacian of* $s, \Delta s$, *is given by*

$$\Delta s = (p-1)\tanh s + (q-1)\coth s .$$

Proof As for the spherical case, $\Delta s(z)$ is $-(m-2) \times$ (mean curvature of $M_{s_0} = s^{-1}(s_0)$) for each $s_0 \in (0,\infty)$, $z \in M_{s_0}$. A calculation similar to that of Lemma 1.3.4 shows that the principal curvatures of M_{s_0} are $-\tanh s_0$ with multiplicity $(p-1)$, and $-\coth s_0$ with multiplicity $(q-1)$.

Equation (1.6.1) now becomes

$$\alpha''(s) + ((p-1)\tanh s + (q-1)\coth s)\,\alpha'(s) - \sinh\alpha(s)\cosh\alpha(s)\left(\frac{a_2}{\sinh^2 s} + \frac{a_1}{\cosh^2 s}\right) = 0,$$

$$\text{with } \alpha(0) = 0 . \qquad (1.6.2)$$

Equation (1.6.2) has a singularity at $s = 0$, we therefore reparametrize it using the substitution $e^u = \sinh s$. The equation then becomes

$$\alpha''(u) + \frac{1}{e^{u}+e^{-u}} \left(pe^{u} + (q-2)(e^{u}+e^{-u})\right)\alpha'(u)$$

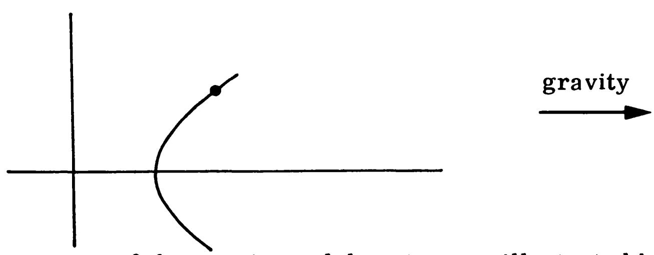
$$- \frac{\sinh \alpha(u) \cosh \alpha(u)}{e^{u}+e^{-u}} \left(\frac{a_2}{e^{u}} + \frac{a_1}{e^{u}+e^{-u}} \right) = 0 ,$$

$$\alpha(-\infty) = 0 . \qquad (1.6.3)$$

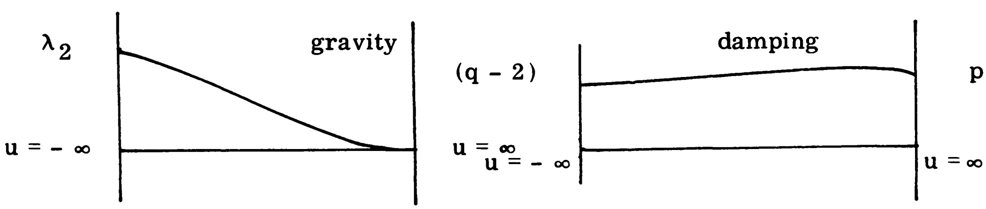
1.7 Solving the equation for hyperbolic spaces.

We provide an outline only of the solution of equation (1.6.3), giving a precise derivation in Chapter 6.

Equation (1.6.3) can be thought of, in some sense, as the equation of a particle constrained to move on a hyperbola in M^2 with a damping force, and with a variable "gravity" acting upon it.



The qualitative nature of the gravity and damping are illustrated by the following graphs:



Consider the situation when u is close to $-\infty$, and so $\alpha(u)$ is close to 0. In this limit $\sinh 2\,\alpha(u) \approx 2\,\alpha(u) \approx \sin 2\,\alpha(u)$. Thus, for large negative time, the qualitative behaviour of the solution of equation (1.6.3) approximates the qualitative behaviour of the solution to Smith's equation.

We choose a time u_0 such that the gravity is greater than 0 at u_0, and so is greater than 0 for all $u \leq u_0$. We choose $\alpha_0 = \alpha(u_0)$ sufficiently close to 0 - note that we have a 1-parameter choice for α_0. The physical moden can now be approximated by a pendulum, and we pick a $\alpha_0'(\alpha_0) = \alpha'(u_0)$ to be the velocity such that the pendulum just reaches 0 as $u \to -\infty$ in backward time (equations of the form $\beta'' = G(u,\beta,\beta')$, where G, $\partial G/\partial\beta$, $\partial G/\partial\beta'$ are continuous, have unique solutions through each point (u_0, β_0, β_0'), [9]). In Chapter 6 we will demonstrate precisely the existence of non-trivial solutions of equation (1.6.3) which exist for all time.

We have of course only established the existence of a non-trivial solution, and have not investigated the behaviour as $u \to \infty$.

We remark that there is a 1-parameter family of solutions depending on the choice of α_0.

Remark 1.7.1 We can write equation (1.3.5) for the spherical case in the form

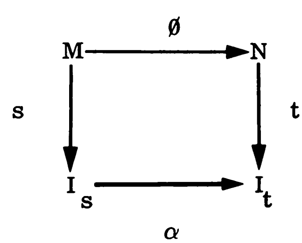
$$\Delta(\alpha(s)) = g(s)\sin 2\alpha(s), \tag{1.7.1}$$

where $s: S^{m-1} \to \mathbb{R}$ and $g(s) = \frac{1}{2}((a_2/\sin^2 s) - (a_1/\cos^2 s))$. Similarly, we can write equation (1.6.1) in the form

$$\Delta(\alpha(s)) = g(s)\sinh 2\alpha(s), \tag{1.7.2}$$

where $s: H^{m-1} \to \mathbb{R}$ and $g(s) = \frac{1}{2}((a_2/\sinh^2 s) + (a_1/\cosh^2 s))$. We remark that equation (1.7.1) bears a resemblance to the well-known sine-Gordon equation, and equation (1.7.2) resembles the sinh-Gordon equation [29].

Remark 1.7.2 The method of this chapter for constructing harmonic maps $\emptyset$ between spheres and between hyperbolic spaces have a common feature, namely symmetry with respect to certain functions $s: M \to \mathbb{R}$ and $t: N \to \mathbb{R}$, where $\emptyset: M \to N$. That is the diagram



commutes, where $s(M) = I_s$ and $t(N) = I_t$ are appropriate intervals. We might ask which functions are suitable as a symmetry of $\emptyset$, in order to reduce the problem of harmonicity to solving a 2^{nd} order ordinary differential equation. There is a class of functions which adapts quite beautifully to our purpose. These shall be the subject of the next chapter.

2 Isoparametric functions

2.1 Definition of isoparametric function

Let (M,g) be a space form (i.e. either the Euclidean sphere S^m, Euclidean space $\mathbb{R}^m$ or hyperbolic space H^m). A smooth function $f: M \longrightarrow \mathbb{R}$ is called *isoparametric* if

$$|df(x)|^2 = \psi_1(f(x)) , \tag{2.1.1}$$

$$\Delta f(x) = \psi_2(f(x)) , \tag{2.1.2}$$

for some smooth functions $\psi_1, \psi_2: \mathbb{R} \longrightarrow \mathbb{R}$.

Such functions were introduced by Cartan [5] in 1938. Their description on Euclidean space and hyperbolic space is relatively trivial, but on the sphere they are rich in geometry. More recently isoparametric functions have been studied in [17, 30, 32, 33, 39, 40].

Lemma 2.1.1 *Let* $f: M \longrightarrow \mathbb{R}$ *be a function satisfying equation (2.1.1), then the integral curves of* ∇f *are geodesics.*

Proof [32] Let $\xi = \nabla f / |df|$, and let $X \in \mathscr{C}(TM)$ be perpendicular to ξ: $g(X,\xi) = 0$. Then $Xf = 0$ so $\xi(Xf) = 0$. Also $\xi f = df(\nabla f)/|df| = \psi_1(f)^{\frac{1}{2}}$ is a function of f so $X(\xi f) = 0$, hence

$$\begin{aligned} 0 &= [X,\xi] f \\ &= (\nabla_X \xi - \nabla_\xi X) f. \end{aligned}$$

Thus $g(\nabla_X \xi - \nabla_\xi X, \xi) = 0$.

Since $1 = g(\xi,\xi)$: $g(\nabla_X \xi, \xi) = 0$, so that $g(\nabla_\xi X, \xi) = 0$. Also $g(X,\xi) = 0$ implies $g(\nabla_\xi X, \xi) = - g(X, \nabla_\xi \xi)$, so that $\nabla_\xi \xi$ is proportional to ξ. But $1 = g(\xi,\xi)$ implies $g(\xi, \nabla_\xi \xi) = 0$. Thus $\nabla_\xi \xi = 0$.

Lemma 2.1.2 Let $f: M \longrightarrow \mathbb{R}$ satisfy equations (2.1.1) and (2.1.2); then if $M_c = f^{-1}(c)$ is a non-singular hypersurface of f; M_c has constant curvature.

Proof Let $i: M_c \longrightarrow M$ be the inclusion map. Then

$$
\begin{aligned}
0 &= \Delta (f \circ i) \\
&= df(\Delta i) + \operatorname{trace} \nabla df(di, di) \\
&= |df| (\text{mean curvature of } M_c) + \Delta f - \nabla df(\xi, \xi),
\end{aligned}
$$

where $\xi = \nabla f / |df|$ (df is non-zero on M_c since M_c is a hypersurface). Now, using Lemma 2.1.1,

$$
\begin{aligned}
\nabla df(\xi, \xi) &= df(\nabla^M_\xi \xi) + \nabla^{\mathbb{R}}_{df(\xi)} df(\xi) \\
&= (\nabla f) |df| .
\end{aligned}
$$

Thus the mean curvature is equal to

$$
= \frac{\nabla f(|df|) - \Delta f}{|df|} , \qquad (2.1.3)
$$

which is a function of f.

Thus if f satisfies equations (2.1.1) and (2.1.2); the level hypersurfaces of f form a parallel family of hypersurfaces of constant mean curvature. Conversely, from equation (2.1.3) and by reversing the proof of Lemma 2.1.1, it is not hard to show that given such a family which are the level hypersurfaces of a function $f: M \longrightarrow \mathbb{R}$, then f satisfies equations (2.1.1) and 2.1.2). Furthermore, one can show

Proposition 2.1.3 [5] Given a parallel family of hypersurfaces of constant mean curvature on M, then the principal curvatures on each level hypersurface are constant on that hypersurface.

We call a hypersurface of M with constant principal curvatures an isoparametric hypersurface.

Example 2.1.4 Let $M = \mathbb{R}^m$ and let $f: \mathbb{R}^m \longrightarrow \mathbb{R}$ be defined by $f(x_1, \ldots, x_m) = x_1^2 + \ldots + x_p^2$, $p \leq m$. Then $|df|^2 = 4f$ and $\Delta f = 2p$. The level hypersurface

$f = s_0^2$ is given by $x_1^2 + \dots x_p^2 = s_0^2$ and is isometric to the cylinder $s_0 . S^{p-1} \times \mathbb{R}^{m-p}$, $s_0 \in (0,\infty)$.

Example 2.1.5 Let $M = S^{m-1}$, and define $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ by $F((x,y)) = |x|^2 - |y|^2$, where we write $\mathbb{R}^m = \mathbb{R}^p \oplus \mathbb{R}^q$, $p+q = m$, $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. Let $f: S^{m-1} \longrightarrow \mathbb{R}$ be the restriction $F|_{S^{m-1}}$. Introduce polar coordinates on S^{m-1} as in Section 1.3; so that each $z \in S^{m-1}$ is expressed as $z = (\cos s\, x, \sin s\, y)$, $x \in S^{q-1}$, $y \in S^{q-1}$. Then $f(z) = \cos^2 s - \sin^2 s = \cos 2s$, so that $df = -2 \sin 2s\, ds$. Thus $|df|^2 = 4 \sin^2 2s = 4(1-f^2)$.

Now $\Delta f = f'' |ds|^2 + f' \Delta s$, but from Lemma 1.3.4 $\Delta s = (q-1) \cot s - (p-1) \tan s$, whence

$$\Delta f = 4((p-2)\cos^2 s - (q-2)\sin^2 s)$$

$$= 2((p-2)(1+f) - (q-2)(1-f)),$$

and equations (2.1.1) and (2.1.2) are both satisfied.

The level hypersurface $M_{s_0} = f^{-1}(\cos 2s_0)$, $s_0 \in (0, \Pi/2)$ is isometric to a product of spheres $\cos s_0\, S^{p-1} \times \sin s_0\, S^{q-1}$.

Example 2.1.6 Let $M = H^{m-1}$. Introduce polar coordinates on H^{m-1} by expressing each point $z \in H^{m-1}$ as $z = (\cosh s\, x, \sinh s\, y)$, $x \in H^{p-1}$, $y \in S^{q-1}$, $m = p+q$ and $s \in [0,\infty)$. Define $f: H^{m-1} \rightarrow \mathbb{R}$ by $f(z) = \cosh 2s$. Then $df = 2 \sinh 2s\, ds$ so that $|df|^2 = 4(f^2 - 1)$. Now $\Delta f = f''(s)|ds|^2 + f'(s)\Delta s$, and from Lemma 1.6.1 $\Delta s = (p-1)\tanh s + (q-1)\coth s$, whence

$$\Delta f = 4((p-2)\cosh^2 s + (q-2)\sinh^2 s)$$

$$= 2((p-2)(f-1) + (q-2)(f+1)),$$

and f is isoparametric.

The level hypersurface $M_{s_0} = f^{-1}(\cosh 2s_0)$, $s_0 \in (0,\infty)$, is isometric to the product $\cosh s_0\, H^{p-1} \times \sinh s_0\, S^{q-1}$.

2.2 Properties of isoparametric functions and Münzner's classification theorem

Let M_c be a level hypersurface of the isoparametric function $f: M \longrightarrow \mathbb{R}$. For $x \in M_c$, let A be the shape operator; $A : T_x M_c \longrightarrow T_x M_c : X \longrightarrow -\nabla_X \xi$, $X \in T_x M_c$, where ξ is the unit normal vector field to M_c. Then the 2nd fundamental form h of M_c, $h \in \mathscr{C}(\odot^2 T^* M_c)$, is defined by $g(AX,Y) = h(X,Y)$ for all $X,Y \in T_x M_c$. The eigenvalues of A are the principal curvatures, $\lambda_1(x), \ldots, \lambda_p(x)$ say; let $S_1(x), \ldots, S_p(x)$ denote the corresponding eigenspaces.

The equation of Codazzi [38] is

$$g(R(X,Y)Z, \xi) = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.2.1}$$

for all $X,Y,Z \in \mathscr{C}(TM_c)$, where R is the curvature tensor of M, and $\xi \in \mathscr{C}(TM)$ is the unit normal vector field to M_c. Since M has constant curvature K say, $R(X,Y)Z = K(g(Y,Z)X - g(X,Z)Y)$ is tangent to M_c, and equation (2.2.1) becomes

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \tag{2.2.2}$$

for all $X,Y,Z \in \mathscr{C}(TM_c)$.

Let $X \in S_i$ and $Y,Z \in S_j$, then, since the λ_j are constant on M_c

$$\begin{aligned}(\nabla_X h)(Y,Z) &= X(h(Y,Z)) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)\\ &= X(\lambda_j g(Y,Z)) - \lambda_j g(\nabla_X Y,Z) - \lambda_j g(Y,\nabla_X Z)\\ &= \lambda_j (\nabla_X g)(Y,Z)\\ &= 0.\end{aligned}$$

Thus

$$\begin{aligned}0 &= (\nabla_Y h)(X,Z)\\ &= Y(h(X,Z)) - h(\nabla_Y X,Z) - h(X,\nabla_Y Z)\\ &= 0 - \lambda_j g(\nabla_Y X,Z) - \lambda_i g(X,\nabla_Y Z)\\ &= (\lambda_j - \lambda_i)\, g(X,\nabla_Y Z).\end{aligned} \tag{2.2.3}$$

Thus, for all $Y,Z \in S_j$: $\Pi(\nabla_Y Z) \in S_j$ where $\Pi: T_X M \longrightarrow T_X M_c$ is projection. We therefore have

Proposition 2.2.1 The distribution S_j, $j = 1, \ldots, p$, is integrable, and the integral submanifolds are totally geodesic in M_c. Furthermore the leaves of S_j are umbilical in M with mean curvature vector parallel to ξ. So in the case $M = S^{m-1}$, the leaves are small spheres in S^{m-1}.

Proof That the distribution is integrable and totally geodesic follows from equation (2.2.3). Since, for $Y, Z \in S_j$

$$\nabla_Y Z = \Pi(\nabla_Y Z) + \lambda_j g(Y,Z)\xi ;$$

the leaves of S_j are umbilical in M with mean curvature parallel to ξ. □

Suppose M is S^{m-1} (the cases $M = \mathbb{R}^m$, H^{m-1} are similar).

Define $\gamma_t : M_c \longrightarrow M$ by

$$\gamma_t(x) = \cos t + \sin t\ \xi_x , \text{ for all } x \in M_c .$$

For each i = 1, 2, ..., p, write $\lambda_i(x) = \cot\theta_i$ for some θ_i depending only on the level hypersurface M_c. Suppose $X \in S_j$, i.e. $X = \beta'(0)$, $\beta(0) = x$, where $\beta(s)$ is a curve in M_c. Then

$$\begin{aligned} d\gamma_{\theta_i}(X) &= \frac{d}{ds}(\gamma_{\theta_i} \circ \beta(s))\Big|_{s=0} \\ &= \cos\theta_i X + \sin\theta_i \frac{d}{ds}\xi_{\beta(s)}\Big|_{s=0} \\ &= \cos\theta_i X + \sin\theta_i (\nabla_X \xi) \\ &= \cos\theta_i X + \sin\theta_i(-\cot\theta_j X) . \end{aligned} \quad (2.2.4)$$

Thus $d\gamma_t(X) = 0$ if and only if $t = \theta_i$ and $X \in S_i(x)$, for some i = 1, 2, ..., p. Call $\gamma_{\theta_i}(M_c)$ a focal variety of f.

Theorem 2.2.2 [32] Each focal variety is a minimal submanifold of S^{m-1}.

Remark 2.2.3 From equation (2.2.4) we see that $d\gamma_t(X)$ is proportional to X for all $X \in S_j$, j = 1, ..., p. Thus $\mathscr{L}_\xi X \in S_j$, where $\mathscr{L}$ denotes Lie derivation on M, and since $\nabla_X \xi = -\lambda_j X$; we see that $\nabla_\xi X = \mathscr{L}_\xi X + \nabla_X \xi \in S_j$, j = 1, ..., p, so the principal curvature distributions are parallel with respect to ξ.

Proposition 2.2.4 [5] If M is S^{m-1}, there are at most two focal varieties. If M is $\mathbb{R}^m$ or H^{m-1} there is at most one.

Let V_1, V_2 denote the focal varieties (if there is only one, then let $V_2 = \emptyset$), and let $\Pi: M\backslash V_2 \longrightarrow V_1$ be the projection map down the integral curves of ξ, then

Proposition 2.2.5 The map $\Pi : M\setminus V_2 \to V_1$ is harmonic. If the number of distinct principal curvatures is less than or equal to two, then Π is a harmonic morphism (c.f. Lemma 1.3.3).

Proof Suppose dim $M = m$, and let $x \in M\setminus V_2$. Choose a frame field $X_1, \ldots, X_m$ about x adapted to the S_j spaces, in the sense that for each j, if dim $S_j = m_j$, a subset $X_{j_1}, \ldots, X_{j_{m_j}}$ of $X_1, \ldots, X_m$ forms an orthonormal basis for S_j at each point, and let $X_m = \xi$. Furthermore, since the integral submanifolds of the S_j distributions are totally geodesic in the level hypersurfaces of f, we can suppose $\nabla_{X_i} X_i$ is proportional to ξ for $i = 1, \ldots, m$.

Now $\Pi = \gamma_{\theta_j}$ for some $j = 1, \ldots, m$, so the horizontal space through x is $\mathscr{H}_x = \bigoplus_{i\neq j} S_i(x)$, and the vertical space $\mathscr{V}_x = S_j(x) \oplus \mathbb{R}\,\xi$. From equation (2.2.4) we see that Π is horizontally conformal if and only if the number of distinct principal curvatures p is equal to 2.

To see that Π is harmonic, we work out trace $\nabla d\Pi$. If $i = 1, \ldots, m-1$ then

$$\begin{aligned}\nabla d\Pi(X_i, X_i) &= -d\Pi(\nabla_{X_i} X_i) + \nabla_{d\Pi(X_i)} d\Pi(X_i) \\ &= 0,\end{aligned}$$

from equation (2.2.4) and since $\nabla_{X_i} X_i$ is proportional to ξ. Clearly $\nabla d\Pi(\xi,\xi) = 0$, so that Π is harmonic. □

On $\mathbb{R}^m$, the number of distinct principal curvatures on each level hypersurface of f is 1, and on H^{m-1} at most 2 [5]. Cartan classified all such families of isoparametric hypersurfaces. He also solved the classification problem for $p = 3$ on S^{m-1} [6], finding that m could be only 2,5,8,14 or 26. Münzner [30] showed that every isoparametric hypersurface is algebraic, and only certain p are allowed; we state his remarkable theorem without proof.

Theorem 2.2.5 [30] Let M be S^{m-1} and M_c an isoparametric hypersurface with p distinct principal curvatures $\lambda_1, \ldots, \lambda_p$ with multiplicities $m_1, \ldots m_p$ respectively. Then

(i) $m_{i+2} = m_i$ - i.e. there are at most 2 distinct multiplicities.

(ii) M_c is the level surface of the restriction to S^{m-1} of a homogeneous polynomial F on $\mathbb{R}^m$ satisfying

$$|\nabla F(x)|^2 = p^2 |x|^{2p-2} \quad (2.2.5)$$

$$\Delta F(x) = d|x|^{p-2} , \quad (2.2.6)$$

where $d = p^2(m_2 - m_1)/2$ if p is even and 0 if p is odd.

(iii) Conversely, any such F defines a family of isoparametric hypersurfaces on S^{m-1}.

(iv) p can only be 1,2,3,4 or 6.

Remark 2.2.6 We will call p the degree of the isoparametric function, hypersurface or family of hypersurfaces.

Remark 2.2.7 The function $f = F|_{S^{m-1}}$ has image $f(S^{m-1}) = [-1,1]$, where F is as in Theorem 2.2.5 (ii). The sets $f^{-1}(-1)$, $f^{-1}(1)$ correspond to the focal varieties.

Remark 2.2.8 If one puts $f = \cos ps$, $s \in [0, \pi/p]$, then s represents the affine parameter such that $\xi = \nabla s$.

Lemma 2.2.9 The Laplacian $\Delta s(x)$ is minus the mean curvature of the level hypersurface $M_{s_0} = f^{-1}(\cos ps_0)$, $x \in M_{s_0}$.

Proof Let $i_{s_0}: M_{s_0} \longrightarrow M$ be the inclusion map. Then

$$\begin{aligned} 0 &= \Delta(s \circ i_{s_0}) \\ &= ds(\Delta i_{s_0}) + \operatorname{trace} \nabla ds(di_{s_0}, di_{s_0}). \end{aligned}$$

But $\xi = \partial/\partial s$ is affine geodesic, hence $\nabla_\xi \xi = 0$; so that $\nabla ds(\xi,\xi) = ds(\nabla_\xi \xi) + \nabla^{\mathbb{R}}_{ds(\xi)} ds(\xi) = 0$ ($ds(\xi)$ is the unit vector along $\mathbb{R}$). That is

$\operatorname{trace} \nabla ds(di_{s_0}, di_{s_0}) = \Delta s$. Thus

$$0 = ds(\Delta i_{s_0}) + \Delta s .$$

But since Δi_{s_0} is proportional to ξ; we conclude that $\Delta s . \xi = - \Delta i_{s_0}$. □

Remark 2.2.10 With respect to the affine parameter s, whose level hypersurfaces we denote by $M_{s_0} = s^{-1}(s_0)$, the principal curvatures are $\lambda_i = -\cot(s_0 + (i-1)\pi/p)$, $i = 1, \ldots, p$, on M_{s_0}, and the corresponding integral small sphere of the distribution $S_i(x)$, $x \in M_{s_0}$, has radius $\sin(s_0 + (i-1)\pi/p)$. Hence we get

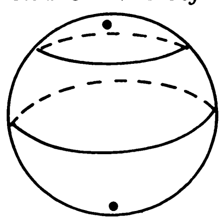
Remark 2.2.11 The mean curvature varies from $-\infty$ to $+\infty$ on S^{m-1} and is continuous, thus, for each family of hypersurfaces on S^{m-1}, there exists one hypersurface with zero mean curvature, i.e. minimal. In fact the minimal hypersurface is $M_{\pi/2p} = f^{-1}(\cos \pi/2) = f^{-1}(0)$.

2.3 Examples of isoparametric functions

Example 2.3.1 The isoparametric families of hypersurfaces of degree 1 are all given by the restriction of linear functions:

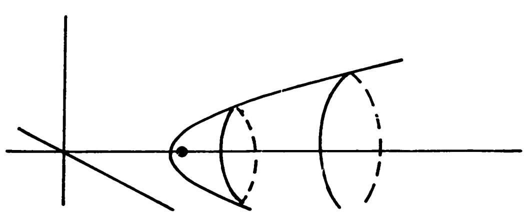
(i) On $\mathbb{R}^m$; define $f: \mathbb{R}^m \to \mathbb{R}$ by $f(x_1, \ldots, x_m) = x_1$. The level hypersurfaces are isometric to $\mathbb{R}^{m-1}$, and there is no focal variety.

(ii) On S^{m-1}; write $z \in S^{m-1}$ as $z = (\cos s, \sin s\, x)$, $x \in S^{m-2}$ and $s \in [0, \pi]$. Define $f: S^{m-1} \to \mathbb{R}$ by $f(z) = \cos s$. The level hypersurfaces are $(m-2)$-dimensional small spheres. Each focal variety is a single point $f^{-1}(1)$ and $f^{-1}(-1)$.

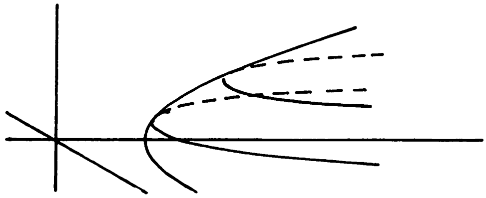


(iii) on H^{m-1} there are three distinct cases.

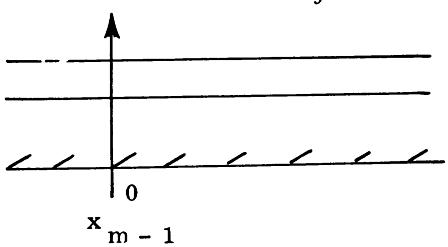
(a) For each $z \in H^{m-1}$, write $z = (\cosh s, \sinh s\, y)$, $y \in S^{m-1}$ and $s \in [0,\infty)$. Let $f: H^{m-1} \to \mathbb{R}$ be defined by $f(z) = \cosh s$. The level hypersurfaces are $(m-1)$-dimensional spheres. There is one focal variety which is a point.



(b) Let $z \in H^{m-1}$ be written as $z = (\cosh s\, x, \sinh s)$, $x \in H^{m-2}$, $s \in [0,\infty)$. Define $f : H^{m-1} \to \mathbb{R}$ by $f(z) = \sinh s$. The level hypersurfaces are "equidistants" from the plane $x_m = 0$ in M^m, and are $(m-2)$-dimensional hyperbolic spaces. There is no focal variety.



(c) Express H^{m-1} as the upper half plane in $\mathbb{R}^{m-1}$ with metric $ds^2 = (dx_1^2 + \dots + dx_{m-1}^2)/x_{m-1}^2$, where $x_1, \dots x_{m-1}$ are standard coordinates on $\mathbb{R}^{m-1}$. Define $f : H^{m-1} \to \mathbb{R}$ by $f(x_1, \dots, x_{m-1}) = x_{m-1}$. The level hypersurfaces are "horospheres" - that is, they are isometric to $(m-2)$-dimensional Euclidean spaces. There is no focal variety.



Example 2.3.2 The isoparametric families of hypersurfaces of degree 2 on $\mathbb{R}^m$, S^{m-1} and H^{m-1} are all given in Examples 2.1.4, 2.1.5 and 2.1.6. In Example 2.1.4 there is one focal variety which is isometric to $\mathbb{R}^{m-p}$. In Example 2.1.5 there are two focal varieties, one is isometric to S^{p-1} and the other is isometric to S^{q-1}. In Example 2.1.6 there is one focal variety isometric to H^{p-1}.

Example 2.3.4 Define $F : \mathbb{R}^{3\nu+2} \to \mathbb{R}$, $\nu = 1,2,3,4$ or 8 by

$$F(u,v,X,Y,Z) = u^3 - 3uv^2 + \frac{3}{2}u(X\bar{X} + Y\bar{Y} - 2Z\bar{Z}) + \frac{3\sqrt{3}}{2}v(X\bar{X} - Y\bar{Y})$$

$$+ \frac{3\sqrt{3}}{2}(XYZ + \bar{Z}\bar{Y}\bar{X}), \qquad (2.3.1)$$

where $u, v \in \mathbb{R}$, $X, Y, Z \in F$, F is one of $\mathbb{R}$, $\mathbb{C}$, quaternions or Cayley numbers, and $\bar{\cdot}$ is conjugation on F. Then $f = F|_{S^{3\nu+1}}$ is an isoparametric function of degree 3 (all such arise in this way [6]). The focal varieties are antipodal, and are isometric to the projective plane $P^2(F)$, embedded as the standard Veronese minimal submanifold in the sphere.

Example 2.3.5 [32] Identify (n+1)-dimensional complex space $\mathbb{C}^{n+1}$ with $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$, by writing a point $z \in \mathbb{C}^{n+1}$, $z = (z_1, \ldots, z_{n+1})$, $z_j = x_j + y_j$, $j = 1, \ldots, n+1$, as $z = x + iy = (x_1, \ldots, x_{n+1}) + i(y_1, \ldots, y_{n+1})$. We shall also write z as (x,y) whenever this is convenient.

Let $F : \mathbb{C}^{n+1} \longrightarrow \mathbb{R}$ be defined by

$$F(z) = (|x|^2 - |y|^2)^2 + 4 <x,y>^2 . \qquad (2.3.2)$$

Then $f = F|_{S^{2n+1}}$ is isoparametric of degree 4 on S^{2n+1}.

Define $i_s : S^1 \times S_{n+1,2} \longrightarrow S^{2n+1}$, where $S_{n+1,2}$ is the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{n+1}$, by

$$i_s(e^{i\theta}, (x,y)) = e^{i\theta}(\cos s\, x + i \sin s\, y) ,$$

where $s \in [0, \Pi/4]$. Then i_s is an immersion which double covers the level hypersurface $M_s = f^{-1}(\sin^2 2s)$. The hypersurface is obtained by the identification

$$(\theta, (x,y)) \sim (\theta + \Pi, (-x,-y)).$$

The principal curvatures of M_s are $-\cot s$, $-\cot(s - \Pi/4)$, $-\cot(s - \Pi/2)$, $-\cot(s - 3\Pi/4)$ with multiplicities n-1, 1, n-1, 1 respectively. There are two focal varieties, one at $s = 0$, which corresponds to the set $\{e^{i\theta}.x\} = S^1 \times S^n/S^0$, and one at $s = \Pi/4$, which corresponds to the set $\{e^{i\theta}(x + iy)/2^{\frac{1}{2}}\}$, i.e. the set $\{(x + iy)/2^{\frac{1}{2}}\} = S_{n+1,2}$. Let us study the Riemannian geometry of M_s in more detail.

Define $S^s_{n+1,2}$ to be the analytic submanifold of $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ given by $S^s_{n+1,2} = \{(x,y) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1};\ |x|^2 = \cos^2 s,\ |y|^2 = \sin^2 s \text{ and } <x,y> = 0\} = \{(\cos s\, x, \sin s\, y) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1};\ (x,y) \in S_{n+1,2}\}$. Let $e_1, \ldots, e_{n+1}$ be the orthonormal basis for $\mathbb{R}^{n+1}$ such that $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the i'th place. Choose $p \in S^s_{n+1,2}$ to be $p = (\cos s\, e_1, \sin s\, e_2)$. Consider the following curves in $\mathbb{R}^{n+1}$:

$$\tilde{\gamma}_i(u) = \cos(u/\cos s)e_1 + \sin(u/\cos s)\, e_i ,$$
$$\tilde{\lambda}_i(u) = \cos(u/\sin s)\, e_2 + \sin(u/\sin s)\, e_i ,\ i = 3, \ldots, n+1 , \qquad (2.3.3)$$

and define the curves through p in $S^s_{n+1,2}$:

$$\gamma_i(u) = (\cos s\, \tilde{\gamma}_i(u),\ \sin s\, e_2)$$

$$\lambda_i(u) = (\cos s\, e_1,\ \sin s\, \tilde{\lambda}_i(u)),\ i = 3, \ldots, n+1$$

$$\mu(u) = (\cos s(\cos u\, e_1 + \sin u\, e_2),\ \sin s\,(-\sin u\, e_1 + \cos s\, e_2)) . \qquad (2.3.4)$$

The tangent space to $S^s_{n+1,2}$ at p is given by

$$T_p S^s_{n+1,2} = \{(v,w)_p;\ \langle v,w\rangle = 0 = \langle w,y\rangle \text{ and } \langle v,y\rangle + \langle x,w\rangle = 0 \text{ where } p = (x,y)\},$$

and the vectors

$$\gamma'(0) = (e_i, 0)_p$$

$$\lambda_i'(0) = (0,e_i)_p \quad , i = 3, \ldots, n+1$$

$$\mu'(0) = (\cos s\, e_2,\ -\sin s\, e_1), \qquad (2.3.5)$$

form an orthonormal basis at p for $T_p S^s_{n+1,2}$.

<u>Lemma 2.3.6</u> <u>The curves</u> (2.3.4) <u>are all geodesic in</u> $S^s_{n+1,2}$ <u>at</u> p - <u>that is</u>, $(\nabla^{\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}}_{\beta'(0)} \beta'(0))_p$ <u>is perpendicular to</u> $T_p S^s_{n+1,2}$, <u>where</u> $\beta(u)$ <u>is one of the curves</u> (2.3.4).

<u>Proof</u> A curve $\beta(u)$, $\beta(0) = p$, in $S^s_{n+1,2}$ is geodesic at p if and only if $\beta''(0)$ is perpendicular to $T_p S^s_{n+1,2}$. For example, $\mu''(0) = (\cos s(-e_1),\ \sin s\,(-e_2))$. Clearly the scalar product of $\mu''(0)$ with the vectors of (2.3.5) is zero. Similarly for the other curves of (2.3.4). □

We study $\tilde{M}_s$, the double cover of M_s, which locally can be described as the set of points $e^{i\theta}(\cos s\ x + i\sin s\ y)$, $s \in (0, \pi/4)$, $(x,y) \in S_{n+1,2}$ and $\theta \in [0,2\pi)$. Fix θ, then $S_{n+1,2}$ is embedded in $\mathbb{C}^{n+1}$ as the manifold $S^s_{n+1,2}$. Fix $\cos s\ x + i\sin s\ y$, then as θ varies, we trace out a great circle of S^{2n+1}. How does this circle intersect the $S^s_{n+1,2}$?

Fix $p = e^{i\theta_0}(\cos s\, x_0 + i \sin s\, y_0) \in \widetilde{M}_s$. Consider the two curves

$$\gamma(u) = e^{i\vartheta(u)}(\cos s\, x_0 + i\sin s\, y_0)$$

$$\delta(u) = e^{i\theta_0}(\cos s\, x(u) + i\sin s\, y(u)), \qquad (2.3.6)$$

both of which are contained in $\widetilde{M}_s$, where $x(0) = x_0, y(0) = y_0$, $\theta(0) = \vartheta_0$ and $(x(u), y(u)) \in S_{n+1,2}$ for all u. These are both curves through p, and without loss of generality $\theta'(0) = 1$. Then

$$\gamma'(0) = i\theta'(0)\, e^{i\theta_0}(\cos s\, x_0 + i\sin s\, y_0)$$

$$= e^{i\theta_0}(-\sin s\, y_0 + i\cos s\, x_0)$$

and

$$\delta'(0) = e^{i\theta_0}(\cos s\, x'(0) + i\sin s\, y'(0)).$$

The Riemannian scalar product

$$<\gamma'(0), \overline{\delta'(0)}> = i(\cos s\, x_0 + i\sin s\, y_0).(\cos s\, x'(0) + i\sin s\, y'(0))$$

$$= i(i\sin s\cos s\, y_0.x'(0) - i\cos s\sin s\, x_0.y'(0)),$$

$$\text{since } x_0.x'(0) = y_0.y'(0) = 0$$

$$= \sin 2s\, x_0.y'(0), \text{ since } x_0.y'(0) + y_0.x'(0) = 0$$

$$= \cos(2s - \Pi/2)\, x_0.y'(0).$$

Thus the angle of incidence of γ and δ is independent of θ_0, and is $2s - \Pi/2$. Furthermore, the plane of incidence at p is spanned by $\mu'(0)$ and $\gamma'(0)$, where μ is as in (2.3.4). We therefore have

Lemma 2.3.7 The level hypersurfaces M_s, locally can be described as an "angular product" of S^1 with $S^s_{n+1,2}$, where the angle of incidence is $2s - \Pi/2$, and the plane of incidence is spanned by the tangent vectors to the curve $\mu(u)$ of (2.3.4) and the curve $\gamma(u)$ of (2.3.6).

Example 2.3.8 [17] An n-tuple $(P_1, \ldots, P_n)$ of symmetric endomorphisms of

$\mathbb{R}^{2l}$ is called a Clifford system if

$$P_iP_j + P_jP_i = 2\,\delta_{ij}\,I\,,\quad i,j = 1,\ \dots,\ n. \tag{2.3.7}$$

Theorem 2.3.9 [17] Given a Clifford system $(P_1, \dots, P_n)$ on $\mathbb{R}^{2l}$ such that $m_1 = n$ and $m_2 = l-n$ are both positive, then

$$F(x) = |x|^4 - 2\sum_i \langle P_ix, x\rangle^2,\ x \in \mathbb{R}^{2l},$$

defines an isoparametric function $f = F|_{S^{2l-1}}$ on S^{2l-1} of degree 4, with the multiplicities of the principal curvatures being (m_1, m_2).

Example 2.3.5 is a special case of this example. This family of examples also includes many where the level hypersurfaces of f are non-homogeneous (they are not the orbits of a subgroup of the orthogonal group) - including the non-homogeneous examples of Ozeki and Takeuchi [33] (which we give explicitly in chapter 8). Furthermore these examples include ones where the focal varieties are also non-homogeneous.

2.4 Generalizing the notion of isoparametric families of hypersurfaces

We wish to generalize the notion of isoparametric families of hypersurfaces to Riemannian manifolds whose curvature is not necessarily constant. There are various possible definitions, but we make one which is suitable for our purpose of constructing harmonic maps.

Let (M,g) be a Riemannian manifold together with a family of hypersurfaces $(M_c)_{c\in I}$, where I is some indexing set, in the sense that there exists a closed nowhere dense subset K of M such that $M\setminus K$ is foliated by hypersurfaces M_c. Suppose that on $M\setminus K$ the following conditions are satisfied:

(i) The locally defined unit normal vector field to the foliation ξ, satisfies $\nabla_\xi \xi = 0$.

(ii) If M_c is a hypersurface in the foliation, then the principal curvatures $\lambda_1, \dots, \lambda_p$ are defined up to a sign. Suppose that for each $i = 1, \dots, p$; λ_i is constant on M_c.

(iii) If S_k is the distribution on M_c corresponding to the principal curvature λ_k, then projection down the integral curves of ξ, $\rho_s : M_s \longrightarrow M_c$, preserves S_k in the sense that, if $X \in S_k(x)$, $x \in M_s$, then $\rho_{s*} X \in S_k(\rho_s(x))$, provided that ρ_s is a diffeomorphism (i.e. we don't encounter a focal variety). Then we call the family of hypersurfaces a generalized family of isoparametric hypersurfaces. In certain circumstances, it may be that the family of hypersurfaces is no longer defined by a function.

Lemma 2.4.1 Let $(M_c)_{c\in I}$ be a generalized family of isoparametric hypersurfaces on M which are the level sets of a function $f: M \longrightarrow \mathbb{R}$, then

$$|df|^2 = \psi_1(f)$$

$$\Delta f = \psi_2(f),$$

for some functions ψ_1 and ψ_2. Call such an f a generalized isoparametric function.

Proof The vector field ξ is defined, up to a sign, by $\xi = \nabla f/|df|$. Let $X \in \ker df$, then, since $g(X,\xi) = 0$ and $\nabla_\xi \xi = 0$; we conclude that $g(\nabla_\xi X, \xi) = 0$, and $\nabla_\xi X \in \ker df$. Also $g(\xi,\xi) = 1$ implies $\nabla_X \xi \in \ker df$, thus $0 = [X,\xi]f = X(\xi f) - \xi(Xf) = X(\xi f)$. But $\xi f = (|df|^2)^{\frac{1}{2}}$, so that $\ker(d(|df|^2)^{\frac{1}{2}}) \supset \ker df$. Since the dimensions are equal and finite; we conclude that $\ker(d(|df|^2)^{\frac{1}{2}}) = \ker df$, and that $|df|^2 = \psi_1(f)$ for some function ψ_1.

Equation (2.1.3) is still valid for non-constant curvature, giving $\Delta f = \psi_2(f)$, for some function ψ_2. □

Remark 2.4.2 Since the curvature term in the Codazzi equation (2.2.1) is now no longer necessarily zero, it is not necessarily true that the S_k distribution on M_c is totally geodesic or integrable.

Example 2.4.3 All isoparametric functions f on S^{m-1} define a generalized isoparametric family of hypersurfaces on $\mathbb{R}P^{m-1}$. In the case when the degree of f is even; the family of isoparametric surfaces on $\mathbb{R}P^{m-1}$ has two focal varieties. In the case when the degree of f is odd; there is just one focal variety.

Example 2.4.4 Express $\mathbb{R}^{2m}$ as $\mathbb{R}^{2p} \oplus \mathbb{R}^{2q}$ where $p + q = m$, and identify $\mathbb{R}^{2p} \oplus \mathbb{R}^{2q}$ with $\mathbb{C}^p \oplus \mathbb{C}^q$ in the obvious way. Consider $F: \mathbb{R}^{2m} \longrightarrow \mathbb{R}$, $F(x,y) = |x|^2 - |y|^2$, $x \in \mathbb{R}^{2p}$, $y \in \mathbb{R}^{2p}$, $y \in \mathbb{R}^{2q}$, whose restriction defines an isoparametric function on S^{2m-1} with focal varieties S^{2p-1} and S^{2q-1}. The function f factors through the action of S^1 on S^{2m-1}, to give a generalized isoparametric function on $\mathbb{C}P^{m-1}$ with focal varieties $\mathbb{C}P^{p-1}$ and $\mathbb{C}P^{q-1}$. The level hypersurfaces are diffeomorphic to $S^{2p-1} \times S^{2q-1}/S^1$.

Remark 2.4.5 The function $f: S^{2n+1} \to \mathbb{R}$ of Example 2.3.5 factors through the action of S^1 on S^{2n+1} to give a well-defined function g on $\mathbb{C}P^n$, with focal varieties $\mathbb{R}P^n$ and $S_{n+1,2}/S^1$. The level hypersurfaces are diffeomorphic to $S_{n+1,2}/S^0$. Then g is a generalized isoparametric function.

Remark 2.4.6 Let $\pi: \mathbb{C}P^3 \to S^4$ be the Hopf map, and let $f: S^4 \to \mathbb{R}$ be the isoparametric function of degree 3 of Example 2.3.4. Define $g: \mathbb{C}P^3 \to \mathbb{R}$ by $g(x) = f(\pi(x))$, for all $x \in \mathbb{C}P$. Then g is a generalized isoparametric function. For example, if $\pi: M \to N$ is a harmonic Riemannian submersion, and N_c is a hypersurface of constant mean curvature in N, then $\pi^{-1}(N_c)$ has constant mean curvature in M [15].

Example 2.4.7 Consider the tangent bundle TM of a Riemannian manifold (M, g). Let π be the projection map $\pi: TM \to M$. Define the metric G on TM to be the Sasaki metric [35], and write D for the Levi-Civita connection on TM. The metric G and connection D have the properties that

(i) the horizontal lift of any geodesic of M is a geodesic of TM.

(ii) Every straight line in the fibre of $\pi: TM \to M$ is a geodesic of TM.

(iii) Every fibre of $\pi: TM \to M$ is totally geodesic in TM.

Define $f: TM \to R$ by $f(x,v) = |v|^2$, where $x \in M$ and $v \in T_xM$. Then $\xi = \nabla f$ is a geodesic vector field in TM. Let $M_c = f^{-1}(c)$ be a hypersurface of TM, $c \neq 0$. Then for $x \in M_c$; let $X \in T_xM_c$ be horizontal with respect to π. Then $D_X\xi = 0$, so X is a principal curvature vector with principal curvature 0 of multiplicity equal to $m = \dim M$. If $X \in T_xM_c$ is vertical with respect to π, then $D_X\xi = X/c$, and X is a principal curvature vector with principal curvature $-1/c$ of multiplicity $m - 1$. Clearly the principal curvature eigenspaces are preserved under projection down the integral curves of ξ. Hence $f: TM \to \mathbb{R}$ is a generalized isoparametric function. Since the horizontal distribution of $\pi: TM \to M$ is integrable if and only if M is flat; the eigenspace distribution corresponding to the principal curvature 0 will not in general be integrable.

3 The stress–energy tensor

3.1 Derivation of the stress-energy tensor

Let (M,g) be a compact Riemannian manifold, and consider an action which, for simplicity, we suppose has the form

$$I(\emptyset) = \int_M L(j^1(\emptyset))\,dx \ , \tag{3.1.1}$$

where $\emptyset$ is a section of a Riemannian fibre bundle $\Pi: E \longrightarrow M$, and L is a function on the bundle of 1-jets of sections of $\Pi: E \longrightarrow M$; $L: J^1(E) \longrightarrow \mathbb{R}$, with L possibly depending on the metric of E.

Example 3.1.1 Let $\emptyset: (M,g) \longrightarrow (N,h)$ be a smooth map, $E = M \times N$. Then $\emptyset$ can be regarded as a section of E. Let $L(j^1(\emptyset)) = |d\emptyset|^2$ be the square of the Hilbert-Schmidt norm of $d\emptyset$.

In general one looks for extremals of the action (3.1.1) with respect to variations of the section $\emptyset$, and one finds that $\emptyset$ is an extremal if and only if $\emptyset$ satisfies the Euler-Lagrange equations

$$\mathscr{E}\mathscr{L}(\emptyset) = 0 \ , \tag{3.1.2}$$

where $\mathscr{E}\mathscr{L}(\emptyset) \in \mathscr{C}(\emptyset^{-1}TE)$ (we write $\mathscr{E}\mathscr{L}$ for the Euler-Lagrange operator on sections). In Example 3.1.1, the extremals are the harmonic maps and the Euler-Lagrange equation for $\emptyset$ is

$$\Delta \emptyset = 0 \ . \tag{3.1.3}$$

Suppose now we vary the metric g. If $g(u)$ is a smooth 1-parameter family of metrics, $g(0) = g$, then $\delta g = \partial g/\partial u\,|_{u=0}$ lies in $\mathscr{C}(\odot^2 T^* M)$.

Proposition 3.1.2 For a fixed section $\emptyset$,

$$\frac{dI(\emptyset)}{du}\Big|_{u=0} = \int_M \langle S_\emptyset , \delta g\rangle\, dx \ ,$$

where $S_\phi \in \mathscr{C}(\odot^2 T^*M)$, called the stress-energy tensor of ϕ, is given by $S_\phi = \partial_g L + \frac{1}{2} g L$, and $<,>$ is the metric induced on $\odot^2 T^*M$ from g (by $\partial_g L$ we mean the section of $\odot^2 T^*M$, which is given in components by $(\partial_g L)_{ab} = \frac{\partial L}{\partial g_{cd}} g_{ac} g_{bd}$ (summing over repeated indices), where $g = g_{ab} dx^a dx^b$ with respect to a local coordinate system (x^a) on M).

Proof Suppose g has a local representation in the form $g = g_{ab} dx^a dx^b$ (summing over repeated indices) with respect to local coordinates (x^a) on M. Then

$$\frac{dI}{du}\Big|_{u=0} = \int_M \frac{\partial L}{\partial g_{ab}} \delta g_{ab} \, dx + \int_M L \frac{\partial (dx)}{\partial g_{ab}} \delta g_{ab} .$$

The volume element dx can be written in the form $dx = (\det g)^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^m$. Then

$$\begin{aligned} \frac{\partial (dx)}{\partial g_{ab}} &= \tfrac{1}{2} (\det g)^{-\frac{1}{2}} (\text{co-factor of } g_{ab}) \, dx^1 \wedge \ldots \wedge dx^m \\ &= \tfrac{1}{2} (\det g)^{-1} (\text{co-factor of } g_{ab}) (\det g)^{\frac{1}{2}} \, dx^1 \wedge \ldots \wedge dx^m \\ &= \tfrac{1}{2} g^{ab} \, dx. \end{aligned}$$

□

Corollary 3.1.3 If ϕ and L are as in Example 3.1.1, then

$$S_\phi = e(\phi) g - \phi^* h .$$

Proof Let (x^a, y^α) be a local coordinate system on $E = M \times N$. These induce coordinates $(x^a, y^\alpha, y^\alpha_a)$ on $J^1(E)$ with respect to which $L(x^a, y^\alpha, y^\alpha_a) = g^{ab} y^\alpha_a y^\beta_b h_{\alpha\beta}$, where $g = g_{ab} dx^a dx^b$ and $h = h_{\alpha\beta} dy^\alpha dy^\beta$ are local representatives of the metrics g,h respectively (summing over repeated indices). Then

$$\frac{\partial L}{\partial g_{ab}} = \left(\frac{\partial g^{ij}}{\partial g_{ab}} \right) y^\alpha_i \, y^\beta_j \, h_{\alpha\beta} .$$

But $g^{ij} g_{jk} = \delta^i_k$, so

$$0 = \frac{\partial g^{ij}}{\partial g_{ab}} g_{jk} + g^{ij} \frac{\partial g_{jk}}{\partial g_{ab}} .$$

Whence

$$\frac{\partial g^{ij}}{\partial g_{ab}} = - g^{il} \frac{\partial g_{lk}}{\partial g_{ab}} g^{jk}$$

$$= - g^{il} \delta^a_l \delta^b_k g^{jk}$$

$$= - g^{ia} g^{jb} .$$

Thus $\partial L/\partial g_{ab} = - y^{\alpha a} y^{\beta b} h_{\alpha\beta}$. Or in coordinate free notation $\partial_g L(\phi) = - \phi^* h$. □

<u>Proposition 3.1.4</u> <u>If</u> X <u>is a smooth vector field on</u> M, <u>then</u>

$$\int_M \langle \mathcal{E}\mathcal{L}(\phi), d\phi(X) \rangle dx - 2 \int_M \nabla^* S_\phi(X) dx = 0 ,$$

<u>where</u> $\nabla^* S_\phi$ <u>denotes the divergence of</u> S_ϕ ; $\nabla^* S_\phi(X) = \sum_i \nabla_{X_i} S_\phi(X_i, X)$, <u>where</u> (X_i) <u>is an orthonormal frame field.</u>

<u>Proof</u> Let ψ_u be the family of diffeomorphisms associated to the vector field X, then for each u

$$I = \int_M L\, dx$$

$$= \int_{\psi_u(M)} L\, dx$$

$$= \int_M \psi_u^*(L\, dx).$$

Therefore $\int_M (L\, dx - \psi_u^*(L\, dx)) = 0$, and so

$$\int_M \mathcal{L}_X (L\, dx) = 0,$$

where $\mathcal{L}_X$ denotes Lie derivation with respect to X. Also

$$\int_M \mathcal{L}_X (L\, dx) = \int_M \langle \mathcal{E}\mathcal{L}(\phi), d\phi(X) \rangle dx + \int_M \langle S_\phi, \mathcal{L}_X g \rangle dx.$$

Now, in coordinates, letting ";" denote covariant differentiation; $(\mathcal{L}_X g)_{ab} = 2X_{(a;b)}$

(the bracket (,) means symmetrization of the indices), so that

$$\int_M (S_\emptyset^{ab}\, \delta g_{ab})\, dx = 2\int_M ((S_\emptyset^{ab} X_a)_{;b} - S^{ab}_{\emptyset;b} X_a)\, dx$$

$$= -2 \int_M S^{ab}_{\emptyset;b} X_a \, dx. \qquad \square$$

Corollary 3.1.5 If $\emptyset$ is an extremal of I, and so satisfies the Euler-Lagrange equations (3.1.2), then

$$\nabla^* S_\emptyset = 0$$

Proof The statement follows immediately from Proposition 3.1.4, since X was arbitrarily chosen. $\square$

Proposition 3.1.6 Suppose (M,g) is any Riemannian manifold (i.e. not necessarily compact), and $\emptyset : (M,g) \longrightarrow (N,h)$ a map. Let $S_\emptyset \in \mathscr{C}(\odot^2 T^* M)$ be given by $S_\emptyset = e(\emptyset) g - \emptyset^* h$, then for all $X \in \mathscr{C}(TM)$,

$$\langle \Delta\emptyset, d\emptyset(X)\rangle = -\nabla^* S_\emptyset(X).$$

Proof In coordinates, as in Corollary 3.1.3,

$$(S_\emptyset)_{ab} = \tfrac{1}{2}(g^{ij} \emptyset^\alpha_i \emptyset^\beta_j h_{\alpha\beta}) g_{ab} - \emptyset^\alpha_a \emptyset^\beta_b h_{\alpha\beta} .$$

Thus

$$(S_\emptyset)_{ab}{}^{;b} = (g^{ij} \emptyset^\alpha_i \emptyset^\beta_{j;a} - \emptyset^{\alpha;b}_a \emptyset^\beta_b - \emptyset^\alpha_a \emptyset^{\beta;b}_b) h_{\alpha\beta} .$$

$$= -\emptyset^\alpha_a \emptyset^{\beta;b}_b h_{\alpha\beta}. \qquad \square$$

Corollary 3.1.7 If $\emptyset : (M,g) \longrightarrow (N,h)$ is harmonic, and $S_\emptyset \in \mathscr{C}(\odot^2 T^*M)$ is given by $S_\emptyset = e(\emptyset) g - \emptyset^* h$, then

$$\nabla^* S_\emptyset = 0.$$

Conversely, if $\emptyset$ is a submersion almost everywhere, and $\nabla^* S_\emptyset = 0$, then $\emptyset$ is harmonic.

3.2 Examples

Example 3.2.1 Suppose $\phi : (M,g) \longrightarrow (N,h)$ is conformal with $\phi^* h = \rho g$ for some smooth function $\rho : M \longrightarrow \mathbb{R}$. Then Corollary 3.1.3 gives

$$S_\phi = \tfrac{1}{2}\rho(m-2)g \quad . \tag{3.2.1}$$

Thus if $m = 2, S_\phi = 0$; and if $m > 2$ and ϕ is harmonic, Corollary 3.1.7 implies ρ is constant, i.e. ϕ is homothetic. This generalizes a result of Hoffman and Osserman [24].

Conversely, if $S_\phi = 0$, then $0 = \text{trace } S_\phi = \frac{1}{2}(m-2)e(\phi)$, whence, if ϕ is non-constant; $m = 2$ and $\phi^* h = e(\phi) g$, i.e. ϕ is conformal. Thus

Proposition 3.2.2 If $\phi : (M,g) \longrightarrow (N,h)$, then $S_\phi = 0$ if and only if $\dim M = 2$ and ϕ is conformal.

Corollary 3.2.3 Suppose $\phi : (S^2, g) \longrightarrow (N,h)$ is a harmonic map of the 2-sphere into a Riemannian manifold N, then ϕ is conformal.

Proof Choose local isothermal coordinates z on S^2, so that $g = \rho(z)\, dz\, d\bar{z}$. Then a tensor $T \in \mathscr{C}(\odot^2 T^* M)$ has a type decomposition of the form $T = T^{(2,0)} + T^{(1,1)} + T^{(0,2)}$, where $T^{(i,j)}$ is spanned by $dz^i d\bar{z}^j$ locally. A calculation shows that $(\phi^* h)^{(1,1)} = e(\phi) g$. Thus $S_\phi = S_\phi^{(2,0)} + S_\phi^{(0,2)}$, where $S_\phi^{(0,2)} = \overline{S_\phi^{(2,0)}}$. That the divergence, $\nabla^* S_\phi = 0$, amounts to $S_\phi^{(2,0)}$ being a quadratic holomorphic differential on S^2. But it is well-known that there are no non-trivial quadratic holomorphic differentials on S^2. Thus $S_\phi = 0$, and Proposition 3.2.2 shows that ϕ is conformal.

Example 3.2.4 Suppose $\phi : (M,g) \longrightarrow (N,h)$ is a Riemannian submersion, i.e. ϕ is a submersion, and the differential is an isometry on the horizontal space. Then

Proposition 3.2.5 (See [12] for a different proof): the map ϕ is harmonic if and only if the fibres are minimal.

Proof Locally, about a point $x \in M$, choose an orthonormal frame field for TM, $X_1, \dots, X_n, X_{n+1}, \dots, X_m$, where the first n vectors are horizontal and the last $n - m$ vectors are vertical. Choose indices $i, j, \dots$ to run from 1 to n; $r, s, \dots$

to run from $n+1$ to m and $a,b,\ldots$ to run from 1 to m, and use the usual summation convention for repeated indices.

Suppose $\emptyset$ is harmonic, and $Y \in \mathscr{C}(TM)$ is arbitrary, then

$$
\begin{aligned}
0 &= (\nabla_{X_a} S_\emptyset)(Y,X_a) \\
&= -(\nabla_{X_a} \emptyset^* h)(Y,X_a) \\
&= -X_a((\emptyset^* h)(Y,X_a)) - \emptyset^* h(\nabla_{X_a} Y,X_a) - \emptyset^* h(Y,\nabla_{X_a} X_a).
\end{aligned}
\tag{3.2.2}
$$

Choose $Y = X_j$ to be horizontal, then

$$
\begin{aligned}
0 &= X_i\, g(X_j,X_i) \\
&= g(\nabla_{X_i} X_j,X_i) + g(X_j,\nabla_{X_i} X_i) \\
&= g(\mathscr{H}\nabla_{X_i} X_j,X_i) + g(X_j,\mathscr{H}\nabla_{X_i} X_i) \\
&= \emptyset^* h(\nabla_{X_i} X_j,X_i) + \emptyset^* h(X_j,\nabla_{X_i} X_i),
\end{aligned}
$$

where $\mathscr{H}$ denotes projection onto the horizontal space. So that

$$0 = \emptyset^* h(Y,\nabla_{X_r} X_r),$$

and the fibre is minimal.

Conversely, if the fibres are minimal, then $\nabla^* S_\emptyset = 0$. □

3.3 The eigenvalue decomposition of the stress-energy tensor

Let $\emptyset : (M,g) \to (N,h)$ be a smooth map, and let $m = \dim M$, $n = \dim N$. Suppose $\emptyset^* h$ has eigenvalues $\nu_1, \ldots, \nu_m$ with respect to g, where $\nu_1 \geq \ldots \geq \nu_m \geq 0$. Let $X_1, \ldots X_m$ be an orthonormal basis on a domain of M adapted to this eigenvalue decomposition, i.e. if T_j is the eigenspace of ν_j, then a subset of $X_1, \ldots X_m$ forms an orthonormal basis of T_j. The energy density of the map $\emptyset$ is given by

$$e(\emptyset) = \sum_{k=1}^{m} \nu_k/2 . \tag{3.3.1}$$

If $P \in \mathscr{C}(\odot^2 T^* M)$, then let $P_{ij} = P(X_i,X_j)$ denote the components of P with respect to $X_1, \ldots, X_m$. Then

$$S_{ij} = e(\emptyset) g_{ij} - (\emptyset^* h)_{ij}$$

$$= 1 \cdot (\emptyset) - \nu_i) \delta_{ij} .$$

Now

$$e(\emptyset) - \nu_i = \Sigma_q \frac{(1 - 2\delta_{iq})}{2} \nu_q ,$$

therefore

$$S_{ij} = \Sigma_q \frac{(1 - 2\delta_{iq})}{2} \nu_q \delta_{ij} . \tag{3.3.2}$$

Example 3.3.1 In Example 3.2.4, when $\emptyset$ is a Riemannian submersion; $\nu_1, \ldots, \nu_n = 1$ and $\nu_{n+1}, \ldots, \nu_m = 0$.

Example 3.3.2 If $\emptyset$ is a harmonic morphism, as in Section 1.3, with dilation $\lambda : M \to \mathbb{R}$, then $\nu_1, \ldots, \nu_n = \lambda^2$, and $\nu_{n+1}, \ldots, \nu_m = 0$.

Example 3.3.3 In Proposition 2.2.5, the eigenspace of S_Π corresponding to non-zero eigenvalues, correspond with the principal curvature spaces S_i, i = 1, ...p, $i \neq j$, where $\Pi = \gamma_{\Theta_j}$. The eigenspace T_k of S_Π with $\nu_k = 0$ corresponds to the kernel of $d\Pi$, i.e. the direct sum of S_j with $\mathbb{R}\,\xi$, where ξ is the unit normal vector field to the isoparametric family.

4 Equivariant theory

4.1 <u>Maps which are equivariant with respect to isoparametric functions</u>.

Let M be a space form upon which is defined an isoparametric function $f: M \to \mathbb{R}$. Let M^* be the union of non-singular hypersurfaces of f with the induced topology, and let $\pi_f: M^* \to M^*/f$ be the canonical projection. Then M^*/f inherits a topology from M^* with respect to which it becomes an open interval. If $\bar{x} \in M^*f$; define a metric $\bar{g}$ at $\bar{x}$ as follows. If $\bar{X} \in T_{\bar{x}}(M^*/f)$ and $x \in \pi_f^{-1}(\bar{x})$, then there exists a unique normal X to $\bar{x}$ at x such that $d\,\pi_f(X) = \bar{X}$; define $\bar{g}(\bar{X},\bar{Y}) = g(X,Y)$.

<u>Lemma 4.1.1</u> <u>The metric $\bar{g}$ defined above is well-defined.</u>

<u>Proof</u> We must check that $\bar{g}$, as defined above, is independent of the choice of $x \in \pi_f^{-1}(\bar{x})$, for all $\bar{x} \in M^*/f$. Let $x,y \in \pi_f^{-1}(\bar{x})$, $\delta(u)$ a curve defining the unique normal X to $\bar{x}$ at x, and $\tilde{\delta}(u)$ be a curve defining the unique normal $\tilde{X}$ to $\bar{x}$ at y, such that $d\,\pi_f(X) = d\,\pi_f(\tilde{X}) = \bar{X}$. Then $d\,\pi_f(X) = (d/du)(\pi_f \circ \delta(u))|_{u=0}$ and $d\,\pi_f(\tilde{X}) = (d/du)(\pi_f \circ \tilde{\delta}(u))|_{u=0}$. Since the level surfaces are parallel; we can choose δ and $\tilde{\delta}$ so that $\pi_f \circ \delta(u) = \pi_f \circ \tilde{\delta}(u)$, for all $u \in (-\epsilon,\epsilon)$ for some $\epsilon > 0$. Thus $g(X,X) = g(\tilde{X},\tilde{X})$ and the result now follows. □

With respect to the metric $\bar{g}$, $\pi_f: (M^*,g) \to (M^*/f,\bar{g})$ becomes a Riemannian submersion. We add in appropriate end points to the interval M^*/f corresponding to the focal varieties of f, to obtain a closed, half-open or open interval I_f, depending on whether there are two, one or no focal varieties respectively. We extend π_f to a map $\pi_f: M \to I_f$. Similarly if g is an isoparametric function on a space form N, we can define $\pi_g: N \to I_g$.

Reparametrize f,g to become unit affine parameters s,t respectively (see Remark 2.2.8) ; so $\xi = \nabla s$ and $\eta = \nabla t$ both have norm 1.

Suppose $\emptyset : M \to N$ is a smooth map such that

$$\begin{array}{ccc} M & \xrightarrow{\emptyset} & N \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ I_s & \xrightarrow[\alpha]{} & I_t \end{array} \qquad (4.1.1)$$

is commutative for some $\alpha : I_s \to I_t$. If, in addition,

$$d\emptyset(\xi)_x = u(x)\eta_{\emptyset(x)}, \text{ for all } x \in M , \qquad (4.1.2)$$

where $u : M \to \mathbb{R}$ is some function, then we shall call $\emptyset$ <u>wavefront preserving (WFP) with respect to the isoparametric functions</u> s <u>and</u> t.

<u>Lemma 4.1.2</u> <u>If</u> $\emptyset : M \to N$ <u>is WFP with respect to the isoparametric functions</u> $s : M \to I_s$ <u>and</u> $t : N \to I_s$, <u>then</u>

$$d\emptyset(\xi)_x = \alpha'(s(x))\eta_{\emptyset(x)}, \text{ for all } x \in M^* ,$$

where $\xi = \nabla s, \eta = \nabla t$ <u>and</u> α <u>is as in Diagram (4.1.1)</u>.

<u>Proof</u> From equation (4.1.2), $d\emptyset(\xi)_x = u(x)\eta_{\emptyset(x)}$. Since $t : N \to I_t$ is a Riemannian submersion and $d\emptyset(\xi)$ is horizontal with respect to t

$$|d\emptyset(\xi)_x|^2 = |dt \circ d\emptyset(\xi)_x|^2 .$$

But

$$\begin{aligned} dt \circ d\emptyset(\xi) &= d\alpha \circ ds(\xi), \text{ from diagram (4.1.1)} \\ &= d\alpha(\partial/\partial s) \\ &= \alpha'(s) , \end{aligned}$$

writing $\partial/\partial s$ for the unit tangent vector to I_s. □

Suppose $\emptyset : M \to N$ is WFP with respect to s and t ; choose a particular hypersurface M_{s_0} of M^* for some fixed s_0, and let $\emptyset_{s_0} = \emptyset|_{M_{s_0}} : M_{s_0} \to N_{\alpha(s_0)}$. Define projections $\rho : M^* \to M_{s_0}$, $\sigma : N^* \to N_{\alpha(s_0)}$ to be the maps which project onto

M_{s_0}, $N_{\alpha(s_0)}$ respectively along the integral curves of ξ, η respectively. Then $\rho_s \equiv \rho|_{M_s} : M_s \to M_{s_0}$, $\sigma_t \equiv \sigma|_{N_t} : N_t \to N_{\alpha(s_0)}$ are both diffeomorphisms. Thus we can define a map $\phi_{s,t} : M_s \to N_t$ by

$$\phi_{s,t}(x) = \sigma_t^{-1} \circ \phi_{s_0} \circ \rho_s(x), \text{ for all } x \in M_s . \tag{4.1.3}$$

The map $\phi_{s,t}$ is not necessarily independent of the choice of s_0; however

<u>Lemma 4.1.3</u> <u>Provided that</u> $\phi(M^*) \subset N^*$ (<u>i.e. there is no</u> s <u>in the interior of</u> I_s <u>with</u> $\alpha(s) \in \partial I_t$ - <u>the end points of</u> I_t), <u>then</u> $\phi_{s,t}$ <u>is well-defined and independent of the choice of</u> s_0.

<u>Proof</u> Choose a hypersurface M_{s_1} of M^* for some fixed s_1, and define $\tilde{\rho}_s : M_s \to M_{s_1}$ to be the projection down the normal geodesics, and similarly for $\tilde{\sigma}_t : N_t \to N_{\alpha(s_1)}$. Define $\tilde{\phi}_{s,t} : M_s \to N_t$ by

$$\tilde{\phi}_{s,t}(x) = \tilde{\sigma}_t^{-1} \circ \phi_{s_1} \circ \tilde{\rho}_s(x), \text{ for all } x \in M_s .$$

Let $x \in M_s$; then the unique normal geodesic γ to M_s through x intersects M_{s_0} at x_0 say, and M_{s_1} at x_1. Suppose $\phi_{s_0}(x_0) = y_0$ and $\phi_{s_1}(x_1) = y_1$. Since, for all $z \in M^*$, $d\phi(\xi)_z = u(z)\eta_{\phi(z)}$; then $\phi(\gamma) = \delta$, where δ is the unique normal geodesic passing through y_0 and y_1. Thus,

$$\begin{aligned} \tilde{\phi}_{s,t}(x) &= \tilde{\sigma}_t^{-1} \circ \phi_{s_1} \circ \tilde{\rho}_s(x) \\ &= \tilde{\sigma}_t^{-1}(y_1) \\ &= \sigma_t^{-1}(y_0) \\ &= \sigma_t^{-1} \circ \phi_{s_0} \circ \rho_s(x) \\ &= \phi_{s,t}(x) . \end{aligned}$$

□

If ϕ is WFP with respect to s and t and satisfies the conditions of Lemma 4.1.3, then we shall call ϕ <u>simply wavefront preserving (S-WFP) with respect to</u> s <u>and</u> t.

<u>Lemma 4.1.4</u> <u>Suppose that</u> ϕ <u>is S-WFP with respect to the isoparametric functions</u>

s and t, then $\emptyset|_{M_{s_1}} = \emptyset_{s_1,\alpha(s_1)}$, for all $s_1 \in \text{int } I_s$.

Proof Let $x \in M_{s_1}$, then the unique normal geodesic γ to M_{s_1} at x intersects M_{s_0} at x_0 say. Also $y = \emptyset(x) \in N_{\alpha(s_1)}$, and there exists a unique normal geodesic δ to $N_{\alpha(s_1)}$ at $\emptyset(x)$ intersecting $N_{\alpha(s_0)}$ at y_0 say (δ is unique since we do not pass through a focal variety).

Since for all $z \in M^*$, $d\emptyset(\xi)_z = u(z)\eta_{\emptyset(z)}$; $\emptyset(\gamma) = \delta$, and $\emptyset(x_0) = y_0$. Thus

$$\emptyset_{s_1,\alpha(s_1)}(x) = c_{\alpha(s_1)}^{-1} \circ \emptyset_{s_0} \circ \rho_{s_1}(x)$$

$$= c_{\alpha(s_1)}^{-1} \circ \emptyset_{s_0}(x_0)$$

$$= c_{\alpha(s_1)}^{-1}(y_0)$$

$$= y .$$

Thus $\emptyset(x) = \emptyset_{s_1,\alpha(s_1)}(x)$ for all $x \in M_{s_1}$. □

Let the hypersurface M_s have distinct principal curvatures $\lambda_1(s), \ldots, \lambda_p(s)$, and N_t have distinct principal curvatures $\mu_1(t), \ldots, \mu_q(t)$. Let $S_k(x)$ be the eigenspace of $\lambda_k(s)$ at $x \in M_s$, $k = 1, \ldots, p$, and $T_j(y)$ the eigenspace of $\mu_j(t)$ at $y \in N_t$, $j = 1, \ldots, q$. Denote the integral submanifolds of $S_k(x)$, $T_j(y)$ by $\mathscr{S}_k(x)$, $\mathscr{T}_j(y)$ respectively.

Let $\emptyset : M \to N$ be S-WFP with respect to s and t. Suppose in addition there is a j_k with

(i) $d\emptyset_{s_0}(S_k(x)) \subset T_{j_k}(\emptyset_{s_0}(x))$, for all $x \in M_{s_0}$ and for all k.

Lemma 4.1.5 If $\emptyset : M \to N$ is S-WFP with respect to s and t, and satisfies condition (i) above, then $d\emptyset_{s,t}(S_k(x) \subset T_{j_k}(\emptyset_{s,t}(x))$, for all $x \in M_s$ and for all k, for some $j_k = 1, \ldots, q$.

Proof Consider the map $\gamma_u : M_s \to M$ of Section 2.2; for all $x \in M_s$

$$\gamma_u(x) = \cos u \; x + \sin u \; \xi_x$$

(we assume that M is a sphere - the other cases are similar). Then ρ_s and $\gamma_{s_0 - s}$ are identical, and equation (2.2.4) shows that $d\rho_s(X)$ is proportional to X. The unit normal at $\gamma_u(x)$, $x \in M_s$, is given by

$$\xi_{\gamma_u(x)} = -\sin u \; x + \cos u \; \xi_x ,$$

and it is now straightforward to compute $\nabla_{d\rho_s(X)} \xi_{\rho_s(x)}$, whence we conclude that $d\rho_s(S_k(x)) = S_k(\rho_s(x))$. Similarly $d\sigma_t^{-1}(T_{j_k}(\phi_{s_0} \circ \rho_s(x))) = T_{j_k}(\sigma_t^{-1} \circ \phi_{s_0} \circ \rho_s(x))$. Hence $d\phi_{s,t}(S_k(x)) \subset T_{j_k}(\phi_{s,t}(x))$.

<u>Corollary 4.1.6</u> <u>For all</u> $x \in M^*$ <u>and for all</u> k ;

$$\phi(\quad_k(x)) \subset \mathcal{T}_{j_k}(\phi(x)),$$

<u>for some</u> $j_k = 1, \ldots, q$.

If $\phi : M \to N$ is S-WFP with respect to s and t; let $\gamma_k(s,t) : M_s \to \mathbb{R}$ be defined by

$$\gamma_k(s,t)(x) = \text{trace}_{S_k(x)} \; h(d\phi_{s,t}, d\phi_{s,t}) ,$$

for all $x \in M_s$, $k = 1, \ldots, p$, where h is the metric on N. Suppose

(i) $d\phi_{s_0}(S_k(x)) \subset T_{j_k}(\phi_{s_0}(x))$, for all $x \in M_{s_0}$.

(ii) $\phi_{s,t} : M_s \to N_t$ is harmonic for all s and t, and

(iii) for each k, $\gamma_k(s,t)(x)$ depends only on s and t.

Then call ϕ <u>S-equivariant with respect to the isoparametric functions</u> s <u>and</u> t. Write $\gamma(s,t) = \sum_{k=1}^{p} \gamma_k(s,t)$ for such a ϕ.

<u>Example 4.1.7</u> Let $\phi : S^{m-1} \to S^{n-1}$ be one of Smith's maps defined in Section 1.3. Clearly ϕ is S-equivariant with respect to isoparametric functions s,t of degree 2 on S^{m-1}, S^{n-1} respectively. Indeed, each level hypersurface M_s is isometric to $\cos s \; S^{p-1} \times \sin s \; S^{q-1}$, and each level hypersurface N_t is isometric

to cost $S^{r-1} \times$ sint S^{s-1}. The map $\phi_{s,t} : M_s \to N_t$ is defined by

$$\phi_{s,t} (\cos s\ x, \sin s\ y) = (\cos t\ g_1(x), \sin t\ g_2(y)) ,$$

where $x \in S^{p-1}$, $y \in S^{q-1}$, $g_1 : S^{p-1} \to S^{r-1}$ is harmonic with $|dg_1|^2 = a_1$ constant and $g_2 : S^{q-1} \to S^{s-1}$ is harmonic with $|dg_2|^2 = a_2$ constant. Thus $\phi_{s,t}$ is harmonic, and $\gamma_1(s,t) = \cos^2 t\ a_1/\cos^2 s$, $\gamma_2(s,t) = \sin^2 t\ a_2/\sin^2 s$ depend only on s and t, and ϕ is S-equivariant.

Similarly the maps between hyperbolic spaces defined in Section 1.6 are S-equivariant with respect to isoparametric functions of degree two.

Theorem 4.1.8 If $\phi : M \to N$ is S-equivariant with respect to isoparametric functions s and t, then ϕ is harmonic if and only if

$$\alpha''(s) + \Delta s\, \alpha'(s) + \sum_{k=1}^{p} \mu_{j_k} \gamma_k = 0 , \qquad (4.1.4)$$

for all $s \in \text{int } I_s$, if and only if

$$\alpha''(s) + \Delta s\, \alpha'(s) - \tfrac{1}{2} d_t \gamma(s, \alpha(s)) = 0 , \qquad (4.1.5)$$

for all $s \in \text{int } I_s$.

Remark 4.1.9 If the condition that a map ϕ be harmonic reduces to solving a second order ordinary differential equation; we will say that reduction occurs.

Remark 4.1.10 A reduction theorem for maps equivariant with respect to group actions is proved by Smith [36] (see [36] for the definition of such maps). We remark that when ϕ is both equivariant with respect to the action of G on M and H on N, G and H are Lie groups, and harmonically r-equivariant in the sense that we have described, then the reduction equation of [36] and equation (4.1.5) above both agree.

Remark 4.1.11 We will give two proofs of Theorem 4.1.8, one direct proof and one using the stress-energy tensor and obtain equations (4.1.4) and (4.1.5) respectively. Afterwards we will show more directly that equations (4.1.4) and (4.1.5) are in fact the same.

First proof of Theorem 4.1.8 We first of all show that there exists a smooth function

$\psi : M^* \to \mathbb{R}$, such that

$$\Delta\phi(x) = \psi(x)\,\eta_{\phi(x)} ,$$

for all $x \in M^*$.

For each level hypersurface M_{s_0} of s, let $i_{s_0}: M_{s_0} \to M$ denote the inclusion map. Then $\phi \circ i_{s_0}$ is harmonic onto its image $N_{\alpha(s_0)}$, thus

$$\Delta(\phi \circ i_{s_0})(x) = \psi_1(x)\,\eta_{\phi(x)} ,$$

for each $x \in M_{s_0}$, where $\psi_1 : M_{s_0} \to \mathbb{R}$ is some function. Now

$$\Delta(\phi \circ i_{s_0}) = d\phi\,(\Delta i_{s_0}) + \operatorname{trace} \nabla d\phi(d i_{s_0}, d i_{s_0}).$$

Since Δi_{s_0} is proportional to ξ ; equation (4.1.2) implies

$$d\phi\,(\Delta\, i_{s_0})(x) = \psi_2(x)\,\eta_{\phi(x)},$$

for all $x \in M_{s_0}$ and for some function $\psi_2 : M_{s_0} \to \mathbb{R}$. Therefore

$$\operatorname{trace} \nabla\, d\phi(d i_{s_0}, d\, i_{s_0})(x) = \psi_3(x)\,\eta_{\phi(x)},$$

for all $x \in M_{s_0}$ and for some function $\psi_3 : M_{s_0} \to \mathbb{R}$.

Furthermore, from Lemma 1.1.1,

$$\begin{aligned}\nabla d\phi\,(\xi,\xi) &= -d\phi\,(\nabla^M_{\xi}\,\xi) + \nabla^N_{d\phi(\xi)}\, d\phi(\xi)\\ &= \nabla^N_{\alpha'(s)\eta}\,\alpha'(s)\eta\\ &= \psi_4\eta ,\end{aligned}$$

for some function $\psi_4 : M^* \to \mathbb{R}$. Thus

$$\begin{aligned}\Delta \phi &= \text{trace } \nabla d\phi \\ &= \text{trace } \nabla d\phi (d i_{s_0}, d i_{s_0}) + \nabla d\phi(\xi,\xi) \\ &= \psi \eta ,\end{aligned}$$

for some function $\psi : M^* \to \mathbb{R}$. We must now work out $h(\Delta \phi, \eta)$.

Let $(X_a)_{1 \le a \le \dim M-1}$ be an orthonormal frame field on a domain in M, which is tangent to the hypersurfaces M_s; thus $(X_a, \xi)_{1 \le a \le \dim M-1}$ forms an orthonormal frame field on a domain of M. Then

$$\begin{aligned}\Delta \phi &= \text{trace } \nabla d\phi \\ &= \sum_a \nabla d\phi (X_a, X_a) + \nabla d\phi (\xi,\xi) ,\end{aligned} \tag{4.1.6}$$

and

$$\begin{aligned}\nabla d\phi(\xi,\xi) &= - d\phi(\nabla_\xi \xi) + \nabla^N_{d\phi(\xi)} d\phi(\xi) \\ &= \nabla^N_{\alpha'(s)\eta} \alpha'(s)\eta \\ &= \alpha'(s)(d(\alpha'(s))(\eta)\eta) \\ &= \alpha'(s) . \alpha''(s) . \frac{1}{\alpha'(s)} \eta \\ &= \alpha''(s)\eta .\end{aligned} \tag{4.1.7}$$

Now

$$\sum_a \nabla d\phi (X_a, X_a) = \Delta(\phi \circ i_{s_0}) - d\phi(\Delta i_{s_0}) .$$

But

$$\begin{aligned} 0 &= \Delta(s \circ i_{s_0}) \\ &= ds(\Delta i_{s_0}) + \text{trace } \nabla ds(d i_{s_0}, d i_{s_0}) \\ &= ds(\Delta i_{s_0}) + \Delta s ,\end{aligned}$$

since $\nabla ds(\xi,\xi) = 0$ (s being a Riemannian submersion). Thus

$$\Delta i_{s_0} = -\Delta s\, \xi \text{ on } M_{s_0}, \tag{4.1.8}$$

and

$$-d\phi(\Delta i_{s_0}) = \alpha'(s)\Delta s\, \eta,$$

by Lemma 4.1.2.

Finally, let $(\bar{X}_a)_{1 \leq a \leq \dim M-1}$ denote the orthonormal basis on M_{s_0} such that $di_{s_0}(\bar{X}_a) = X_a$ for each a. For simplicity of notation write $<X,Y> = h(X,Y)$ for all $X,Y \in \mathscr{C}(TN)$. Then, for each a,

$$0 = <d(\phi \circ i_{s_0})(\bar{X}_a), \eta>;$$

therefore, taking the covariant derivative with respect to $d\phi_{s_0}(\bar{X}_a)$, where we write $\phi_{s_0} = \phi \circ i_{s_0}$,

$$0 = \sum_a <\nabla^N_{d\phi_{s_0}(\bar{X}_a)} d\phi_{s_0}(\bar{X}_a), \eta> + <d\phi_{s_0}(\bar{X}_a), \nabla^N_{d\phi_{s_0}(\bar{X}_a)}\eta>$$

$$= \sum_a <\nabla d\phi_{s_0}(\bar{X}_a, \bar{X}_a) + d\phi_{s_0}(\nabla_{\bar{X}_a}\bar{X}_a), \eta> + <d\phi_{s_0}(\bar{X}_a), \nabla^N_{d\phi_{s_0}}(\bar{X}_a)\eta>$$

$$= <\Delta\phi_{s_0}, \eta> + \sum_a <d\phi_{s_0}(\bar{X}_a), -\mu_{j_{k_a}} d\phi_{s_0}(\bar{X}_a)>,$$

summing over the a, where we suppose $X_a \in S_{k_a}$ for each a. Thus

$$<\Delta\phi_{s_0}, \eta> = \sum_k \mu_{j_k}\gamma_k, \tag{4.1.9}$$

and from equation (4.1.6)

$$<\Delta\phi, \eta> = \alpha''(s) + \alpha'(s)\Delta s + \sum_k \mu_{j_k}\gamma_k.$$

Therefore the map ϕ is harmonic if and only if equation (4.1.4) holds. □

<u>Second proof of Theorem 4.1.8</u> Write the metric g on M^* in the form $g = ds^2 + g_s$, where g_s is the induced metric on M_s. Similarly express h on N^* as $h = dt^2 + h_t$, where h_t is the induced metric on N_t. Then

$$S_\phi = e(\phi) g - \phi^* h$$

$$= \tfrac{1}{2}(\gamma(s,t) + |\alpha'(s)|^2)(ds^2 + g_s) - \phi^*(dt^2 + h_t) , \qquad (4.1.10)$$

where $t = \alpha(s)$. If $(X_a)_{1 \le a \le \dim M-1}$ is the orthonormal basis chosen as in the first proof, then for each b

$$\nabla^* S_\phi(X_b) = \nabla^*(e(\phi) g)(X_b) - \nabla^*(\phi^* h)(X_b)$$

$$= -\nabla^*(\phi^* h_t)(X_b)$$

$$= -X_a(\phi^* h_t(X_a, X_b)) + \phi^* h_t(\nabla_{X_a} X_a, X_b) + \phi^* h_t(X_a, \nabla_{X_a} X_b)$$

$$- (\phi^* h_t(\xi, X_b)) + \phi^* h_t(\nabla_\xi \xi, X_b) + \phi^* h_t(\xi, \nabla_\xi X_b)$$

$$= -X_a(\phi^* h_t(X_a, X_b)) + \phi^* h_t(\nabla_{X_a} X_a, X_b) + \phi^* h_t(X_a, \nabla_{X_a} X_b) , \qquad (4.1.11)$$

summing over repeated indices. On the other hand, ϕ_{s_0} is harmonic onto its image, so

$$\bar\nabla^* S_{\phi_{s_0}} = 0,$$

where $\bar\nabla$ is the connection on M_{s_0}. Now

$$S_{\phi_{s_0}} = \tfrac{1}{2}\gamma(s_0, t_0) g_{s_0} - \phi_{s_0}^* h, \text{ where } t_0 = \alpha(s_0) ,$$

$$= \tfrac{1}{2}\gamma(s_0, t_0) g_{s_0} - \phi_{s_0}^* h_{t_0}, \text{ since } \phi_{s_0}^*(dt^2) = 0.$$

Thus

$$0 = -\bar\nabla^*(\phi_{s_0}^* h_{t_0})(\bar X_b)$$

$$= -\bar X_a(\phi_{s_0}^* h_{t_0}(\bar X_a, \bar X_b)) + \phi_{s_0}^* h_{t_0}(\bar\nabla_{\bar X_a} \bar X_a, \bar X_b) + \phi_{s_0}^* h_{t_0}(\bar X_a, \bar\nabla_{\bar X_a} \bar X_b), \qquad (4.1.12)$$

where $(\bar{X}_a)_{1 \le a \le \dim M-1}$ is the frame field of the first proof. Also

$$\bar{\nabla}\, di_{s_0}(\bar{X}_c, \bar{X}_d) = - di_{s_0}(\bar{\nabla}_{\bar{X}_c} \bar{X}_d) + \nabla^M_{X_c} X_d, \text{ for all } c, d,$$

from Lemma 1.1.1. Since i_{s_0} is an isometric immersion; $\bar{\nabla}\, di_{s_0}(\bar{X}, \bar{Y})$ is perpendicular to $di_{s_0}(Z)$ for all $X, Y, Z \in \mathscr{C}(TM_{s_0})$, so that

$$< di_{s_0}(\bar{\nabla}_{\bar{X}_c} \bar{X}_d), di_{s_0}(\bar{X}_e) > = < \nabla_{X_c} X_d, X_e > \text{ for all } c, d, e .$$

Substituting this into equation (4.1.12) shows that the right hand side of equation (4.1.11) is zero, and $\nabla^* S_\emptyset$ is zero on vectors tangent to the level hypersurfaces of s. It remains to evaluate $\nabla^* S_\emptyset$ on ξ.

$$\nabla^* S_\emptyset(\xi) = \tfrac{1}{2} d\gamma(\xi) + \alpha''(s)\alpha'(s) - \nabla^*(\emptyset^* dt^2 + \emptyset^* h_t)(\xi)$$

$$= \tfrac{1}{2} d_s\gamma + \tfrac{1}{2} d_t\gamma . (dt/ds) + \alpha''(s)\alpha'(s)$$

$$- \nabla^*(\emptyset^* dt^2)(\xi) - \nabla^*(\emptyset^* h_t)(\xi) . \tag{4.1.13}$$

We work out $\nabla^*(\emptyset^* dt^2)(\xi)$. Consider $\nabla^*(ds^2)$; for any $X \in \mathscr{C}(TM)$;

$$\nabla^*(ds^2)(X) = (\nabla_{X_a} ds^2)(X_a, X) + (\nabla_\xi ds^2)(\xi, X)$$

$$= X_a(ds^2(X_a, X)) - ds^2(\nabla_{X_a} X_a, X) - ds^2(X_a, \nabla_{X_a} X)$$

$$+ (\nabla_\xi ds^2)(\xi, X)$$

$$= - ds(\nabla_{X_a} X_a). ds(X) + \xi(ds(\xi). ds(X)) - ds(\xi). ds(\nabla_\xi X) .$$

On the other hand

$$\Delta s = \nabla^*(ds)$$

$$= (\nabla_{X_a} ds)(X_a) + (\nabla_\xi ds)(\xi)$$

$$= \xi(ds(\xi)) - ds(\nabla_{X_a} X_a)$$

$$= - ds(\nabla_{X_a} X_a) .$$

Therefore $\nabla^*(ds^2)(\xi) = \Delta s$ and $\nabla^*(ds^2)(X_a) = 0$ for each a, so that

$$\nabla^*(ds^2) = \Delta s\, ds.$$

Now $\emptyset^* dt^2 = \alpha'(s)^2 ds^2$; therefore

$$\nabla^* |d\alpha|^2 ds^2) = \nabla_{X_a}(\alpha'(s)^2 ds^2)(X_a) + \nabla_\xi(\alpha'(s)^2 ds^2)(\xi)$$

$$= \nabla_{X_a}(\alpha'(s)^2).ds^2(X_a) + \nabla_\xi(\alpha'(s)^2).ds^2(\xi)$$

$$+ \alpha'(s)^2 \Delta s\, ds$$

$$= 2\alpha''(s)\alpha'(s)ds + \alpha'(s)^2 \Delta s\, ds.$$

We now evaluate $-\nabla^*(\emptyset^* h_t)(\xi)$:

$$-\nabla^*(\emptyset^* h_t)(\xi) = \emptyset^* h_t(X_a, \nabla_{X_a}\xi)$$

$$= -\sum_a \lambda_{k_a} \emptyset^* h_t(X_a, X_a)$$

$$= -\sum_k \lambda_k \gamma_k .$$

Thus equation (4.1.13) becomes

$$\nabla^* S_\emptyset(\xi) = \tfrac{1}{2} d_s\gamma + \tfrac{1}{2} d_s\gamma.(dt/ds) - \alpha''(s)\alpha'(s)$$
$$- \Delta s(\alpha'(s))^2 - \sum_k \lambda_k \gamma_k. \qquad (4.1.14)$$

We therefore obtain the equation

$$\alpha''(s) + \Delta s \alpha'(s) - \tfrac{1}{2} d_t\gamma - \frac{1}{2\alpha'(s)}(d_s\gamma - 2\sum_k \lambda_k\gamma_k) = 0, \qquad (4.1.15)$$

as a condition of harmonicity. Proposition 4.1.12 below will show that the last term in equation 4.1.15 is zero, and that we have indeed established that $\emptyset$ is harmonic if and only if equation (4.1.5) holds. □

Proposition 4.1.12 If $\emptyset : M \to N$ is S-equivariant with respect to isoparametric functions s and t, then

(i) $d_s \gamma_k - 2\lambda_k \gamma_k = 0$

(ii) $d_t \gamma_k + 2\mu_{j_k} \gamma_k = 0$,

for each $k = 1, \ldots, p$.

Proof From Corollary 4.1.6, we see that for all $x \in M^*$, $\emptyset|_{\mathscr{S}_k(x)} : \mathscr{S}_k(x) \to \mathscr{T}_{j_k}(\emptyset(x))$

has constant energy density. Without loss of generality assume that M is S^{m-1} and N is S^{n-1}, so that $\mathscr{S}_k$ and $\mathscr{T}_j$ are small spheres (the other cases are similar).

Fix a level hypersurface M_{s_0}, and fix $x_0 \in M_{s_0}$, and consider $\emptyset|_{\mathscr{S}_k(x_0)}$: $\mathscr{S}_k(x_0) \to \mathscr{T}_{j_k}(\emptyset(x_0))$. Rescale the map to obtain a map $\psi : S^k \to S^{n_{j_k}}$, where m_k is the multiplicity of λ_k, and n_j the multiplicity of μ_j. Then $|d\psi|^2 = a_k$ = constant.

From Remark 2.2.10, $\mathscr{S}_k(x)$ is a small sphere of radius $\sin(s-(k-1)\Pi/p)$ for $x \in M_s$, and $\mathscr{T}_{j_k}(y)$ is a small sphere of radius $\sin(t-(j_k-1)\Pi/q)$ for $y \in N_t$. We therefore find that

$$\gamma_k(s,t) = a_k \cdot \sin^2(t-(j_k-1)\Pi/q) / \sin^2(s-(k-1)\Pi/p).$$

Therefore

$$d_s \gamma_k(s,t) = -\frac{2\sin(s-(k-1)\Pi/p)\cos(s-(k-1)\Pi/p)}{\sin^4(s-(k-1)\Pi/p)} \cdot \sin^2(t-(j_k-1)\Pi/q)\, a_k$$

$$= -2\cot(s-(k-1)\Pi/p)\gamma_k(s,t)$$

$$= 2\lambda_k \gamma_k ,$$

and

$$d_t \gamma_k(s,t) = \frac{2\sin(t-(j_k-1)\Pi/q)\cos(t-(j_k-1)\Pi/q\,.\,\gamma_k}{\sin^2(t-(j_k-1)\Pi/q}$$

$$= 2\cot(t-(j_k-1)\Pi/q)\,\gamma_k$$

$$= -2\mu_{j_k}\gamma_k\,. \qquad \square$$

Remark 4.1.13 In fact condition (iii) can be removed in the definition of S-equivariance on account of Proposition 4.1.14 below. However, we retain it in the statement of Theorem 4.1.8 in order to simplify the exposition.

Proposition 4.1.14 If $\emptyset : M \to N$ is S-WFP, harmonic and satisfies conditions (i) and (ii), then $\sum_k \mu_{j_k}\gamma_k (= d_t\gamma|_{t=\alpha(s)})$ depends only on s.

Proof By the conditions on $\emptyset$,

$$<d(j_t o\, \emptyset_{s,t}), \eta> = 0\,.$$

Let (X_a) be a local orthonormal basis on M_s, adapted to the principal curvature spaspaces. Then, for each a,

$$0 = \nabla_{X_a} <d(j_t o\, \emptyset_{s,t})(X_a), \eta>$$

$$= <\nabla^N_{d(j_t o\, \emptyset_{s,t})(X_a)} d(j_t o \emptyset_{s,t})(X_a), \eta>$$
$$+ <d(j_t o \emptyset_{s,t})(X_a), \nabla^N_{d(j_t o\, \emptyset_{s,t})(X_a)}\eta>$$

$$= <\nabla d j_t(d\emptyset_{s,t}(X_a), d\emptyset_{s,t}(X_a)), \eta>$$
$$+ <d(j_t o \emptyset_{s,t})(X_a), -\mu_{j_{k(a)}} d(j_t o \emptyset_{s,t})(X_a)> \qquad (4.1.16)$$

where $X_a \in S_{k(a)}$ say.

On the other hand $\emptyset$ is harmonic, so $\Delta\emptyset = 0$. But

$$0 = \Delta\emptyset$$
$$= \text{trace}\,\nabla d\emptyset(di_s, di_s) + \nabla d\emptyset(\xi,\xi)\,,$$

and since $\nabla d\emptyset(\xi,\xi) = A(s,t)\eta$, where A depends only on s and t $(t = \alpha(s))$ (c.f. Lemma 4.1.2), then

$$\text{trace } \nabla d\emptyset(di_s, di_s) = B(s,t)\,\eta, \ (t = \alpha(s))$$

with B depending only on s and t. Now

$$\Delta(\emptyset \circ i_s) = d\emptyset(\Delta i_s) + \text{trace } \nabla d\emptyset(di_s, di_s)$$

$$= C(s,t)\eta\ ,\ (t = \alpha(s))$$

where C depends only on s and t (since $d\emptyset(\Delta i_s) = D(s,t)\,\eta$ $(t = \alpha(s))$, with D depending only on s and t). Since $\emptyset \circ i_s = j_t \circ \emptyset_{s,t}$ $(t = \alpha(s))$;

$$\Delta(j_t \circ \emptyset_{s,t}) = E(s,t)\eta\ ,\ (t = \alpha(s)),$$

with E depending only on s and t , i.e.

$$d\,j_t(\Delta\emptyset_{s,t}) + \text{trace } \nabla d\, j_t(d\,\emptyset_{s,t}, d\,\emptyset_{s,t}) = E(s,t)\,\eta\,.$$

But $\emptyset_{s,t}$ is harmonic, so that

$$\text{trace } dj_t(d\,\emptyset_{s,t}, d\,\emptyset_{s,t}) = E(s,t)\,\eta\ .$$

Whence, from equation (4.1.16), we see that $\sum_k \mu_{j_k}\gamma_k$ depends only on s and t, with $t = \alpha(s)$. □

We now wish to remove the "S" condition on $\emptyset$; i.e. we want to allow α with $\alpha(s) \in \partial I_t$ for $s \in \text{int } I_s$. Furthermore, we want to allow the possibility that $\emptyset$ covers N several times.

Suppose N is S^{n-1}, and $t: S^{n-1} \to \mathbb{R}$ is an isoparametric function of degree q. Let $f: S^{n-1} \to [-1,1]$ be the restriction of the standard homogeneous polynomial F of Münzner's theorem (Theorem 2.2.5) giving s. Call $\emptyset : M \to S^{n-1}$ <u>wavefront preserving (WFP)</u> if

(i)
$$\begin{array}{ccccc} M & & \xrightarrow{\quad\emptyset\quad} & & S^{n-1} \\ {\scriptstyle s}\downarrow & & & & \downarrow{\scriptstyle f} \\ I_s & \xrightarrow{\ \alpha\ } & \mathbb{R} & \xrightarrow[\cos q]{} & [-1,1] \end{array}$$
, commutes,

and (ii) $d\emptyset(\nabla s)_x = u(x)\nabla t_{\emptyset(x)}$, for some $u: M \to \mathbb{R}$, for all $x \in M$, where $\cos q: \mathbb{R} \to [-1,1]$ is given by $\cos q\,(t) = \cos qt$, for all $t \in \mathbb{R}$. Then Lemma 4.1.2 is still valid. If N is $\mathbb{R}^n$ or H^{n-1} we remove the function cosq, and Diagram (4.1.1) applies. We may also wish to restrict $\alpha: I_s \to \mathbb{R}$ to have fixed boundary conditions, and to have values in a certain domain $D \subset \mathbb{R}$ (for example, we may wish to find harmonic maps covering the sphere twice using isoparametric functions of degree 2, in which case we would suppose $\alpha: [0,\Pi/2] \to [0,\Pi]$).

Suppose in addition, that we can define a map $\emptyset_{s,t}: M_s \to N_f^{-1}(\cos q\,t)$, for all $s \in \text{int}\ I_s$, and for all $t \in D\backslash\alpha(\partial I_s)$, such that $\emptyset_{s,\alpha(s)} = \emptyset|_{M_s}$, and the curves $\delta_s(t)(x) = \emptyset_{s,t}(x)$, for each $s \in \text{int}\ I_s$ and for each $x \in M_s$, satisfy

$$(d/dt)\ \delta_s(t)(x) = v_s(t)\eta_{\emptyset(x)}\ ,$$

for some function $v_s: M^* \to \mathbb{R}$ depending only on s.

Suppose for all $x \in M^*$, (i) $d\emptyset(S_k(x)) \subset T_{j_k}(\emptyset(x))$, for all k, and for some j_k.

<u>Lemma 4.1.14</u> <u>If</u> $\emptyset: M \to S^{n-1}$ <u>is WFP with respect to</u> s <u>and</u> t, <u>and satisfies condition (i) above, then</u> $d\emptyset_{s,t}(S_k(x)) \subset T_{j_k}(\emptyset_{s,t}(x))$ <u>for all</u> $x \in M_s$ <u>and for all</u> $k = 1, \ldots, p$.

<u>Proof</u> From the proof of Lemma 4.1.5; it is clear that projection along the integral curves of ξ,η preserves the eigenspaces S_k, T_j respectively, provided that we do not project through a focal variety. However, for all $s \in \text{int}\ I_s$, and for all $t \in D\backslash\alpha(\partial I_s)$, there exists a $\tilde{s} \in \text{int}\ I_s$ with a well-defined projection $\rho_s: M_s \to M_{\tilde{s}}$ along integral curves of ξ, and a well-defined projection $\sigma_t: N_t \to N_{\alpha(\tilde{s})}$ along integral curves of η. This is because all $\tilde{s} \in \text{int}\ I_s$ are allowed, and there exists at least one such $\tilde{s}$ with $N_{\alpha(\tilde{s})}$ close enough to N_t. Hence the map $\emptyset_{s,t}$ preserves the principal curvature eigenspaces. □

If furthermore (ii) $\emptyset_{s,t}$ is harmonic for all $s \in I_s$ and for all $t \in D\backslash(\alpha(\partial I_s))$, and (iii) $\gamma_k(s,t)(x) = \text{trace}_{S_k(x)}\, h(d\emptyset_{s,t}, d\emptyset_{s,t})$ depends only on s and t, for each $k = 1, \ldots, p$; we call $\emptyset$ <u>equivariant with respect to</u> s <u>and</u> t.

Now locally, we can consider an equivariant map $\emptyset$ as being S-equivariant, by considering t as varying in an interval of the form $[l\,\Pi/q, (l+1)\,\Pi/q]$ (in the case N is a sphere), for some $1 = 0, \ldots, q-1$. Since the proofs of Theorem 4.1.8 are local in nature (one works in a sufficiently small domain about each point); it is clear that Theorem 4.1.8 is also true for equivariant maps. Thus

Theorem 4.1.15 If $\phi : M \to N$ is equivariant with respect to isoparametric functions s and t, then ϕ is harmonic if and only if

$$\alpha''(s) + \Delta s\, \alpha'(s) + \sum_{k=1}^{p} \mu_{j_k} \gamma_k = 0\,, \tag{4.1.4}$$

for all $s \in \operatorname{int} I_s$, if and only if

$$\alpha''(s) + \Delta s\, \alpha'(s) - \tfrac{1}{2} d_t\, \gamma(s, \alpha(s)) = 0\,, \tag{4.1.5}$$

for all $s \in \operatorname{int} I_s$.

Remark 4.1.16 Equations (4.1.4) and (4.1.5) are valid for $s \in \operatorname{int} I_s$; it is conceivable (and often the case) that the equations become singular for $s \in \partial I_s$, that is $\lim_{s \to \partial I_s} \Delta s$ and $\lim_{s \to \partial I_s} (\Sigma\, \mu_{j_k} \gamma_k)$ are infinite. In this case, one looks for a re-parametrization of the equations so as to remove the singularities.

Remark 4.1.17 Given $\phi : M^* \to N^*$ which is S-equivariant with α satisfying equation (4.1.4); it is sometimes possible to extend ϕ to a smooth map $\phi : M \to N$. For example, this is the case for the Smith maps of Section 1.3.

4.2 Generalized equivariant maps between Riemannian manifolds.

Let M,N be connected Riemannian manifolds each admitting a generalized family of isoparametric hypersurfaces $(M_c)_{c \in I}$, $(N_d)_{d \in J}$ respectively. Let M^* be the union of the hypersurfaces of M; $M^* = \bigcup_{c \in I} M_c$, then M^* has a topology induced from that of M. Let $\pi : M^* \to I$ be the projection, then I inherits a topology from M^* with respect to which it becomes a set of open intervals. As before we can define a metric on I with respect to which π becomes a Riemannian submersion (M^* has the metric induced from M). We can add in appropriate end points to the intervals I corresponding to focal varieties, to obtain a closed, half-closed or open interval (or possibly a circle in the case that M is a torus or Klein bottle c.f. Section 5.1 (iii)). Similarly we can define N^* and the projection map $\sigma : N^* \to J$.

Suppose that the generalized family of isoparametric hypersurfaces $(M_c)_{c \in I}$ is defined by a function f on M, and furthermore, that f admits a reparametrization $s = s(f)$ such that $|\nabla s|^2 = 1$, then we say that the generalized family of isoparametric hypersurfaces admits a unit speed reparametrization.

Let $\emptyset : M \to N$ be a map between Riemannian manifolds which admit generalized families of isoparametric hypersurfaces $(M_c)_{c \in I}$, $(N_d)_{d \in J}$ respectively, both of which admit unit speed reparametrizations. The notions of $\emptyset$ being wavefront preserving and equivariant can be defined as in Section 4.1 (by making the definitions locally). Lemmas 4.1.2, 4.1.3, 4.1.4 and 4.1.5 all remain valid. Similarly the proofs of Theorem 4.1.8 (replacing equation (4.1.5) with equation (4.1.15)) remain valid, and we obtain

Theorem 4.2.1 If $\emptyset : M \to N$ is equivariant with respect to the generalized isoparametric functions $f : M \to R$ and $g : N \to \mathbb{R}$, where f and g admit unit speed reparametrizations s and t respectively, then $\emptyset$ is harmonic if and only if

$$\alpha''(s) + \Delta s\, \alpha'(s) + \sum_{k=1}^{p} \mu_{j_k} \gamma_k = 0 , \tag{4.2.1}$$

for all $s \in \operatorname{int} I_s$, if and only if

$$\alpha''(s) + \Delta s\, \alpha'(s) - \tfrac{1}{2} d_t \gamma(s, \alpha(s))$$

$$- \frac{1}{2\alpha'(s)} \left(d_s \gamma - 2 \sum_{k=1}^{p} \lambda_k \gamma_k\right) = 0 , \tag{4.2.2}$$

for all $s \in \operatorname{int} I_s$.

Remark 4.2.2 Proposition 4.1.12 is no longer necessarily valid; however it can be proved directly that equations (4.2.1) and (4.2.2) are equivalent.

Proposition 4.2.3 With the notations defined above and in Section 4.1; if $\emptyset$ is equivariant;

$$\alpha'(s) \mu_{j_k} \gamma_k = \lambda_k \gamma_k - \tfrac{1}{2} d \gamma_k(s,t)(\xi) , \tag{4.2.3}$$

for each $k = 1, \ldots, p$.

Proof Let $(X_a)_{1 \le a \le \dim M - 1}$ be a local orthonormal frame field on M, and suppose that $(X_{k_i})_{1 \leq i \leq m_k}$ forms an orthonormal basis for S_k at each point (m_k = dim S_k).

Then, for each k,

$$\lambda_k \gamma_k = - \sum_i \phi^* h(X_{k_i}, \nabla^M_{X_{k_i}} \xi)$$

$$= - \sum_i < d\phi(X_{k_i}), d\phi(\nabla^M_{X_{k_i}} \xi)>$$

$$= \sum_i < d\phi(X_{k_i}), \nabla d\phi(X_{k_i}, \xi) - \nabla^N_{d\phi(X_{k_i})} d\phi(\xi)>$$

$$= \alpha'(s)\, \mu_{j_k} \gamma_k + \sum_i <d\phi(X_{k_i}), \nabla d\phi(X_{k_i}, \xi)> .$$

But

$$\sum_i <d\phi(X_{k_i}), \nabla d\phi(X_{k_i}, \xi)> = \sum_i <d\phi(X_{k_i}), d\phi(\nabla_\xi X_{k_i}) + \nabla_\xi d\phi(X_{k_i})> .$$

Since the functions are isoparametric, projection down the integral curves of ξ preserves the S_k spaces. Thus, if $\delta(u)$ is an integral curve of ξ, then we can choose our frame field $(X_{k_i})_{i = 1, \ldots, m_k}$ such that $\delta(u)_* X_{k_i}$ is proportional to X_{k_i}. Thus $\mathscr{L}_\xi X_{k_i}$ is proportional to X_{k_i} for each i; but $\nabla_{X_{k_i}} \xi = \lambda_k X_{k_i}$, therefore $\nabla_\xi X_{k_i}$ is proportional to X_{k_i}.

But $1 = g(X_{k_i}, X_{k_i})$, thus $\nabla_\xi X_{k_i} = 0$. Thus we obtain

$$\sum_i <d\phi(X_{k_i}), \nabla d\phi(X_{k_i}, \xi)> = \sum_i <d\phi(X_{k_i}), \nabla_\xi d\phi(X_{k_i})>$$

$$= \tfrac{1}{2} d\gamma_k(\xi). \qquad \square$$

Since $d\gamma_k(\xi) = d_s \gamma_k + d_t \gamma_k (dt/ds)$; we see that equations (4.2.1) and (4.2.2) are indeed equivalent.

Remark 4.2.4 Remark 4.1.17 applies equally for maps ϕ satisfying equation (4.2.1).

The theory of this chapter is best illustrated when applied to specific examples. With each example, the qualitative features of the above theory can be very different. We shall see how to adapt Theorem 4.1.8 in the next chapter. We shall leave the application of Theorem 4.2.1 until Chapter 9, where we consider deformations of metrics.

5 Examples of equivariant maps

5.1 Maps from Euclidean space to the sphere

First of all we prove a lemma which will be useful throughout this chapter.

Lemma 5.1.1 Suppose $f : M \to \mathbb{R}$ is a smooth function on a Riemannian manifold M with $|df|^2 = \psi_1(f)$, $\Delta f = \psi_2(f)$ for some smooth functions ψ_1 and ψ_2. Then, given an equation of the form

$$\alpha''(f)\,|df|^2 + \alpha'(f)\Delta f = \psi(f)\ , \tag{5.1.1}$$

for some smooth function ψ; under reparametrization $u = u(f)$, $u'(f) \neq 0$, equation (5.1.1) remains invariant - that is, writing $\beta(u) = \alpha(f(u))$; equation (5.1.1) becomes

$$\beta''(u)\,|du|^2 + \beta'(u)\,\Delta u = \psi(u)\ ,$$

where $\psi(u) = \psi(f(u))$.

Proof We can write f as a function of $u, f = f(u)$:

$$\beta'(u) = \alpha'(f)\,f'(u)$$

$$\beta''(u) = \alpha''(f)\,f'(u)^2 + \alpha'(f)f''(u)\ .$$

Also

$$|df|^2 = f'(u)^2\,|du|^2$$

$$\Delta f = f''(u)\,|du|^2 + f'(u)\Delta u\ .$$

Therefore

$$\alpha''(f)|df|^2 + \alpha'(f)\Delta f = \frac{1}{f'(u)^2}\left(\beta''(u) - \beta'(u)\frac{f''(u)}{f'(u)}\right) f'(u)^2\,|du|^2$$

$$+ \frac{\beta'(u)}{f'(u)}\left(f''(u)\,|du|^2 + f'(u)\,\Delta u\right).$$

$$= \beta''(u)\,|du|^2 + \beta'(u)\,\Delta u\ .$$ □

Example 5.1.2 Let $\emptyset : \mathbb{R}^m \to S^m$ be defined by

$$\emptyset(s.x) = (\cos \alpha(s), \sin \alpha(s).x) ,$$

where $x \in S^{m-1}$, $s \in [0,\infty)$ and $\alpha(0) = 0$. The map $\emptyset$ is equivariant with respect to the isoparametric functions s and t, where $s(s_0.x) = s_0$, for all $s_0 \in [0,\infty)$ and $x \in S^{m-1}$, and $t((\cos t_0, \sin t_0.x) = t_0$, for all $t_0 \in [0,\pi]$ and $x \in S^{m-1}$. The square of the norm of the derivative of s and the Laplacian of s are given by

$$|ds|^2 = 1, \quad \Delta s = (m-1)/s , \tag{5.1.2}$$

and $\gamma(s,t) = |d\emptyset_{s,t}|^2$ is given by

$$\gamma(s,t) = (m-1). \sin^2 t/s^2 . \tag{5.1.3}$$

From Theorem 4.1.8; $\emptyset$ is harmonic if and only if

$$\alpha''(s) + \frac{(m-1)}{s}. \alpha'(s) - \frac{(m-1)}{s^2}. \sin \alpha(s) \cos \alpha(s) = 0. \tag{5.1.4}$$

This equation is singular at $s = 0$, and we remove the singularity with a suitable substitution. Using Lemma 5.1.1 we make the substitution $u = u(s)$, where u is given by $e^u = s$. Then

$$|du|^2 = 1/e^{2u} ,$$

and

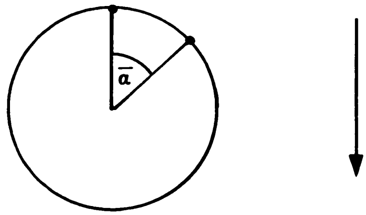
$$\Delta u = (m-2)/e^{2u} ,$$

whence equation (5.1.4) becomes

$$\alpha''(u) + (m-2)\alpha'(u) - \tfrac{1}{2}(m-1). \sin 2\alpha(u) = 0 , \tag{5.1.5}$$

where $u \in (-\infty,\infty)$, and as $u \to -\infty$; $\alpha(u) \to 0$.

Equation 5.1.5 is a well-known equation - namely the equation of a pendulum with constant gravity and constant damping. The variable u measures time, and $\bar{\alpha} = 2\alpha$ is the angle the pendulum makes with the upward vertical.



We require an exceptional trajectory satisfying equation (5.1.5), such that the pendulum is just standing vertically upwards at time $u = -\infty$.

The solutions of the pendulum equation are well-known, see for example [23, p.183 & p.196]. Consider first the exceptional case; $m = 2$. Then the damping is zero - a good treatment of this case is to be found in [4, Section 6.3]. For each $u_0 \in (-\infty, \infty)$, there is a unique solution to equation (5.1.5) with initial conditions $\alpha(u_0) = \alpha_0$ and $\alpha'(u_0) = \alpha'_0$, for all $\alpha_0 \in [0, 2\Pi]$ and for all $\alpha'_0 \in \mathbf{R}$. Choose $\alpha_0 = \Pi/2$, then if

(i) $\alpha'_0 = 1$; the solution $\alpha : \mathbf{R} \to \mathbf{R}$ is strictly monotonic and has image $(0, \Pi)$ with $\lim_{u \to \infty} \alpha(u) = \Pi$ and $\lim_{u \to -\infty} \alpha(u) = 0$. In fact $\alpha(u) = \sin^{-1} \tanh(u - u_0)$ - this solution is called critical.

(ii) $\alpha'_0 < 1$, then $\alpha : \mathbf{R} \to \mathbf{R}$ is oscillatory with image contained in a closed subinterval of $(0, \Pi)$.

(iii) $\alpha'_0 > 1$, then the solution α is strictly monotonic increasing and surjective onto $\mathbf{R}$.

In order to obtain a smooth harmonic map we choose the critical solution (i), giving a map $\phi : \mathbf{R}^2 \to S^2$, covering S^2 except for the one point $(-1, 0)$. Note in fact that we have a 1-parameter family of maps corresponding to the choice of α_0. That the map ϕ so constructed is smooth at $0 \in \mathbf{R}^2$ is demonstrated in Chapter 6.

In the case when $m > 2$; the damping is non-zero. For each $u_0 \in (-\infty, \infty)$, and for each $\alpha(u_0) = \alpha_0$; there is again a critical solution with $\lim_{u \to -\infty} \alpha(u) = 0$. However, in this case $\lim_{u \to \infty} \alpha(u) = \Pi/2$, corresponding to the pendulum hanging straight down. In fact the pendulum does not reach the upward vertical on its first upward swing, and performs decreasing oscillations about the downward vertical position. As in the case when $m = 2$, the map ϕ is smooth across the point $0 \in \mathbf{R}^m$.

Such solutions have been considered by J.C. Wood [42], in connection with the Dirichlet problem for the Euclidean disc, where he shows that there is no map from the Euclidean disc to the sphere of the above form with $\alpha_0 \geq \pi/2 + \sin^{-1}\tanh(K_m)$, where $K_m = (m-1)^{\frac{1}{2}}/(m-2)$.

Similar considerations apply to the map $\phi : \mathbb{R}^m \to S^n$ given by

$$\phi(s.x) = (\cos\alpha(s), \sin\alpha(s).g(x)), \ \alpha(0) = 0$$

for all $x \in S^{m-1}$, $s \in [0,\infty)$, and where $g: S^{m-1} \to S^{n-1}$ is harmonic of constant energy density. The corresponding reduction equation has the same form as equation (5.1.5) but with a different value for the gravity. The qualitative behaviour of the solutions is the same as for the case when g is the identity map.

Example 5.1.3 Let $\phi : \mathbb{R}^m \to S^m$ be defined, for each integer k, by

$$\phi((x, sy)) = (\cos\alpha(s).e^{ikx}, \sin\alpha(s).y), \qquad (5.1.6)$$

with $\alpha(0) = 0$, where $x \in \mathbb{R}$, $y \in S^{m-2}$, $m \geq 3$ and $s \in [0,\infty)$. Then ϕ is equivariant with respect to the isoparametric functions s and t, where $s((x, s_0 y)) = s_0$, for all $s_0 \in [0,\infty)$, $x \in \mathbb{R}$ and $y \in S^{m-2}$, and $t((\cos t_0.u, \sin t_0.v)) = t_0$, for all $t_0 \in [0,\pi/2]$, $u \in S^1$ and $v \in S^{m-2}$. The square of the norm of the derivative of s and the Laplacian of s are given by

$$|ds|^2 = 1, \qquad \Delta s = (m-2)/s, \qquad (5.1.7)$$

and $\gamma(s,t)$ is given by

$$\gamma(s,t) = k^2\cos^2 t + (m-2)\sin^2 t/s^2. \qquad (5.1.8)$$

From Theorem 4.1.8, ϕ is harmonic if and only if

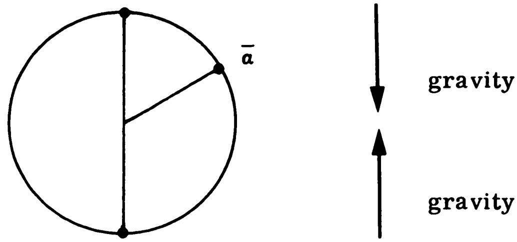
$$\alpha''(s) + \frac{(m-2)}{s}\alpha'(s) - \frac{((m-2)-k^2).}{s^2}\sin\alpha(s)\cos\alpha(s) = 0. \qquad (5.1.9)$$

This equation is singular when $s = 0$, and so, using Lemma 5.1.1, we make the substitution $u = u(s)$ where $e^u = s$, to give the equation

$$\alpha''(u) + (m-3)\alpha'(u) - \tfrac{1}{2}((m-2) - k^2.e^{2u}).\sin 2\alpha(u) = 0, \qquad (5.1.10)$$

where $u \in (-\infty,\infty)$ and $\lim_{u\to-\infty}\alpha(u) = 0$.

Equation (5.1.10) is that of a pendulum with constant damping and variable gravity. As in Example 5.1.2, u measures time and $\bar{\alpha} = 2\alpha$ is the angle the pendulum makes with the upward vertical.



For each $u_0 \in (-\infty, \infty)$, there exists a unique solution $\alpha(\alpha_0, \alpha_0')$ to equation (5.1.10) with prescribed initial conditions $\alpha(u_0) = \alpha_0$ and $\alpha'(u_0) = \alpha_0'$, for all $\alpha_0 \in (0, \Pi/2)$ and for all $\alpha_0' \in \mathbb{R}$. We look for exceptional solutions such that the pendulum is just standing vertically upwards at time $u = -\infty$. We demonstrate the existence of such non-trivial solutions in Chapter 6.

Note that we have a 1-parameter family of solutions depending on our choice of α_0. Different choices of α_0 lead to qualitatively different solutions. To see this we first of all state a comparison theorem for second order equations.

<u>Theorem 5.1.4</u> [9] <u>Let</u> p_i' <u>and</u> k_i <u>be continuous on</u> $[a,b]$, $i = 1,2$, <u>and let</u>

$$0 < p_2(u) \le p_1(u)$$

$$k_2(u) \ge k_1(u), \qquad u \in [a,b] .$$

<u>Let</u> L_i <u>be the operator</u> $L_i v = (p_i v')' + k_i v$. <u>If</u> f_i <u>is a solution of</u> $L_i f_i = 0$, <u>let</u> $w_i = \tan^{-1}(f_i/p_i f_i')$. <u>If</u> $w_2(a) \ge w_1(a)$, <u>then</u> $w_2(u) \ge w_1(u)$ <u>for all</u> $u \in [a,b]$. <u>Moreover, if</u> $k_2(u) > k_1(u)$ <u>on</u> (a,b), <u>then</u> $w_2(u) > w_1(u)$ <u>for</u> $a < u \le b$.

To apply this, write equation (1.3.7) in the form

$$\alpha''(u) + h(u)\,\alpha'(u) + g(u)\sin 2\,\alpha(u) = 0 .$$

We can write this in divergence form:

$$L\,\alpha = (p\,\alpha')' + k_\alpha\,\alpha = 0 ,$$

where

$$p(u) = \exp\left(\int_{u_0}^{u} (h(s)\,ds\right)$$

$$k_\alpha(u) = g(u) \sin 2\,\alpha(u)\, p(u)/\alpha(u) .$$

<u>Proposition 5.1.5</u> <u>If</u> α_0 <u>is chosen close enough to</u> 0 ; <u>the solution</u> α <u>never covers the sphere</u>.

<u>Proof</u> Let u_1 be the time such that

$$(m-3)^2 = 4(k^2.e^{2u_1} - (m-2)) - A^2 ,$$

for some fixed $A > 0$ (such a u_1 always exists), and let $\alpha_1 = \alpha(u_1)$. Then α_1 depends continuously on α_0 and α_0' (c.f. [9]). As $\alpha_0 \to 0$; $\alpha_0' \to 0$ and $\alpha_1 \to 0$, since in the limit $\alpha_0 = 0$, $\alpha_0' = 0$ the solution is trivial. Thus there exists α_0 such that $\alpha_1 < \Pi/4$. Consider the equation

$$\delta''(u) + b\,\delta'(u) + c\,\delta(u) = 0 , \qquad (5.1.11)$$

with $b = (m-3)$ and $c = k^2.e^{2u_1} - (m-2)$. Given the initial conditions $\delta(u_1) = \alpha(u_1)$ and $\delta'(u_1) = \alpha'(u_1)$, this has a periodic solution with period $\Gamma = 2\Pi/(4c - b^2) = 2\Pi/A$.

Choose α_0 such that there exists u_2 with $u_2 > u_1$, $\Gamma < u_2 - u_1$ and $\alpha_2 = \alpha(u_2) < \Pi/2$. Then over the interval $[u_1, u_2]$, we can use the comparison theorem (Theorem 5.1.4). Since the gravity of the equation (5.1.10) dominates that of equation (5.1.11) over the interval $[u_1, u_2]$; Theorem 5.1.4 shows that $w_1(u) \geq w_2(u)$ for $u \in [u_1, u_2]$, where $w_1 = \tan^{-1}(\alpha/p_1\alpha')$ and $w_2 = \tan^{-1}(\delta/p_2\delta')$, where p_1 and p_2 are defined as in Theorem 5.1.4. In particular w_1 passes through $\Pi/2$ if w_2 does ; i.e. α' has become zero and the solution turns. Since the gravity is now increasing, the solution α will make decreasing oscillations, and $\lim_{u \to \infty} \alpha(u) = 0$. □

In contrast to Example 5.1.2 we can show that

<u>Proposition 5.1.6</u> <u>If</u> $(m-3)^2 < 4(m-2)$, i.e. $m = 3, \ldots, 7$; <u>then there exist solutions covering the whole sphere</u>.

<u>Proof</u> We take it as given that for each $\alpha_0 \in (0, \Pi/2)$, there is an α_0' with $\alpha(\alpha_0, \alpha_0')$ asymptotic to 0 as $u \to -\infty$. This will be shown in Lemma 6.1.4 and Lemma 6.1.5. Furthermore, the solution α is strictly decreasing as u decreases from u_0. We

show that α_0' is bounded away from 0 as $\alpha_0 \to \pi/2$. For in this case, the solution $\alpha(\alpha_0, \alpha_0')$ will continue past $\pi/2$ as u increases from u_0 if α_0 is chosen close enough to $\pi/2$.

Suppose there is a sequence $\alpha_0^n \to \pi/2$ such that $\alpha_0'^n \to 0$. Since in the limit $\alpha_0 = \pi/2$, $\alpha_0' = 0$ the solution is trivial; we can assume u_0 to be arbitrarily small, i.e. for any u_1, $\alpha(\alpha_0^n, \alpha_0'^n)(u_1) \to \pi/2$ as $\alpha_0^n \to \pi/2$ (the solution depends continuously on the initial conditions). Let $\alpha = \alpha(\alpha_0^n, \alpha_0'^n)$ be a solution of equation (5.1.10), where n is chosen sufficiently large. Make two substitutions into equation (5.1.10): firstly let $\delta = \pi/2 - \alpha$, and secondly let $v = -u$. Equation (5.1.10) then becomes

$$\delta''(v) = (m-3)\delta'(v) - \tfrac{1}{2}((m-2) - k^2 e^{-2v}).\sin 2\,\delta(v)$$

Consider the equation of the harmonic oscillator:

$$\beta''(v) = (m-3)\beta'(v) - M\,\beta(v) ,$$

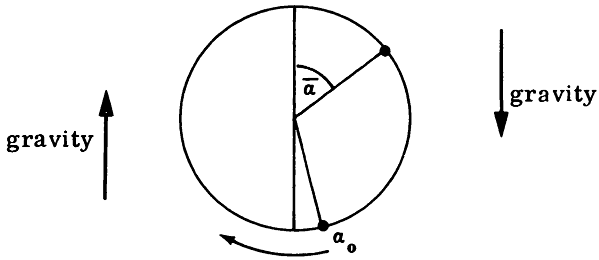
where M is chosen such that $((m-2) - k^2 e^{2u}) \sin 2\,\delta / 2\delta - M > 0$ on a suitable interval $[v_0, v_1]$ with the period of $\beta < v_1 - v_0 (v_0 = u_0)$. Let β have the same initial data as δ at v_0, and let $w = \delta'\beta - \delta\beta'$. Then

$$w' = \delta\beta \left[(m-3)\left(\frac{\delta'}{\delta} - \frac{\beta'}{\beta} \right) + \left(M - ((m-2) - e^{-2v}) \frac{\sin 2\delta}{2\delta} \right)\right]$$

$$w(v_0) = 0 , \qquad w'(v_0) = 0 .$$

Provided $\delta'/\delta \le \beta'/\beta$ (i.e. $w \le 0$) with $\delta.\beta > 0$, we have $\omega' < 0$. Also as long as $\omega' < 0$ we have $w \le 0$. Now look at the first zero of β on $[v_0, v_1]$. We must have $\beta'\delta > 0$ here since $w < 0$, which implies δ has a zero on $[v_0, v_1]$. This is impossible since δ is strictly increasing and > 0. □

If we continue following the qualitative behaviour of the solution considered in the proof of Proposition 5.1.6, we see that since the gravity changes for $u > u_0$, the pendulum now moves under the influence of an upward gravity force past $\alpha = \pi$. The gravity now increases and continues to have the same sign. If α_0' is small enough, the pendulum will perform ever decreasing oscillations about $\alpha = \pi$.



Conceivably, there may be a certain α_0' such that the trajectory is exceptional and reaches $3\pi/2$ as $u \to \infty$. If α_0' is large enough the pendulum could make several rotations before tending to its equilibrium position.

Similar conditions apply to maps $\emptyset : \mathbb{R}^m \to S^n$ of the form

$$\emptyset((x, s.y)) = (\cos \alpha(s).e^{ikx}, \sin \alpha(s).g(x)), \ \alpha(0) = 0 ,$$

where k is an integer, $s \in [0,\infty)$, $x \in \mathbb{R}$, $y \in S^{m-2}$ and $g : S^{m-2} \to S^{n-2}$ is harmonic of constant energy density.

Again all the above constructed harmonic maps are smooth across the focal varieties (c.f. Chapter 6).

<u>Example 5.1.7 [11]</u> Consider the map $\emptyset : \mathbb{R}^2 \to S^2$ given by

$$\emptyset((x,s)) = (\cos \alpha(s).e^{ikx}, \sin \alpha(s)), \alpha(0) = 0 , \tag{5.1.12}$$

where k is a non-zero integer, $s \in (-\infty,\infty)$ and $x \in \mathbb{R}$. Then $\emptyset$ is equivariant with

$$|ds|^2 = 1 , \qquad \Delta s = 0 ,$$

and

$$\gamma(s,t) = k^2 . \cos^2 t .$$

Equation (4.1.5) becomes

$$\alpha''(s) = k^2 . \sin \alpha(s). \cos \alpha(s) . \tag{5.1.13}$$

This equation has a periodic solution in terms of elliptic functions, and the map $\emptyset$ factors to produce a harmonic map from the torus to the sphere.

5.2 Maps from hyperbolic space to the sphere

<u>Example 5.2.1</u> Let $\emptyset : H^m \to S^m$ be defined by

$$\emptyset((\cosh s, \sinh s.x)) = (\cos\alpha(s), \sin\alpha(s).x), \; \alpha(0) = 0 . \tag{5.2.1}$$

where $x \in S^{m-1}$ and $s \in [0,\infty)$. Then $\emptyset$ is equivariant with respect to the isoparametric functions s and t, where $s: H^m \to \mathbb{R}$ is given by $s((\cosh s_0, \sinh s_0.x)) = s_0$, and $t: S^m \to \mathbb{R}$ is given by $t((\cos t_0, \sin t_0.x)) = t_0$. From Lemma 1.6.1, we obtain

$$|ds|^2 = 1 , \qquad \Delta s = (m-1).\coth s ,$$

and $\gamma(s,t)$ is given by

$$\gamma(s,t) = (m-1).\sin^2 t/\sinh^2 s .$$

From Theorem 4.1.8, $\emptyset$ is harmonic if and only if

$$\alpha''(s) + (m-1).\coth s.\alpha'(s) - \frac{(m-1)}{\sinh^2 s}.\sin\alpha(s)\cos\alpha(s) = 0 . \tag{5.2.2}$$

This equation has a singularity at $s = 0$, and so we make the substitution $u = u(s)$, where $e^u = \sinh s$. Then

$$|du|^2 = (e^u + e^{-u})/e^u ,$$

and

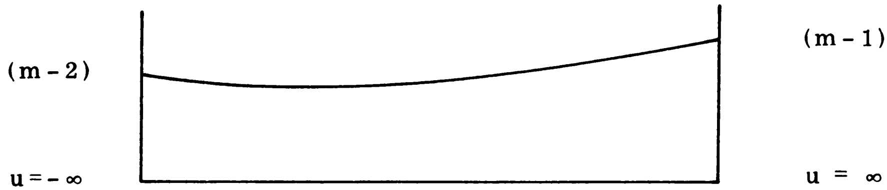
$$\Delta u = (e^u + (m-2)(e^u + e^{-u}))/e^u ,$$

whence, from Lemma 5.1.1 the reduction equation becomes

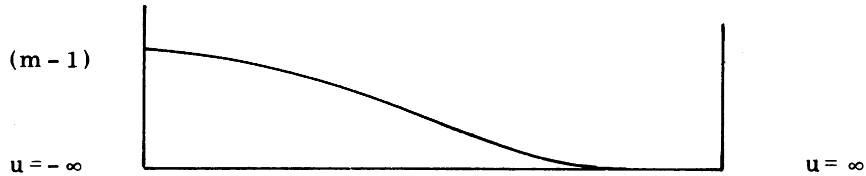
$$\alpha''(u) + \frac{1}{e^u + e^{-u}}(e^u + (m-2)(e^u + e^{-u}))\alpha'(u) - \frac{(m-1)}{e^u(e^u + e^{-u})}\sin\alpha(u)\cos\alpha(u) = 0 , \tag{5.2.3}$$

where $u \in (-\infty,\infty)$ and $\lim_{u\to-\infty} \alpha(u) = 0$.

Equation (5.2.3) is that of a pendulum with variable gravity and variable damping. The graph of the damping has the form :



and the graph of the gravity has the form



We look for an exceptional solution with the pendulum just standing up on end at time $u = -\infty$. As for the Euclidean case; we have a 1-parameter family of such solutions with the resulting map smooth across the point $s = 0$. However, the asymptotic development of these solutions as $u \to \infty$ is not known.

Similar considerations apply to maps $\phi : H^m \to S^n$ of the form

$$\phi((\cosh s, \sinh s.x)) = (\cos\alpha(s), \sin\alpha(s).g(x)), \alpha(0) = 0 , \qquad (5.2.4)$$

where $s \in [0,\infty)$, $x \in S^{m-1}$ and $g : S^{m-1} \to S^{n-1}$ is harmonic of constant energy density.

Example 5.2.2 Let $\phi : H^m \to S^m$ be defined, for each integer k, by

$$\phi((\cosh s.x, \sinh s.y)) = (\cos\alpha(s).e^{ikx}, \sin\alpha(s).y) , \qquad (5.2.5)$$

$\alpha(0) = 0$, where $x \in H^1$, $y \in S^{m-2}$ and $s \in [0,\infty)$. Then ϕ is equivariant with respect to the isoparametric functions s and t, where $s : H^m \to \mathbb{R}$ is given by $s((\cosh s_0.x, \sinh s_0.y)) = s_0$, for all $s_0 \in [0,\infty)$, $x \in H^1$, $y \in S^{m-2}$, and $t : S^m \to \mathbb{R}$ is given by $t(\cos t_0 u, \sin t_0.v)) = t_0$, for all $t_0 \in [0, \pi/2]$, $u \in S^1$ and $v \in S^{m-2}$. From Lemma 1.6.1 we obtain

$$|ds|^2 = 1 , \qquad \Delta s = \tanh s + (m-1)\coth s ;$$

also $\gamma(s,t)$ is given by

$$\gamma(s,t) = k^2\cos^2 t/\cosh^2 s + (m-2)\sin^2 t/\sinh^2 s.$$

From Theorem 4.1.8, $\emptyset$ is harmonic if and only if

$$\alpha''(s) + (\tanh s + (m-2)\coth s)\,\alpha'(s) - \sin\alpha(s)\cos\alpha(s)\left(\frac{m-1}{\sinh^2 s} - \frac{k^2}{\cosh^2 s}\right) = 0 . \tag{5.2.6}$$

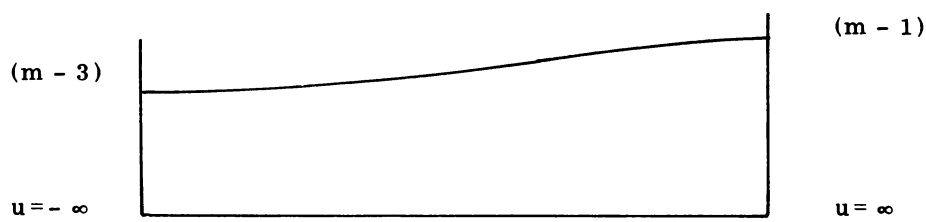
This equation has a singularity at $s = 0$, and so we make the substitution $u = u(s)$ where $e^u = \sinh s$, to give the equation

$$\alpha''(u) + \frac{1}{e^u + e^{-u}} \cdot (2e^u + (m-3)(e^u + e^{-u}))\,\alpha'(u)$$

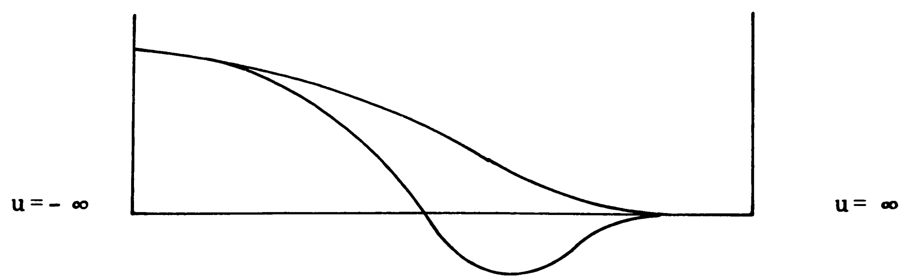
$$- \frac{\sin\alpha(u)\cos\alpha(u)}{e^u + e^{-u}} \cdot \left(\frac{m-1}{e^u} - \frac{k^2}{e^u + e^{-u}}\right) = 0 , \tag{5.2.7}$$

where $u \in (-\infty, \infty)$ and $\lim_{u \to -\infty} \alpha(u) = 0$.

Equation (5.2.7) is the equation of a pendulum with variable gravity and damping. The damping is given by the following graph:



and the gravity by the graph



As in Example 5.1.3, we have an interesting 1-parameter family of non-trivial solutions with the pendulum just standing up on end at time $u = -\infty$, and smooth across the point $s = 0$. By similar arguments to those of Proposition 5.1.5 we can show that, provided k is large enough, there exist solutions which turn before $\alpha(u) = \pi/4$ is reached. However, the qualitative aspect of the various solutions as $u \to \infty$ is not yet known.

5.3 Maps from sphere to sphere.

Example 5.3.1 The maps of Smith described in Section 1.3 are examples of equivariant maps between spheres. Then $\phi : S^{m-1} \to S^{n-1}$ is given by

$$\phi((\cos s.x\ ,\ \sin s.y)) = (\cos\alpha(s).g_1(x), \sin\alpha(s).g_2(y)) , \qquad (5.3.1)$$

with $\alpha(0) = 0$ and $\alpha(\pi/2) = \pi/2$, where $x \in S^{p-1}$, $y \in S^{q-1}$, $p + q = m$, $s \in [0,\pi/2]$ and $g_1 : S^{p-1} \to S^{r-1}$, $g_2 : S^{q-1} \to S^{s-1}$ are both harmonic with $|dg_i|^2 = a_i$ constant, $i = 1,2$. Using Lemma 1.3.4 we obtain

$$|ds|^2 = 1 , \qquad \Delta s = (q-1)\cot s - (p-1)\tan s .$$

Also

$$\gamma(s,t) = \frac{\cos^2 t}{\cos^2 s} . a_1 + \frac{\sin^2 t}{\sin^2 s} . a_2 .$$

From Theorem 4.1.8 we conclude that ϕ is harmonic if and only if equation (1.3.5) is satisfied. A more detailed consideration of this example is to be found in Sections 1.3 and 1.4.

Example 5.3.2 Let $f : S^{2n+1} \to \mathbb{R}$ be the isoparametric function of Example 2.3.5. Recall that the level surfaces M_s are parametrized by the sets $\{e^{i\theta}(\cos s.x + i\sin s.y) \in \mathbb{C}^{n+1} ;\ \theta \in [0,2\pi], (x,y) \in S_{n+1,2}\}$, for each $s \in [0,\pi/4]$; in fact M_s is isometric to $S^1 \times S^s_{n+1,2}/S^0$, where $S^s_{n+1,2}$ is the analytic submanifold of $\mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1}$ defined by the set

$$S^s_{n+1,2} = \{(\cos s.x,\ \sin s.y) \in \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1} ;\ (x,y) \in S_{n+1,2}\} .$$

Define $\phi : S^{2n+1} \to S^2$ by

$$\phi : (e^{i\theta}(\cos s.x + i\sin s.y)) = (\cos\alpha(s), \sin\alpha(s).e^{2ik\theta}), \tag{5.3.2}$$

where k is some non-zero integer, $\alpha(0) = 0$ and $\alpha(\Pi/4) = \Pi$. Then ϕ is S-equivariant with respect to the isoparametric functions s and t, where $s: S^{2n+1} \to \mathbb{R}$ is given by $s(e^{i\theta}(\cos s_0.x + i\sin s_0.y)) = s_0$, for all $s_0 \in [0, \Pi/4]$, $\theta \in [0, 2\Pi]$ and $(x,y) \in S_{n+1,2}$, and $t: S^2 \to \mathbb{R}$ is given by $t((\cos t_0, \sin t_0.u)) = t_0$, for all $t_0 \in [0, \Pi]$ and $u \in S^1$. The principal curvatures of M_s are $-\cot s$, $-\cot(s - \Pi/4)$, $-\cot(s - \Pi/2)$, $-\cot(s - 3\Pi/4)$ with multiplicities $n-1, 1, n-1, 1$ respectively, hence

$$\Delta s = -(n-1)\tan s + (n-1)\cot s + \frac{1 - \tan s}{1 + \tan s} - \frac{1 + \tan s}{1 - \tan s}. \tag{5.3.3}$$

Also

$$\gamma(s,t) = 4k^2 . \sin^2 t .$$

From Theorem 4.1.8 we conclude that ϕ is harmonic if and only if

$$\alpha''(s) + \left(-(n-1)\tan s + (n-1)\cot s + \frac{1 - \tan s}{1 + \tan s} - \frac{1 + \tan s}{1 - \tan s}\right)\alpha'(s)$$

$$- 4k^2 \sin\alpha(s)\cos\alpha(s) = 0, \tag{5.3.4}$$

with $\alpha(0) = 0$, and $\alpha(\Pi/4) = \Pi$. This is singular at $s = 0, \Pi/4$, and so make the substitution $u = u(s)$, where $e^u = \tan s/(1 - \tan s)$. Then

$$|du|^2 = (1 + 2e^u + 2e^{2u})/e^{2u}, \text{ and} \tag{5.3.5}$$

$$\Delta u = \frac{((1+e^u)^2 + e^{2u})}{e^{2u}} \left[2(1+e^u) - \frac{1+2e^u}{e^u} - \frac{(n-1)e^u}{1+e^u} \right.$$

$$\left. + (n-1)\frac{(1+e^u)}{e^u} + \frac{1}{1+2e^u} - (1+2e^u) \right]. \tag{5.3.6}$$

Using Lemma 5.1.1, equation (5.3.4) becomes

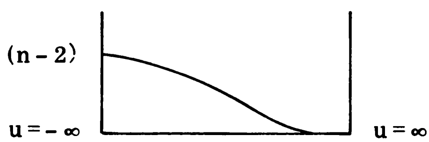
$$\alpha''(u) + \frac{e^u}{(1+2e^u+2e^{2u})}\left[2(1+e^u) - \frac{1+2e^u}{e^u} - \frac{(n-1)e^u}{1+e^u} + (n-1)\frac{(1+e^u)}{e^u}\right.$$

$$\left. + \frac{1}{1+2e^u} - (1+2e^u)\right]\alpha'(u) - \frac{4k^2e^{2u}\sin\alpha(u)\cos\alpha(u)}{(1+2e^u+2e^{2u})^2} = 0 \ , \qquad (5.3.7)$$

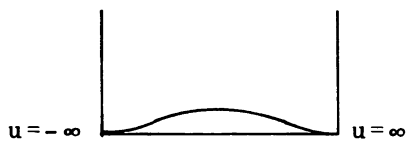
where $u \in (-\infty, \infty)$, $\lim_{u\to-\infty} \alpha(u) = 0$ and $\lim_{u\to\infty} \alpha(u) = \Pi$.

Equation (5.3.7) is that of a pendulum with variable gravity and damping.

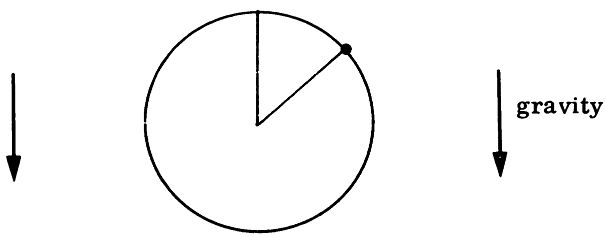
Damping.



Gravity.



As before $\bar{\alpha} = 2\alpha$ measures the angle the pendulum makes with the upward vertical.



We look for an exceptional trajectory such that the pendulum starts at the top at time $u = -\infty$; makes one circuit, and just reaches the top again at time $u = \infty$. It is not known whether or not the equation has such a solution - we would certainly expect a solution in the case $n = 2$ due to the symmetric appearance of the gravity and damping. It has been pointed out by R. Wood that these maps are all homotopically trivial.

Example 5.3.3 Let $f: S^{2n+1} \to \mathbb{R}$ be the isoparametric function of Example 5.3.2. Consider the case when $n = 3$, and define a map $\phi^s: S^s_{4,2} \to S^2$ by

$$\phi^s((x,y)) = \frac{2}{\sin 2s}(-x_1y_2 + y_2y_1 - x_3y_4 + x_4y_3, -x_1y_3 + x_3y_1 - x_4y_2 + x_2y_4,$$

$$- x_1y_4 + x_4y_1 - x_2y_3 + x_3y_2), \quad (5.3.8)$$

where $x = (x_1, \ldots, x_4)$, $y = (y_1, \ldots, y_4) \in \mathbb{R}^4$, $|x|^2 = \cos^2 s$, $|y|^2 = \sin^2 s$ and $<x,y> = 0$. Then

$$|\phi^s|^2 = (4/\sin^2 2s)(|x|^2 |y|^2 - <x,y>^2),$$

so that $\phi^s(S^s_{4,2}) \subset S^2$. Furthermore, one can show that ϕ^s is onto (see Example 8.2.1).

Consider the point $p = (\cos s(1,0,0,0), \sin s(0,1,0,0)) \in S^s_{4,2}$, and consider the following curves through p in $S^s_{4,2}$ (see Example 2.3.5):

$$\begin{cases} \beta_1(u) = (\cos s(\cos u,0,0,\sin u), \sin s(0,\cos u,\sin u,0)) \\ \beta_2(u) = (\cos s(\cos u,0,\sin u,0), \sin s(0,\cos u,0,\sin u)) \\ \beta_3(u) = (\cos s(\cos u,0,0,\sin u), \sin s(0,\cos u,-\sin u,0)) \\ \beta_4(u) = (\cos s(\cos u.0,\sin u,0), \sin s(0,\cos u,0,-\sin u)) \\ \mu(u) = (\cos s(\cos u,\sin u,0,0), \sin s(-\sin u,\cos u,0,0)), \end{cases} \quad (5.3.9)$$

The tangent vectors to these curves at $u = 0$ span $T_pS^s_{4,2}$ and are all orthonormal. The acceleration vectors at $u = 0$ are all perpendicular to $T_pS^s_{4,2}$ in $T_p(\mathbb{R}^4 \oplus \mathbb{R}^4)$, i.e. the curves are geodesic at p (that is $\nabla^{S^s_{4,2}}_{\beta'(0)} \beta'(0) = 0$ when β is one of the curves (5.3.9)). Under ϕ^s the curves map to

$$\begin{cases} \alpha_1(u) = \phi^s \circ \beta_1(u) = (-\cos 2u, -\sin 2u, 0) \\ \phi^s \circ \beta_2(u) = (-1,0,0) \\ \phi^s \circ \beta_3(u) = (-1,0,0) \\ \alpha_4(u) = \phi^s \circ \beta_4(u) = (-\cos 2u, 0, \sin 2u) \\ \phi^s \circ \mu(u) = (-1,0,0). \end{cases} \quad (5.3.10)$$

From this we deduce that ϕ^s is a harmonic Riemannian submersion, and in particular

ϕ^s has constant energy density equal to 4.

Define a map $\phi : S^7 \to S^3$ by

$$\phi(e^{i\theta}(\cos s.x + i\sin s.y)) = (\cos\alpha(s), \sin\alpha(s).\phi^s((\cos sx, \sin sy)) \quad ,$$

(5.3.11)

with $\alpha(0) = 0$ and $\alpha(\Pi/4) = \Pi$. Then ϕ is S-equivariant with respect to the isoparametric functions s and t, where $s: S^7 \to \mathbb{R}$ is given by $s(e^{i\theta}(\cos s_0.x + i\sin s_0.y)) = s_0$, for all $s_0 \in [0, \Pi/4]$, $\theta \in [0, 2\Pi]$ and $(x,y) \in S_{4,2}$, and $t: S^3 \to \mathbb{R}$ is given by $t((\cos t_0, \sin t_0.u)) = t_0$, for all $t_0 \in [0, \Pi]$ and $u \in S^2$. The function $\gamma(s,t)$ is given by

$$\gamma(s,t) = 8\sin^2 t .$$

From Theorem 4.1.8 we conclude that ϕ is harmonic if and only if equation (4.1.5) is satisfied; this equation is precisely the same as equation (5.3.7) with $n=3$ and $k = (2)^{\frac{1}{2}}$. It is not known whether this equation has a solution or not. Neither do we know which homotopy class of $\Pi_7(S^3) = \mathbb{Z}_2$, ϕ would represent.

Similarly we can define submersions from $S^s_{8,2}$ to S^4 and from $S^s_{15,2}$ to S^8, to give maps representing a class in $\Pi_{15}(S^5) = \mathbb{Z}_6$ and $\Pi_{29}(S^9)$ respectively .

Example 5.3.4 Let $f: S^{2n+1} \to \mathbb{R}$ be the isoparametric function of Example 5.3.2. Consider $S^s_{4,2}$ and the map $\phi^s: S^s_{4,2} \to \mathbb{R}$ of Example 5.3.3. Define $\phi: S^7 \to S^4$, for each integer k. by

$$\phi(e^{i\theta}(\cos s.x + i\sin s.y)) = (\cos\alpha(s).e^{2ik\theta}, \sin\alpha(s).\phi^s(\cos s.x, \sin sy)) ,$$

(5.1.12)

where $\alpha(0) = 0$ and $\alpha(\Pi/4) = \Pi/2$. Then ϕ is S-equivariant with respect to the isoparametric functions s and t, where $s: S^7 \to \mathbb{R}$ is as in Example 5.3.3, and $t: S^4 \to \mathbb{R}$ is given by $t(\cos t_0.u, \sin t_0.v) = t_0$, for all $t_0 \in [0, \Pi/2]$, $u \in S^1$ and $v \in S^2$.

Let $p \in S^7$ be the point $p = e^{i0}(\cos s e_1 + i\sin s e_2)$, where $(e_i)_{i=1,\dots,4}$ are the standard orthonormal vectors in $\mathbb{R}^4$. Consider the geodesic curve ν in S^7; $\nu(u) = e^{iu}(\cos s e_1 + i\sin s e_2)$. Then, in Example 2.3.5, we saw that $\nu(u)$ intersects $S^s_{4,2}$ at an angle of $2s - \Pi/2$ and in the plane spanned by $\nu'(0)$ and $\mu'(0)$, where μ is defined as in Example 2.3.5 (c.f. Lemma 2.3.7). By choosing an orthonormal basis to M_s at p defined by tangent vectors to the curves (5.3.9); we find

$$\gamma(s,t) = \frac{\cos^2 t . 4k^2}{\cos^2 2s} + 8 \sin^2 t . \tag{5.3.13}$$

From Theorem 4.1.8; ϕ is harmonic if and only if

$$\alpha''(s) + (-(n-1)\tan s + (n-1)\cot s + \frac{1-\tan s}{1+\tan s} - \frac{1+\tan s)}{1-\tan s}\,\alpha'(s)$$

$$- \sin\alpha(s)\cos\alpha(s). \left(8 - \frac{4k^2}{\cos^2 2s}\right) , \tag{5.3.14}$$

where $n = 3$, $s \in [0, \Pi/4]$, $\alpha(0) = 0$ and $\alpha(\Pi/4) = \Pi/2$. This equation has singularities at $s = 0$, $\Pi/4$, and we reparametrize it using the parameter $u = u(s)$ where $e^u = \tan s/(1-\tan s)$. Using Lemma 5.1.1 we obtain the equation

$$\alpha''(u) + \frac{e^u}{(1+2e^u+2e^{2u})}\left[2(1+e^u) - \frac{1+2e^u}{e^u} - \frac{(n-1)e^u}{1+e^u} + (n-1)\frac{1+e^u}{e^u} \right.$$

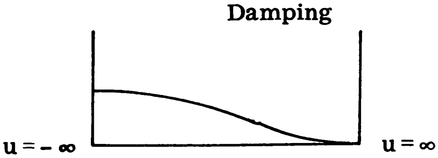
$$\left. + \frac{1}{1+2e^u} - (1+2e^u) \right] \alpha'(u) - \sin\alpha(u)\cos\alpha(u)\times$$

$$\left[\frac{8e^{2u}}{(1+2e^u+2e^{2u})^2} - \frac{4k^2e^{2u}}{(1+2e^u)^2}\right] , \tag{5.3.15}$$

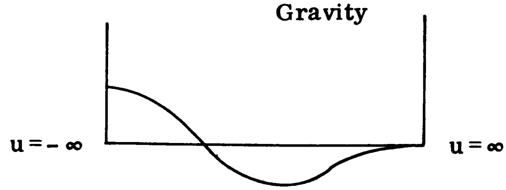
where $u \in (-\infty, \infty)$, $\lim_{u\to-\infty} \alpha(u) = 0$ and $\lim_{u\to\infty} \alpha(u) = \Pi/2$.

Equation (5.3.15) is that of a pendulum with variable gravity and damping ($\bar{\alpha} = 2\alpha$ measures the angle the pendulum makes with the upward vertical).

Damping



Gravity



The gravity has a negative component only if $k = -1, 1$. We would require an exceptional solution with the pendulum just standing up on end when $u = -\infty$, and hanging straight down when $u = \infty$. We would expect such a solution for $k = -1, 1$, since then the gravity has a balancing effect.

Example 5.3.5 (This example was suggested to me by H. Karcher):

Let $f: S^{m-1} \to [-1,1]$ be an isoparametric function on S^{m-1}, defined as the restriction of a homogeneous polynomial F of degree p on $\mathbb{R}^m$ (c.f. Theorem 2.2.5). If the multiplicities of the distinct principal curvatures are equal; the polynomial is harmonic. Then we can normalize F such that

$$|\nabla F(x)|^2 = |x|^{2p-2}, \text{ for all } x \in \mathbb{R}^m , \tag{5.3.16}$$

In this case the map $\emptyset = \nabla F|_{S^{m-1}}$ defines a harmonic polynomial map from S^{m-1} to S^{m-1}. That discovery was made by R. Wood [43]. Suppose now that the multiplicities of the distinct principal curvatures are not necessarily equal.

Proposition 5.3.6 Let $\emptyset$ be defined as above, where F is now any polynomial satisfying the conditions of Theorem 2.2.5, then $\emptyset$ is equivariant with respect to the isoparametric functions f on the domain sphere and f on the range sphere.

Proof (due to H. Karcher) We can express each point $x \in \mathbb{R}^m$ in the form

$$x = r.e ,$$

where $r \in [0,\infty)$ and $e \in S^{m-1}$. Then $F(r.e) = r^p f(e)$, and so

$$(1/p)\nabla F(e) = f(e).e + (1/p)\nabla f(e) . \tag{5.3.17}$$

Let $M_0 = f^{-1}(0)$ be the minimal hypersurface (any choice of hypersurfaces will do). Let $e(s) = e.\cos s + \xi_e \sin s$, where $e \in M_0$, ξ_e is the unit normal to M_o at e, and $s \in [-\pi/2p, \pi/2p]$. Then

$$f(e(s)) = \sin ps , \tag{5.3.18}$$

and

$$\nabla f(e(s)) = p\cos ps.e'(s) . \tag{5.3.19}$$

Also

$$e'(s) = -e.\sin s + \xi_e.\cos s , \tag{5.3.20}$$

so

$$< \nabla f, e' > = p.\cos ps \ . \qquad (5.3.21)$$

and

$$(1/p)\nabla F(e(s)) = \sin ps.e(s) + \cos ps.e'(s)$$
$$= \sin(p-1)s.e + \cos(p-1)s.\ \xi_e \ . \qquad (5.3.22)$$

The result now follows from equation (5.3.22). □

Remark 5.3.7 If we let $s \in [-\pi/2p, \pi/2p]$ represent the affine geodesic parameter, so that $f = \cos p(s+\pi/2p)$. Then $s = -\pi/2p$ and $s = \pi/2p$ correspond to the focal varieties, and $s = 0$ corresponds to the minimal hypersurface. Now

$$\emptyset(e(s)) = \sin(p-1)s.e + \cos(p-1)s.\ \xi_e$$
$$= \cos(-(p-1)s + \pi/2).e + \sin(-(p-1)s + \pi/2.\ \xi_e \ . \qquad (5.3.23)$$

So that under the map $\emptyset$, the minimal hypersurface is mapped onto a focal variety and (a) if p is odd, both focal varieties are mapped onto a single focal variety, and (b) if p is even, both focal varieties are mapped to themselves. If we allow a re parametrization of the map as in Theorem 4.1.15, then the function α is such that α: $[-\pi/2p, \pi/2p] \to [\pi/2p, (2p-1)\pi/2p]$ with $\alpha(-\pi/2p) = (2p-1)\pi/2p$ and $\alpha(\pi/2p) = \pi/2p$.

From Theorem 4.1.15, $\emptyset$ is harmonic if and only if the corresponding reduction equation is satisfied. In the case when the multiplicities of the distinct principal curvatures are equal and F is harmonic, then $\emptyset$ is harmonic and equation (4.1.5) is satisfied. In fact from equation (5.3.23) one sees that $\alpha(s) = -(p-1)s + \pi/2$ is the solution. More generally, suppose $p=4$. We will try to find an α: $[-\pi/8, \pi/8] \to [\pi/8, 7\pi/8]$ with $\alpha(-\pi/8) = 7\pi/8$ and $\alpha(\pi/8) = \pi/8$ satisfying equation (4.1.5). We remark that equation (4.1.5) is no longer that of a simple pendulum.

Recall from Remark 2.2.10 that the principal curvatures are $-\cot(s+\pi/8)$, $-\cot(s-\pi/8)$, $-\cot(s-3\pi/8)$ and $-\cot(s-5\pi/8)$ (remembering that s is now parametrized in the interval $[-\pi/8, \pi/8]$ instead of $[0, \pi/4]$), with multiplicities m_1, m_2, m_1, m_2 respectively. Thus

$$\gamma(s,t) = m_1 \frac{\sin^2(t+\Pi/8)}{\sin^2(s+\Pi/8)} + m_2 \frac{\sin^2(t-\Pi/8)}{\sin^2(s-\Pi/8)}$$

$$+ m_1 \frac{\sin^2(t-3\Pi/8)}{\sin^2(s-3\Pi/8)} + m_2 \frac{\sin^2(t-5\Pi/8)}{\sin^2(s-5\Pi/8)} , \qquad (5.2.24)$$

and equation (4.1.5) becomes

$$\alpha''(s) + (-m_1 \tan(s+\Pi/8 + m_1 \cot(s+\Pi/8)$$

$$+ \frac{1-\tan(s+\Pi/8)}{1+\tan(s+\Pi/8)} - \frac{1+\tan(s+\Pi/8}{1-\tan(s+\Pi/8)}) \alpha'(s)$$

$$- m_1 \sin(\alpha(s) + \Pi/8)\cos(\alpha(s) + \Pi/8) \left[\frac{1}{\sin^2(s+\Pi/8)} - \frac{1}{\cos^2(s+\Pi/8)} \right]$$

$$- m_2 \sin(\alpha(s) - \Pi/8)\cos(\alpha(s) - \Pi/8) \left[\frac{1}{\sin^2(s-\Pi/8)} - \frac{1}{\cos^2(s-\Pi/8)} \right]$$

$$= 0 , \qquad (5.3.25)$$

with $s \in [-\Pi/8, \Pi/8]$, $\alpha(-\Pi/8) = 7\Pi/8$ and $\alpha(\Pi/8) = \Pi/8$.

We now reparametrize this equation, using the parameter u defined by $e^u = \tan(s+\Pi/8)/(1-\tan(s+\Pi/8))$, so $u \in (-\infty, \infty)$. A computation shows that

$$|du|^2 = (1 + 2e^u + 2e^{2u})^2/e^{2u} ,$$

and,

$$\Delta u = \frac{((1+e^u)^2 + e^{2u})}{e^u} \left[2(1+e^u) - \frac{1+2e^u}{e^u} - m_1 \cdot \frac{e^u}{1+e^u} \right.$$

$$\left. + m_1 \frac{(1+e^u)}{e^u} + m_2 \frac{1}{1+2e^u} - m_2(1+2e^u) \right] .$$

Using Lemma 5.1.1, equation (5.3.25) becomes,

$$\alpha''(u) + \frac{e^u}{((1+e^u)^2+e^{2u})}\left[2(1+e^u) - \frac{1+2e^u}{e^u} - m_1 \frac{e^u}{1+e^u} + m_1 \frac{(1+e^u)}{e^u} + \frac{m_2}{1+2e^u} - m_2(1+2e^u)\right]\alpha'(u)$$

$$- m_1 \frac{\sin(\alpha(u) + \Pi/8)\cos(\alpha(u) + \Pi/8)}{e^{-u} + 2 + 2e^u}\left[e^{-u} - \frac{e^u}{(1+e^u)^2}\right]$$

$$- m_2 \frac{\sin(\alpha(u) - \Pi/8)\cos(\alpha(u) - \Pi/8)}{e^{-u} + 2 + 2e^u}\left[2e^u - \frac{2e^u}{(1+2e^u)^2}\right] = 0 .$$

(5.3.26)

At first sight this equation looks decidedly unpleasant, but if we look at the qualitative features, we find it has much in common with the Smith equation described in Section 1.3.

Express the equation in the form

$$\alpha''(u) = h(u)\alpha'(u) + g_1(u)\sin(\alpha(u) + \Pi/8)\cos(\alpha(u) + \Pi/8) + g_2(u)\sin(\alpha(u) - \Pi/8)\cos(\alpha(u) - \Pi/8) . \qquad (5.3.27)$$

Then, for u near $-\infty$,

$$h(u) = -(m_1 - 1) - (m_1 - 2)\,0(e^u) + (m_1 + m_2)\,0(e^{2u})$$

$$g_1(u) = m_1 - 0(e^u)$$

$$g_2(u) = 0(e^{3u}) .$$

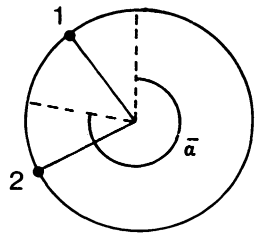
where by "$0(f(u))$ near $u = -\infty$", for some smooth non-zero function f, we mean that $0(f(u))/f(u)$ is bounded and positive for u near $-\infty$. Also for u near $+\infty$,

$$h(u) = (m_2 - 1) + (m_2 - 2)\,0(e^{-u}) - (m_1 + m_2)\,0(e^{-2u})$$

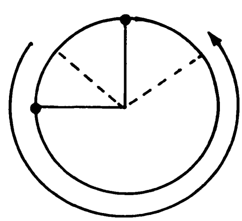
$$g_1(u) = 0(e^{-3u})$$

$$g_2(u) = m_2 - 0(e^{-u}) .$$

Equation (5.3.27) can be thought of as that of a compound pendulum consisting of two penduli fixed at right angles. Each pendulum is acted on by a different varying gravity force (g_1 and g_2), and the system has a varying damping force acting upon it. The position of the system is described by the angle $\bar{\alpha} = 2\alpha$, located between the two penduli.



Label the two penduli 1 and 2 as in the above diagram. Then one wishes to find an exceptional trajectory of the system with $\bar{\alpha}$ just starting off at time $-\infty$ at the angle $7\,\Pi/4$, and just reaching the angle $\Pi/4$ at time $u = +\infty$.



Note that at time $u=-\infty$, the mass of the system lies entirely in pendulum 1. Then the mass transfers to pendulum 2 as time progresses, and then finally at time $u=\infty$, the mass of the system lies entirely in pendulum 2.

One can attempt to solve the equation in a way similar to the one Smith uses for equation (1.3.7) in [36] - this was described at the beginning of Section 1.4. One fixes the time u_0 when $g_1(u_0) = g_2(u_0)$, and then the idea is to manipulate the initial conditions $\alpha_0 = \alpha(u_0)$ and $\alpha_0' = \alpha'(u_0)$. However, the method is very long-winded and involves several comparisons with other equations. The more elegant method used by Smith to solve equation (1.3.7) in [37] adapts to solve equation (5.3.26), and in Chapter 6 we show that equation (5.3.26) has a solution asymptotic to $7\Pi/8$, $\Pi/8$ as u tends to $-\infty, \infty$ respectively, if either

(i) $m_1 = m_2$,

or

(ii) $(m_1 - 1)^2 < 4m_1$ and $(m_2 - 1)^2 < 4m_2$, i.e. $m_1, m_2 = 1, \ldots, 5$. (5.3.28)

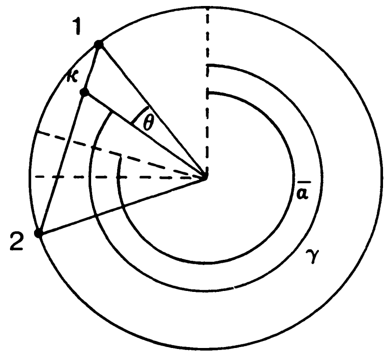
As for Smith's equation we call (ii) the damping conditions. Note how conditions (i) and (ii) compare with the damping conditions described at the beginning of Section 1.4. Furthermore, in Chapter 6, we show that such a solution to equation (5.3.26) yields a smooth harmonic map between spheres. We therefore have

Theorem 5.3.8 Let $F: \mathbb{R}^m \to \mathbb{R}$ be a homogeneous polynomial defining an isoparametric function of degree 4 on S^{m-1}, then $\emptyset = \nabla F|_{S^{m-1}} : S^{m-1} \to S^{m-1}$ is homotopic to a harmonic map, provided either of the above conditions (i) or (ii) are satisfied.

Remark 5.3.9 There are several examples of inhomogeneous families of isoparametric hypersurfaces of degree 4 with m_1 and m_2 satisfying the damping conditions (see, for example, [16]). Theorem 5.3.8 then gives us examples of harmonic maps equivariant with respect to inhomogeneous isoparametric functions. This illustrates that equivariance with respect to isoparametric functions is of some importance in the context of harmonic maps, i.e. we have generalized from the context of homogeneous hypersurfaces.

Remark 5.3.10 Similar considerations apply when $p = 6$. Indeed one can again show that if m_1 and m_2 satisfy (5.3.28), then the equivariant map described in Proposition 5.3.6 is homotopic to a harmonic map. However, we leave the details of this calculation to be described elsewhere.

We briefly consider an alternative way of describing the physical motion represented by equation (5.3.27). Transform the function α to a new function γ which represents the centre of mass of the compound pendulum represented by equation (5.3.27):



Referring to the diagram, we see that

$$\Theta = \tan^{-1}(x/(2^{\frac{1}{2}} - x)).$$

Taking moments about pendulum 1 we obtain

$$(g_1 + g_2)x = 2^{\frac{1}{2}} . g_2$$

i.e.

$$x = \frac{2^{\frac{1}{2}} g_2}{g_1 + g_2} .$$

So $\Theta = \tan^{-1}(g_2/g_1)$, and

$$\gamma = 2\alpha + \Pi/4 - \tan^{-1}(g_2/g_1) . \qquad (5.3.29)$$

Furthermore, a computation shows that

$$\tfrac{1}{2}g_1\sin(2\alpha + \pi/4) + \tfrac{1}{2}g_2\sin(2\alpha - \pi/4) = \tfrac{1}{2}(g_1^2 + g_2^2)^{\frac{1}{2}}\sin\gamma$$

$$\alpha' = \tfrac{1}{2}(\gamma' + \frac{g_1 g_2' - g_2 g_1'}{g_1^2 + g_2^2})$$

and $$\alpha'' = \tfrac{1}{2}(\gamma'' + \frac{(g_1^2 + g_2^2)(g_1 g_2'' - g_2 g_1'') - 2(g_1' + g_2')(g_1 g_2' - g_2 g_1')}{(g_1^2 + g_2^2)^2}) .$$

Whence equation (5.3.27) becomes

$$\gamma''(u) = h(u)\gamma'(u) + (g_1(u)^2 + g_2(u)^2)^{\frac{1}{2}} \sin\gamma(u) + p(u) , \qquad (5.3.30)$$

where,

$$p = \frac{2(g_1' + g_2')(g_1 g_2' - g_2 g_1') - (g_1^2 + g_2^2)(g_1 g_2'' - g_2 g_1'')}{(g_1^2 + g_2^2)^2}$$

$$+ \frac{h(g_1 g_2' - g_2 g_1')}{g_1^2 + g_2^2} \quad .$$

This is the equation of a simple pendulum with varying damping, acted upon by a variable gravity force, and with a force p of varying modulus acting tangentially to the circular motion. One can check that $\lim_{u \to \infty} p(u) = \lim_{u \to -\infty} p(u) = 0$. Furthermore, if u is sufficiently large $p(u) > 0$ and $p(u) = 0(e^{-3u})$. A smooth harmonic map will be given by a solution of equation (5.3.30) which is asymptotic to 2Π, 0 as $u \to -\infty, \infty$ respectively.

6 On certain ordinary differential equations of the pendulum type

6.1 Existence of solutions

The equations that arise in the various examples of Chapter 1 and Chapter 5 behave in a similar way to equation (1.3.7) of Smith close to the asymptotic limit $u = -\infty$. We adopt the methods used by Smith in [36] to investigate these equations. Equation (5.3.26) is dealt with separately using results of Hartman [21]. These results were used by Smith to solve equation (1.3.7) in [37]. We use this method to solve both equation (1.3.7) and equation (5.3.26).

First of all we list the equations to be considered, illustrating the various qualitative features of each one.

(i)
$$\alpha''(u) + \frac{1}{e^u + e^{-u}}\left((q-2)e^{-u} - (p-2)e^u\right)\alpha'(u)$$
$$+ \frac{1}{e^u + e^{-u}} \sin\alpha(u)\cos\alpha(u)\,(a_1 e^u - a_2 e^{-u}) = 0 , \qquad (6.1.1)$$

where $u \in (-\infty, \infty)$, $\alpha(-\infty) = 0$ and $\alpha(\infty) = \Pi/2$.

This equation arises with the Smith construction for harmonic maps from sphere to sphere. The constants a_1 and a_2 are given by $a_1 = k(k + p - 2)$ and $a_2 = 1(1+q-2)$, where k is the degree of a homogeneous polynomial map $g_1 : S^{p-1} \to S^{r-1}$ and 1 is the degree of a homogeneous polynomial map $g_2 : S^{q-1} \to S^{s-1}$ (c.f. Corollary 1.1.6 and Section 1.3). However, in Chapter 9 we will consider deformations of metrics, and then the special relationship between the values of the gravity and damping need no longer hold. We will therefore suppose simply that a_1 and a_2 are positive numbers.

Equation (6.1.1) is that of a pendulum with variable damping acted upon by a force of variable gravity as was described in Section 1.4. Write the equation in the form

$$\alpha''(u) = h(u)\alpha'(u) + g(u)\sin\alpha(u)\cos\alpha(u) , \qquad (6.1.2)$$

where, near $u = -\infty$,

$$h(u) = -(q-2) + 0(e^{2u})$$

and

$$g(u) = +a_2 - 0(e^{2u}) .$$

By "$0(f(u))$ near $u = -\infty$", for some smooth non-zero function f, we mean that $0(f(u))/f(u)$ is bounded and positive for u near $-\infty$. Also near $u = +\infty$,

$$h(u) = (p-2) - 0(e^{-2u})$$

and

$$g(u) = -a_1 + 0(e^{-2u}) .$$

(ii)

$$\alpha''(u) + \frac{1}{e^u + e^{-u}} (pe^u + (q-2)(e^u + e^{-u}))\alpha'(u)$$

$$- \frac{\sinh \alpha(u) \cosh \alpha(u)}{e^u + e^{-u}} \left(\frac{a_1}{e^u + e^{-u}} + \frac{a_2}{e^u} \right) = 0 , \qquad (6.1.3)$$

where $u \in (-\infty, \infty)$ and $\alpha(-\infty) = 0$.

This is the hyperbolic analogue of case (i) (c.f. Section 1.6). Here we assume that $a_1 \geq 0$, and $a_2 > 0$. Write this equation in the form

$$\alpha''(u) = h(u)\alpha'(u) + g(u) \sinh \alpha(u) \cosh \alpha(u) , \qquad (6.1.4)$$

where, near $u = -\infty$,

$$h(u) = -(q-2) - 0(e^{2u})$$

and

$$g(u) = a_2 + 0(e^{2u}) .$$

(iii)

$$\alpha''(u) + (m-2)\alpha'(u) - (m-1) \sin \alpha(u) \cos \alpha(u) = 0 , \qquad (6.1.5)$$

$$\alpha''(u) + (m-3)\alpha'(u) - ((m-2) - k^2 e^{2u}) \sin \alpha(u) \cos \alpha(u) = 0 , \qquad (6.1.6)$$

where $u \in (-\infty, \infty)$ and $\alpha(-\infty) = 0$. Both these equations arise in the construction of harmonic maps from Euclidean space to sphere. The first has been well studied (see, for example, [23]), representing a pendulum with constant damping acted upon by a force of constant gravity. If we write the second of these equations in the form

$$\alpha''(u) = h(u)\alpha'(u) + g(u)\sin\alpha(u)\cos\alpha(u) , \tag{6.1.7}$$

then, for u near $-\infty$;

$$h(u) = -(m-3)$$

and

$$g(u) = (m-2) - 0(e^{2u}) .$$

(iv)

$$\alpha''(u) + \frac{1}{e^{u}+e^{-u}}\left(e^{u} + (m-2)(e^{u}+e^{-u})\right)\alpha'(u)$$

$$- \frac{(m-1)}{e^{u}(e^{u}+e^{-u})}\sin\alpha(u)\cos\alpha(u) = 0 , \tag{6.1.8}$$

$$\alpha''(u) + \frac{1}{e^{u}+e^{-u}}\left(2e^{u} + (m-3)(e^{u}+e^{-u})\right)\alpha'(u)$$

$$- \frac{\sin\alpha(u)\cos\alpha(u)}{e^{u}+e^{-u}}\left[\frac{m-1}{e^{u}} - \frac{k^{2}}{e^{u}+e^{-u}}\right] , \tag{6.1.9}$$

where $u \in (-\infty,\infty)$ and $\alpha(-\infty) = 0$. These equations arise in the construction of harmonic maps from hyperbolic space to sphere. We can express them both in the form

$$\alpha''(u) = h(u)\alpha'(u) + g(u)\sin\alpha(u)\cos\alpha(u) , \tag{6.1.10}$$

where, for u near $-\infty$;

$$h(u) = -b - 0(e^{2u})$$

and

$$g(u) = (m-1) - 0(e^{2u}) .$$

with b equal to $(m-2)$ in the first case, and $(m-3)$ in the second.

(v)

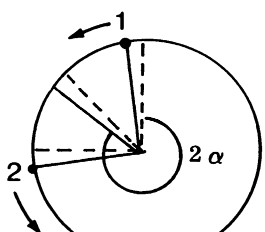
$$\alpha''(u) + \frac{e^u}{(1+e^u)^2 + e^{2u}} \left[2(1+e^u) - \frac{1+2e^u}{e^u} - \frac{m_1 e^u}{1+e^u} + \frac{m_1(1+e^u)}{e^u} + \frac{m_2}{1+2e^u} - m_2(1+2e^u) \right] \alpha'(u)$$

$$- m_1 \frac{\sin(\alpha(u) + \pi/8)\cos(\alpha(u) + \pi/8)}{e^{-u} + 2 + 2e^u} \left[e^{-u} - \frac{e^u}{(1+e^u)^2} \right].$$

$$- m_2 \frac{\sin(\alpha(u) - \pi/8)\cos(\alpha(u) - \pi/8)}{e^{-u} + 2 + 2e^u} \left[2e^u - \frac{2e^u}{(1+2e^u)^2} \right] = 0 .$$

(6.1.11)

where $u \in (-\infty, \infty)$, $\alpha(-\infty) = 7\pi/8$ and $\alpha(\infty) = \pi/8$. This equation arises in Example 5.3.5 for harmonic maps from sphere to sphere. It is now no longer the equation of a simple pendulum, but a compound pendulum, consisting of two penduli fixed together, separated by an angle of $\pi/2$ - the system having a variable damping and variable gravities acting on each pendulum distinctly.



Write this equation in the form

$$\alpha''(u) = h(u)\alpha'(u) + g_1(u)\sin(\alpha(u) + \pi/8)\cos(\alpha(u) + \pi/8) + g_2(u)\sin(\alpha(u) - \pi/8)\cos(\alpha(u) - \pi/8) \qquad (6.1.12)$$

Then for u near $-\infty$:

$$h(u) = -(m_1-1) - (m_1-2)\,0(e^{u}) + (m_1+m_2)\,0(e^{2u}) ,$$

$$g_1(u) = m_1 - 0(e^{u})$$

and

$$g_2(u) = 0(e^{3u}) ,$$

and for u near ∞ ;

$$h(u) = (m_2-1) + (m_2-2)\,0(e^{-u}) - (m_1+m_2)\,0(e^{-2u}) ,$$

$$g_2(u) = m_2 - 0(e^{-u})$$

and

$$g_1(u) = 0(e^{-3u}) .$$

Note also that $g_1(u), g_2(u) > 0$, for all $u \in (-\infty,\infty)$.

Consider a general equation which includes (i) - (iv) as special cases:

$$\alpha''(u) = h(u)\alpha'(u) - g(u)f(\alpha(u))f'(\alpha(u)) , \tag{6.1.13}$$

where f is an analytic function with the property that $f(x) - x = \pm 0(x^3)$. Also $h(u)$ and $g(u)$ are smooth functions on $(-\infty,\infty)$ and for u near $-\infty$;

$$h(u) = -b \pm 0(e^{2u})$$

and

$$g(u) = a \pm 0(e^{2u}) ,$$

for some constants a and b.

Now $f'(x) = 1 \pm 0(x^2)$, so that as x increases from 0, so does $f(x)$; let x_0 be the first x greater than 0 such that $f(x_0)f'(x_0) = 0$, or if no such x exists; let x_0 be an arbitrary positive number. Choose a time u_0 such that $g(u) > 0$ for all $u \le u_0$. By the uniqueness theorem for ordinary differential equations the above equation has a unique solution through $(u_0, \alpha_0, \alpha_0')$ which we denote by $\alpha(\alpha_0, \alpha_0')$ - this exists for all time. The following parallels closely the methods of Smith [36].

Let $A(\alpha_0) = \{\alpha_0' \in \mathbb{R} ; \alpha(\alpha_0, \alpha_0')$ decreases monotonically to zero in finite time as u decreases from $u_0\}$. Then this set is bounded below by 0. Choose $\alpha_0'(\alpha_0) = \inf A(\alpha_0)$.

<u>Lemma 6.1.1</u> $\alpha_0'(x)$ <u>is a well defined function on</u> $(0, x_0)$.

<u>Proof</u> Since $A(\alpha_0)$ is bounded below by 0; it suffices to show that $A(\alpha_0)$ is non-empty, i.e. given $\alpha_0 \in (0,x_0)$; we wish to find an α_0' such that $\alpha(\alpha_0,\alpha_0')$ decreases monotonically to 0 in finite time as u decreases from u_0.

As long as $\alpha'(v) \geq 0$ on $[u,u_0]$ for some $u < u_0$, we see from equation (6.1.13) that, provided also $\alpha \geq 0$;

$$\alpha''(v) \leq c_1 + c_2\alpha'(v)$$

on $[u, u_0]$ for some positive constants c_1 and c_2. Integrating this inequality over $[u,u_0]$ yields

$$\alpha'(u_0) - \alpha'(u) \leq c_1(u_0 - u) + c_2(\alpha_0 - \alpha(u))$$

$$\leq c_1(u_0 - u) ,$$

i.e.

$$- \alpha'(u) \leq - \alpha_0' + c_1(u_0-u) .$$

Choose α_0' such that

$$- \alpha_0' + c_1(u_0 - u) \leq \frac{-\alpha_0}{(u_0-u)} < 0 .$$

Then, for all v, $u \leq v \leq u_0$:

$$- \alpha'(v) \leq - \alpha_0' + c_1(u_0-v) \leq - \alpha_0' + c_1(u_0- u)$$

$$\leq - \alpha_0/(u_0- u) .$$

Integrating again over $[u,u_0]$ yields

$$- (\alpha(u_0) - \alpha(u)) \leq - \alpha_0 .$$

i.e. $\alpha(u) \leq 0$, and α has decreased monotonically to 0 in finite time as required. □

<u>Lemma 6.1.2</u> α_0' <u>is strictly positive on</u> $(0, x_0)$.

<u>Proof</u> Given $\alpha_0 \in (0,x_0)$, suppose $\alpha_0(\alpha_0) = 0$, then at u_0,

$$\alpha''(u_0) = g(u_0)f(\alpha_0)f'(\alpha_0) > 0 ,$$

and so $\alpha''(u) > 0$ for u sufficiently close to u_0. Thus $\alpha(u)$ initially increases as u both increases and decreases from u_0. Therefore there cannot be elements α_0' in $A(\alpha_0)$ arbitrarily close to zero, whence $\alpha_0'(\alpha_0) > 0$. □

<u>Lemma 6.1.3</u> $A(\alpha_0)$ <u>is open for all</u> $\alpha_0 \in (0, x_0)$.

<u>Proof</u> Suppose $\alpha_0' \in A(\alpha_0)$ and $\alpha(\alpha_0, \alpha_0')$ arrives at 0 (as time travels backwards) for the first time at time u. Then $\alpha'(u) > 0$. For $\alpha'(u) \geq 0$ and if $\alpha'(u) = 0$ with $\alpha(u) = 0$, it follows from the uniqueness theorem for ordinary differential equations that $\alpha \equiv 0$. Hence α decreases through 0 with positive derivative. Also $\alpha'(v) > 0$ for $u < v < u_0$, since if $\alpha'(v) = 0$, then $\alpha''(v) > 0$ and α would not be monotonic. Again from the proof of Lemma 6.1.2 we see that $\alpha'(u_0) > 0$. Hence $\alpha' > 0$ on $[u - \epsilon, u_0]$ for some $\epsilon > 0$, and so any function which is close enough to α in the C^1 norm will decrease monotonically to 0 on this interval in backward time. Therefore points near α_0' are in $A(\alpha_0)$ and $A(\alpha_0)$ is open. □

<u>Lemma 6.1.4</u> <u>For all</u> $\alpha_0 \in (0, x_0)$, $\alpha(\alpha_0, \alpha_0'(\alpha_0))$ <u>is strictly decreasing as</u> u <u>decreases from</u> u_0 <u>and is asymptotic to</u> 0 <u>as</u> $u \to -\infty$.

<u>Proof</u> We show that $\alpha' > 0$ on $(-\infty, u_0]$, and that $0 < \alpha$ on $(-\infty, u_0]$. For this will show that α decreases to some asymptotic value $\alpha_{-\infty}$, with $0 \leq \alpha_{-\infty} < x_0$. But then as $u \to -\infty$, $\alpha'(u), \alpha''(u) \to 0$, so equation (6.1.13) implies that the only possible value for $\alpha_{-\infty}$ is 0.

Assume that one of these two conditions is violated. One must go wrong first, for if both are violated simultaneously, the uniqueness theorem implies that we have the trivial solution. On the other hand neither goes wrong at u_0 from Lemma 6.1.2. Suppose $\alpha'(u) = 0$ for $u < u_0$ (for the first time), and $0 < \alpha(u) < x_0$, then $\alpha''(u) > 0$ and α increases as time decreases from u. But, by the definition of α_0', there are functions arbitrarily close to α on $[u - \epsilon, u_0]$ which are strictly decreasing or go past 0 on this interval - a contradiction.

Suppose now that $\alpha(u) = 0$, and $\alpha' > 0$ on $[u, u_0]$. Then $\alpha_0'(\alpha_0) \in A(\alpha_0)$ - this is not possible since $A(\alpha_0)$ is open and $\alpha_0'(\alpha_0) = \inf A(\alpha_0)$. □

We now demonstrate a local uniqueness result.

Lemma 6.1.5 Provided that u_0 is chosen sufficiently close to $-\infty$ that $h(u) < 0$ for all $u \leq u_0$, and α_0 is chosen sufficiently close to 0 that if $0 < x < y \leq \alpha_0$, then $f(x)f'(x) < f(y)f'(y)$ (i.e. ff' is an increasing function), then $\alpha_0'(\alpha_0)$ is the unique initial derivative for which we get a solution of the desired form in backward time. Specifically, if $0 < \beta_0' < \alpha_0'(\alpha_0)$ then the associated solution $\alpha(\alpha_0, \beta_0')$ must eventually start to increase before reaching 0, and if $\beta_0' > \alpha_0'(\alpha_0)$, then $\beta_0' \in A(\alpha_0)$.

Remark 6.1.6 The conditions of the above Lemma are always satisfied for any u_0 and α_0 in the case of equation (6.1.3), thus we have a global uniqueness result in that case.

Proof (of Lemma 6.1.5) Let $\beta_0' < \alpha_0'(\alpha_0)$ and write $\alpha_1 = \alpha(\alpha_0, \alpha_0'(\alpha_0))$ and $\alpha_2 = \alpha(\alpha_0, \beta_0')$. Then $\alpha_1(u_0) = \alpha_2(u_0)$ and $\alpha_1'(u_0) > \alpha_2'(u_0)$. Suppose that $\alpha_1'(u_0) - \alpha_2'(u_0) = \epsilon > 0$. Define the function $e(u) = (\alpha_1 - \alpha_2)'(u)$. Then $e(u_0) = \epsilon$, and

$$\begin{aligned} e'(u_0) &= \alpha_1''(u_0) - \alpha_2''(u_0) \\ &= h(u_0)(\alpha_1'(u_0) - \alpha_2'(u_0)) + g(u_0)(f(\alpha_1)f'(\alpha_1) - f(\alpha_2)f'(\alpha_2))(u_0) \\ &= h(u_0)(\alpha_1'(u_0) - \alpha_2'(u_0)) < 0 \ . \end{aligned}$$

Suppose there exists a $u < u_0$, with $e'(u) = 0$. Then

$$\begin{aligned} h(u)\alpha_1'(u) &+ g(u)f(\alpha_1(u))f'(\alpha_1(u)) \\ &= h(u)\alpha_2'(u) + g(u)f(\alpha_2(u))f'(\alpha_2(u)) \ , \end{aligned}$$

i.e.

$$h(u)(\alpha_1'(u) - \alpha_2'(u)) = g(u)(f(\alpha_2(u))f'(\alpha_2(u)) - f(\alpha_1(u))f'(\alpha_1(u))) \ .$$

But $\alpha_1'(u) > \alpha_2'(u)$ and $\alpha_1(u) < \alpha_2(u)$, so that the left hand side of this expression is less than 0, and the right hand side is greater than 0 as ff' is an increasing function – a contradiction; so $e'(u) < 0$ for all $u \leq u_0$, which implies that $e(u) > \epsilon$ for all $u \leq u_0$. But $\lim_{u \to -\infty} e(u) = 0 - \lim_{u \to -\infty} \alpha_2'(u) > \epsilon$. Thus α_2' has become negative in finite time, and α_2 starts to increase (for decreasing time) before reaching 0.

If on the other hand $\beta_0' > \alpha_0'(\alpha_0)$, then clearly $\alpha_2 = \alpha(\alpha_0, \beta_0')$ cannot turn before reaching 0, since then the trajectories of $\alpha_1 = \alpha(\alpha_0, \alpha_0'(\alpha_0))$ and α_2 in phase space $\mathbb{R}^2$ (the curves $u \to (\alpha(u), \alpha'(u)) \subset \mathbb{R}^2$) would cross - an impossibility. Hence the only possibility is that $\beta_0' \in A(\alpha_0)$ or α_2 is asymptotic to 0. But the above arguments show that if α_2 is asymptotic to 0, then α_1 must turn in finite time, i.e. $\alpha_0'(\alpha_0) \in A(\alpha_0)$ - a contradiction. Hence $\beta_0' \in A(\alpha_0)$. □

Corollary 6.1.7 If the conditions of Lemma 6.1.5 are satisfied then $\alpha_0'(\alpha_0)$ is a continuous function of α_0.

Proof Suppose we have a sequence $\alpha_0^n \to \alpha_0$ but $\alpha_0'(\alpha_0^n) \leq \alpha_0'(\alpha_0) - \epsilon$ for some $\epsilon > 0$. Let $\beta_0' = \alpha_0'(\alpha_0) - \epsilon/2$. Then from Lemma 6.1.5 we see that

(i) $\alpha(\alpha_0^n, \beta_0')$ decreases monotonically to 0 in finite time for all n (since $\alpha_0'(\alpha_0^n) < \beta_0'$), and

(ii) $\alpha(\alpha_0, \beta_0')$ eventually begins to increase (as time decreases) before reaching 0.

Letting $\alpha_0^n \to \alpha_0$ this yields a contradiction. □

Corollary 6.1.8

$$\alpha_0'(\alpha_0) \to 0 \quad \text{as} \quad \alpha_0 \to 0 .$$

In case (i) the problem is to find a solution asymptotic to $\Pi/2$ as $u \to \infty$. In his Thesis Smith's method was to derive the unique velocity $\alpha_0'^{-}(\alpha_0)$ such that $\alpha(\alpha_0, \alpha_0'^{-}(\alpha_0))$ is asymptotic to 0 as $u \to -\infty$ as above, and similarly to derive the unique velocity $\alpha_0'^{+}(\alpha_0)$ such that $\alpha(\alpha_0, \alpha'^{+}(\alpha_0))$ is asymptotic to $\Pi/2$ as $u \to \infty$. The damping conditions ensure $\alpha_0'^{-}(\alpha_0)$ and $\alpha_0'^{+}(\alpha_0)$ are bounded away from 0 as α_0 tends to $\Pi/2$, 0 respectively. Then Corollary 6.1.7 and Corollary 6.1.8 show there exists an α_0 with $\alpha_0'^{-}(\alpha_0) = \alpha'^{+}(\alpha_0)$ yielding the required solution. Later, in [34], Smith gave a more sophisticated method of solving case (i) by applying results of Hartman [21]. This method can be applied to solve equation (6.1.11) of case (v) and we use it now to solve both case (i) and case (v).

Suppose we are given an equation of the form

$$\alpha'' = F(u, \alpha, \alpha'), \tag{6.1.14}$$

where $F(u, \alpha, \alpha')$ is a continuous function on the set $E(p,R) = \{(u, \alpha, \alpha');$

$0 \leq u \leq p$, $|\alpha| \leq R$, α' arbitrary$\}$.

Theorem 6.1.9 [21], [31] *Suppose* F *is as above, and*

(i) $F(u,R,0) \geq 0$, $F(u,-R,0) \leq 0$, *for* $0 \leq u \leq p$, *and*

(ii) $|F| \leq \phi(|\alpha'|)$ *where* $\phi(s)$, $0 \leq s < \infty$ *is a positive continuous function satisfying* $\int_0^{\infty} \frac{s\,ds}{\phi(s)} = \infty$. *Let* α_0, α_p *be such that* $|\alpha_0|, |\alpha_p| \leq R$. *Then equation* (6.1.14) *has at least one solution in* $E(p,R)$ *satisfying* $\alpha(0) = \alpha_0$, $\alpha(p) = \alpha_p$. *Furthermore*, $|\alpha'| \leq C$, *where* C *is a constant depending only on* ϕ, R *and* p.

Write equation (6.1.1) in the form

$$\alpha'' = F(u,\alpha,\alpha'),$$

then $|F| \leq C_0(1+|\alpha'|)$ for some constant C_0 and $F(u,0,0) = F(u,\Pi/2,0) = 0$. Adjusting the range of α (to $[0,\Pi/2]$ instead of $[-R,R]$) and the time scale (to $[-T,T]$ instead of $[0,p]$), we can apply Theorem 6.1.9 to yield

Lemma 6.1.10 *Given* $T > 0$ *there is a solution* α_T *of equation (6.1.1) satisfying*

$$\alpha_T(-T) = 0, \quad \alpha_T(T) = \Pi/2,$$

$$0 < \alpha_T(u) < \Pi/2 \text{ and } |\alpha_T'| \leq C \text{ on } (-T,T), \qquad (6.1.15)$$

with C *depending only on* C_0.

Proof It suffices to show that $0 < \alpha_T(u) < \Pi/2$ on $(-T,T)$. We know from Theorem 6.1.9 that $0 \leq \alpha_T(u) \leq \Pi/2$ on $(-T,T)$. If $u_0 \in (-T,T)$ has $\alpha_T(u_0) = 0$, in order that α_T remains in $[0,\Pi/2]$ (before or after u_0); $\alpha_T'(u_0) = 0$. But equation 6.1.1 now implies $\alpha_T''(u_0) = 0$. The uniqueness theorem then implies that $\alpha_T \equiv 0$ - a contradiction. □

We now consider a sequence $T = T(n) \to \infty$. From standard equicontinuity arguments (c.f. Hartman [21], Chapter 1, Section 2) $\alpha_{T(n)}$ converges in C^2 on compact sets to a solution α_0 of equation (6.1.1) with $0 \leq \alpha_0 \leq \Pi/2$, and with $|\alpha_0'|$ bounded.

Lemma 6.1.11 *Provided the damping conditions (1.4.1) hold, the solution* α_0 *of equation (6.1.1) described above is non-trivial.*

Proof Suppose $\alpha_0(u) \equiv 0$ for all $u \in \mathbb{R}$. Writing equation (6.1.1) in the form

$$\alpha''(u) = h(u)\alpha'(u) + g(u)\sin\alpha(u)\cos\alpha(u) ,$$

we see that there is a solution $\alpha = \alpha_{T(n)}$ of equation (6.1.1) satisfying conditions (6.1.15) which is very small out to a large value of u, say $u_1 < T(n)$. Then $g(u_1)$ is close to $-a_1$. Choose a_1^- near a_1 so that $a_1^- + g\sin\alpha\cos\alpha/\alpha < 0$ on a suitable interval $[u_0, u_1]$ and also that the equation

$$\beta'' = (p-2)\beta' - a_1^- \beta ,$$

has non-real roots and period less than $u_1 - u_0$. Let β have the same initial data as α at u_0, and let $w = \alpha'\beta - \alpha\beta'$. Then

$$w' = \alpha\beta\,[h(u)\alpha'/\alpha - (p-2)\beta'/\beta)$$
$$+ (a_1^- + g(u)\sin\alpha\cos\alpha/\alpha)] ,$$

$$w(u_0) = 0 \quad \text{and} \quad w'(u_0) < 0 .$$

Provided $\alpha'/\alpha \leq \beta'/\beta$ (i.e. $w \leq 0$) with $\alpha, \beta > 0$ we have $w' < 0$. Also as long as $w' < 0$ we have $w \leq 0$. Now look at the first zero of β on $[u_0, u_1]$. We must have $\beta'\alpha > 0$ here since $w \leq 0$ - this is impossible if $\alpha > 0$ on $[u_0, u_1]$. Hence α has a zero on this interval, contradicting (6.1.15). Therefore $\alpha_0 \equiv 0$ is impossible. Similarly $\alpha_0 = \Pi/2$ is impossible. □

Lemma 6.1.12 If α is a non-trivial solution of equation (6.1.1) with $0 \leq \alpha \leq \Pi/2$ and $|\alpha'| \leq C$, then $\alpha'(u) > 0$ whenever $|u|$ is sufficiently large, and α has the asymptotic limits $0, \Pi/2$ as $u \to -\infty, \infty$ respectively.

Proof Suppose u_0 is close to $+\beta$ and $p-2 > 0$, then we can assume $h(u) > 0$ and $g(u) < 0$ for all $u \geq u_0$. Then if $\alpha'(u_0) \leq 0$ $\alpha''(u_0) < 0$. Thus there exists a subsequent time $u_1 > u_0$ such that $\alpha'(u_1) = -\epsilon < 0$. Then α'' remains negative as long as α' remains negative, so that $\alpha' \leq -\epsilon$ until α has decreased to 0 and become negative - a contradiction. Hence $\alpha'(u_0) > 0$.

Suppose $p-2 = 0$. Then if $\alpha'(u) \leq 0$ (u large), α'' must become negative before α' changes sign. Hence $\alpha' < 0$ for all subsequent u. In fact, since α' is bounded, it follows that α'' must become negative and remain so at least until α decreases to some small value. However, using the same comparison which showed that α was non-trivial leads to a contradiction.

Thus $\alpha'(u) > 0$ for u sufficiently large, and α has an asymptotic limit $\leq \Pi/2$. The same argument used in the proof of Lemma 6.1.4 now shows the limit must in fact equal $\Pi/2$. The case when u is close to $-\infty$ is similar. □

We now study case (v). Consider equation (6.1.12); express this in the form

$$\alpha'' = F(u,\alpha,\alpha').$$

Then $|F| \leq C_0(1 + |\alpha'|)$ for some C_0 and

$$F(u, 7\Pi/8, 0) = -g_2(u)/2 < 0,$$

$$F(u, \Pi/8, 0) = g_1(u)/2 > 0.$$

Applying Theorem 6.1.9, adjusting the ranges of α and time (remembering that $7\Pi/8$ corresponds to $-R$ and $\Pi/8$ corresponds to R), we have

Lemma 6.1.13 Given $T > 0$, there is a solution α_T of equation (6.1.11) with

$$\alpha_T(-T) = 7\Pi/8, \quad \alpha_T(T) = \Pi/8,$$

$$7\Pi/8 > \alpha_T(u) > \Pi/8 \text{ and } |\alpha_T'| \leq C \text{ on } (-T,T), \tag{6.1.16}$$

with C depending only on C_0.

Proof It suffices to show that $7\pi/8 > \alpha_T(u) > \pi/8$ on $(-T,T)$. We know from Theorem 6.1.9 that $7\pi/8 \geq \alpha_T(u) \geq \pi/8$ on $(-T,T)$. Suppose $u_0 \in (-T,T)$ is such that $\alpha_T(u_0) = 7\pi/8$; in order that α_T remains in $[7\pi/8, \pi/8]$ (before and after u_0) we must have $\alpha'_T(u_0) = 0$. Then equation (6.1.11) implies that $\alpha''_T(u_0) = -g_2(u_0)/2 < 0$, and α decreases past $7\pi/8$ as u increases from u_0 - a contradiction. Similarly we cannot have $\alpha_T(u_0) = \pi/8$. □

As before we consider a sequence $T = T(n) \to \infty$, and by equicontinuity arguments $\alpha_{T(n)}$ converges in C^2 on compact sets to a solution α_0 of equation (6.1.11) with $7\Pi/8 \geq \alpha_0 \geq \Pi/8$ with α_0' bounded. Clearly α_0 cannot be constant since $\sin(2\alpha - \pi/4)$ and $\sin(2\alpha + \pi/4)$ cannot be 0 simultaneously.

Lemma 6.1.14 Let α be the non-trivial solution of equation (6.1.12) described above. Then provided the damping conditions (5.3.28) hold; as $u \to \infty, \alpha_0(u) \to \Pi/8$ and as $u \to -\infty$, $\alpha_0(u) \to 7\Pi/8$.

Proof Unlike case (i), overdamping in this case could cause the solution to be asymptotic to $\Pi/2$.

Consider the case when $u \to \infty$. We claim there are three possibilities:

(a) $\lim_{u \to \infty} \alpha_0(u) = 5\pi/8$,

(b) $\lim_{u \to \infty} \alpha_0(u) = \pi/8$,

(c) $2(\alpha_0 - \pi/8)$ continually oscillates about the downward vertical π.

Of course (b) and (c) could both happen. Case (c) could conceivably occur if $\lim_{u \to \infty} h(u) = 0 (m_2 = 1)$. We will show that the damping conditions ensure that (b) is the only possible case.

Make the substitution $\gamma_0 = 2\alpha_0 - \pi/4$, then equation (6.1.12) becomes

$$\gamma_0''(u) = h(u)\gamma_0'(u) + g_1(u)\cos\gamma_0(u) + g_2 \sin\gamma_0(u) \qquad (6.1.17)$$

where $3\pi/2 \geq \gamma_0 \geq 0$. Suppose $3\pi/2 \geq \gamma_0(u) \geq \pi$ for all $u \geq u_0$, for some large u_0. Then $\gamma_0'(u) > 0$ for $u \geq u_0$, since if $\gamma_0'(u) \leq 0$, together with the fact that $\sin\gamma_0 \leq 0$ and $\cos\gamma_0 \leq 0$, equation (6.1.17) shows that $\gamma_0'' < 0$ at least until γ_0 has passed through π. But if $\gamma_0'(u) > 0$; $\lim_{u \to \infty} \gamma_0(u)$ exists and lies between $3\pi/2$ and π; equation (6.1.7) shows this is impossible. Hence γ_0 must pass through π and continue to do so every time γ_0 enters the range $(\pi, 3\pi/2)$.

Similarly if $\pi/2 \geq \gamma_0(u) \geq 0$ for all $u \geq u_0$ for some large u_0. Then $\gamma_0'(u) < 0$ for $u \geq u_0$, since if $\gamma_0'(u) \geq 0$; equation (6.1.17) shows that $\gamma_0'' > 0$ at least until γ_0 passes through $\pi/2$. But if $\gamma_0'(u) < 0$ for $u \geq u_0$; $\lim_{u \to \infty} \gamma_0(u)$ exists and equation (6.1.17) shows that this limit must equal 0.

If on the other hand $\pi \geq \gamma_0(u) \geq \pi/2$ for all $u \geq u_0$ for some large u_0. Then since $g_1(u) \to 0$ as $u \to \infty$, the only possibility is that $\lim_{u \to \infty} \gamma_0(u) = \pi$. For, if $\gamma_0' > 0$ for $u \geq u_0$; γ_0' cannot become negative until γ_0'' has become negative. And γ_0 must become increasingly close to π for this to occur. Inspection of equation (6.1.17) then shows γ_0 must remain close to π, or pass outside of $[\pi/2, \pi]$.

The only other possibility is that γ_0 continually oscillates about the downward vertical π, showing that one of (a), (b) or (c) hold.

If we assume either (a) or (c) holds, then there is a solution $\gamma = \gamma_T$ of equation (6.1.17) which passes close to π for large $u \leq T$. The idea is to compare γ with the solution of the equation with constant gravity and constant damping, to show that γ

cannot possibly equal 0 at time T.

Let us briefly consider the pendulum equation.

$$\beta''(u) = M\beta'(u) - m \sin\beta(u) \quad , \qquad (6.1.18)$$

where M and m are constant, and we measure the angle β in a clockwise direction from the downward vertical. Suppose $M < 0$ and $m > 0$, then the damping resists the motion, and a generic solution performs decreasing oscillations about the downward vertical. Indeed, comparison with the equation of a harmonic oscillator:

$$\omega''(u) = M\omega'(u) - m\,\omega(u) \quad ,$$

for small β, shows that β must oscillate continually about the downward vertical if $M^2 - 4m < 0$. If on the other hand $M > 0$ and $m > 0$, then the damping increases the acceleration in the direction of the motion. By making the substitution $u \to -u$, equation (6.1.18) becomes

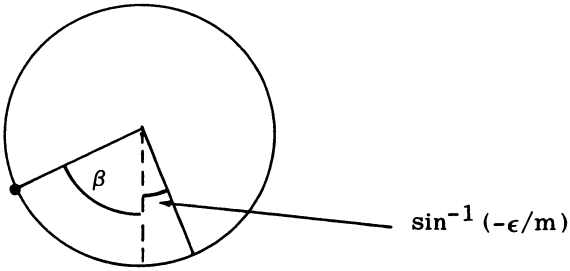
$$\beta''(u) = -M\beta'(u) - m \sin\beta(u) \quad ,$$

so that a generic solution β performs decreasing oscillations in backward time, i.e. β performs increasing oscillations as u increases. In this case β is either asymptotic to the position of unstable equilibrium Π, or β eventually moves around the circle in a single direction.

Now consider the equation

$$\beta''(u) = M\beta'(u) - m \sin\beta(u) - \epsilon \quad , \qquad (6.1.19)$$

where M, m, ϵ are all constant greater than 0, and we assume ϵ to be small. This represents the motion of a pendulum with constant gravity and damping and with a force of constant modulus tangent to the circular trajectory. A particular solution of equation (6.1.19) (which is approximately linear for small β) is $\beta(u) = \sin^{-1}(-\epsilon/m)$. And a generic solution performs decreasing oscillations about the equilibrium position $\beta = \sin^{-1}(-\epsilon/m)$ in backward time. So as u progresses a generic solution performs increasing oscillations about $\sin^{-1}(-\epsilon/m)$, and either tends to a position of unstable equilibrium at $\Pi + \sin^{-1}(-\epsilon/m)$, or eventually moves around the circle in a single direction.



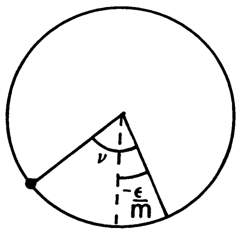
For a detailed account of equation (6.1.19) see [34, Ch.6, Sec.2].

Perform the substitution $\nu = \gamma - \Pi + \theta_0$ in equation (6.1.17), where $\theta_0 = \sin^{-1}(\epsilon/m)$, and the constants in equation (6.1.19) have yet to be chosen. Then equation (6.1.17) becomes

$$\nu''(u) = h(u)\,\nu'(u) - g_1(u)\cos(\nu - \theta_0) - g_2(u)\sin(\nu - \theta_0)\,. \qquad (6.1.20)$$

Similarly put $\eta = \beta + \theta_0$ into equation (6.1.19), which then becomes

$$\eta''(u) = M\,\eta'(u) - m\sin(\eta - \theta_0) - \epsilon\,. \qquad (6.1.21)$$



First of all assume that $m_2 = 1$ so that $\lim_{u \to \infty} h(u) = 0$. Assume also that ν passes through 0 for large u, say u_0. Let u_1 be the last time this occurs (since $\nu(T) = -\Pi + \theta_0$ this must exist). Let ν and η have the same initial conditions at u_1. Define the Wronskian $w = \nu'\eta - \nu\eta'$. Then

$$w' = \nu''\eta - \nu\eta''$$

$$= \nu\eta\left[\left(\frac{h\nu'}{\nu} - M\frac{\eta'}{\eta}\right) + \left(-g_2\frac{\sin(\nu-\theta_0)}{\nu} + m\frac{\sin(\eta-\theta_0)}{\eta}\right)\right]$$

$$w(u_1) = 0\,, \qquad w'(u_1) = 0\,.$$

Also, at u_1,

$$w'' = \eta'(\nu'' - \eta'')$$

$$= \eta'[(h-M)\eta' - g_1 \cos(-\theta_0) - g_2 \sin(-\theta_0) + m \sin(-\theta_0) + \epsilon] .$$

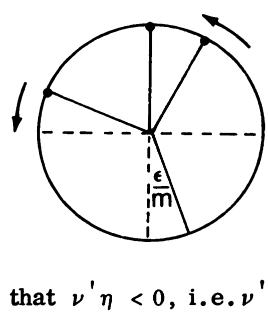
We now make careful choices of the constants M, m and ϵ. Remembering that u_0 can be chosen arbitrarily large, we choose M to be small so that $h(u) - M < 0$, m so that $-g_2(u) + m < 0$ and $\epsilon = g_1(u_0)$, where u_0 is chosen sufficiently large that $g_1(u) < g_1(u_0)$, for all $u \geq u_0$. Then, since $\eta'(u_1) < 0$, we must have $w''(u_1) > 0$. Therefore w' becomes < 0 immediately after u_1. Also, since $(\nu'' - \eta'')(u_1) > 0$; $\nu' - \eta' > 0$ and $\nu - \eta > 0$ immediately after u_1. That is $0 > \nu > \eta$. Now the function $x \rightarrow \sin x/x$ is increasing on $[-\Pi, 0]$, so $x \rightarrow \sin x/(x+a)$ is increasing on $[-\Pi, 0]$, i.e. $x \rightarrow \sin(x-a)/x$ is increasing on $[-\Pi + a, a]$. Thus

$$\sin(\nu - \theta_0)\nu \geq \sin(\eta - \theta_0)/\eta > 0 ,$$

immediately after u_1. Whence

$$- g_2 \frac{\sin(\nu - \theta_0)}{\nu} + m \frac{\sin(\eta - \theta_0)}{\eta} < 0 .$$

Now if $\nu'/\nu < \eta'/\eta$ with $\nu', \eta' < 0$ and $0 > \nu > \eta$, then $\nu' < \eta'$. So $0 > \nu > \eta$ continues, and w' continues to be < 0 at least until η' becomes zero for the first time. This must occur before η reaches $-\Pi + \theta_0$, since η is bounded above by $\Pi/2 + d$ on the previous swing, for some small constant d, and ϵ and M can be made arbitrarily small.



Then $w < 0$ here, so that $\nu'\eta < 0$, i.e. $\nu' > 0$ since $\eta < 0$. So ν' has become zero. But then equation (6.1.19) gives

$$\nu'' = - g_1 \cos(\nu - \theta_0) - g_2 \sin(\nu - \theta_0)$$

$$> - \epsilon \cos(\nu - \theta_0) - m \sin(\nu - \theta_0)$$

$$> 0 ,$$

provided $0 > \nu > -\pi/2 - 2\theta_0$. So that ν' then becomes positive. Now ν' cannot change sign before ν'', but ν'' remains positive at least until ν has passed through 0 again. This contradicts the definition of u_1.

Suppose now that $m_2 > 1$, and (c) occurs. Let $\nu_0 = \gamma_0 - \pi + \theta_0$, so ν_0 satisfies equation (6.1.20). Then a comparison of ν_0 with η similar to the above shows that either $\lim_{u \to \infty} \nu_0 = -\pi + \theta_0$, i.e. case (b) holds, or ν_0 passes through $\pi/2 + \theta_0$ or $-\pi + \theta_0$ in finite time (since η performs increasing oscillations) - contradiction.

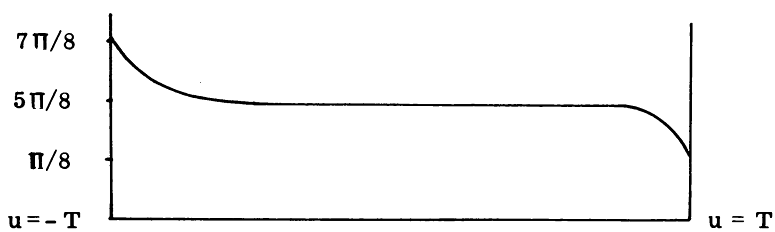
There remains the possibility that ν remains close to 0, but always remains < 0 for large u. In this case, comparison with the equation

$$\beta'' = M\beta' - m\beta ,$$

exactly as for Lemma 6.1.11, shows that ν must subsequently pass through 0. Thus (a) and (c) are impossible, and case (b) holds. Similarly, when $u \to -\infty$, $\alpha_0(u) \to 7\pi/8$. □

<u>Remark 6.1.15</u> If the damping conditions do not hold, then it is conceivable that the above methods of solving equation (6.1.2) and equation (6.1.12) lead in the first case to a trivial solution and in the second case to a solution asymptotic to $5\pi/8$. In the first case the graph of α_T for large T has the form

And in the second case the graph has the form



It is clear that if $\alpha_0 = \lim_{n \to \infty} \alpha_{T(n)}$, then $\alpha_0 \equiv 0$ in the first case and

$\lim_{u \to \infty} \alpha_0(u) = 5\pi/8$ in the second case.

6.2 Asymptotic estimates

In this section we prove the following.

Theorem 6.2.1 *Let α be a solution of any of the equations of cases (ii) - (iv) of Section 6.1, which we express in the form*

$$\alpha''(u) = h(u)\alpha'(u) + g(u)f(\alpha(u))f'(\alpha(u)) \quad .$$

If $a = \lim_{u \to -\infty} g(u)$ *and* $-b = \lim_{u \to -\infty} h(u)$, $a, b > 0$; *let* l *be the number such that* $a = l(l+b)$, *and assume that* $l \geq 1$. *Then for* u *close to* $-\infty$,

$$\text{(i)} \quad \alpha(u) = 0(e^{lu}) \; ,$$

$$\text{(ii)} \quad \alpha(u) = 0(e^{lu}) \; ,$$

and

$$\text{(iii)} \quad \alpha'(u) - l\alpha(u) = 0(e^{(l+2)u}) \; .$$

We demonstrate the theorem by proving a series of lemmas, dealing with each case separately. Again we follow closely the methods used by Smith in [36] for case (i).

Lemma 6.2.2 *If* α *is a solution of equation (6.1.3), then eventually* (u close to $-\infty$)

$$\alpha'(u) \geq (l - 0(e^{2u}))\sinh \alpha(u) \quad ,$$

where l *is the positive number satisfying* $a_2 = l(l + q - 2)$.

Proof Choose $u_1 \leq u_0$ such that $h(u_1)$ and $g(u_1)$ are close to their asymptotic values of $-(q-2)$ and a_2 respectively. For $u \leq u_1$, let $l(u)$ be the solution near l of the equation

$$(l(u)^2 - l(l + q - 2))/l(u) = h(u) = -(q-2) - 0(e^{2u}) \; .$$

Then $l(u) = l - 0(e^{2u})$. We wish to show $\alpha'(u) \geq l(u)\sinh \alpha(u)$.

Let β be the solution of the equation

$$\begin{cases} \beta'(v) = l(u)\sinh \beta(v) \\ \beta(u) = \alpha(u) \end{cases}$$

Then β is asymptotic to 0 as $v \to -\infty$. For $v \leq u$;

$$\beta''(v) = l(u)^2 \sinh \beta(v) \cosh \beta(v)$$
$$= h(u) \cosh \beta(v)\, \beta'(v) + a_2 \sinh \beta(v) \cosh \beta(v)$$
$$< h(v)\, \beta'(v) + a_2 \sinh \beta(v) \cosh \beta(v) . \qquad (6.2.8)$$

Now, suppose $\alpha'(u) < \beta'(u)$, then equation (6.2.8) shows that $\alpha''(u) > \beta''(u)$. Let $u_2 < u$ be the first time for which either $\alpha(u_2) = \beta(u_2), \alpha'(u_2) = \beta'(u_2)$ or $\alpha''(u_2) = \beta''(u_2)$. But if $\alpha'' > \beta''$ and $\alpha' < \beta'$ on $(u_2, u]$ with $\alpha(u) = \beta(u)$, then $\alpha'(u_2) < \beta'(u_2)$ and $\alpha(u_2) > \beta(u_2)$. Hence the only possibility is $\alpha''(u_2) = \beta''(u_2)$. However, $\alpha(u_2) > \beta(u_2)$ implies $\sinh \alpha \cosh \alpha > \sinh \beta \cosh \beta$ at u_2. Equation (6.2.8) shows this is impossible. Hence $u_2 = \infty$. Thus $\beta' - \alpha'$ is non-decreasing on $(-\infty, u]$ - a contradiction, since both $\alpha'(u)$ and $\beta'(u)$ tend to 0 as $u \to -\infty$. □

<u>Lemma 6.2.3</u> <u>Eventually</u> (u <u>close to</u> $-\infty$)

$$\alpha'(u) \le (1 + 0(e^{2u})) \sinh \alpha(u) \cosh \alpha(u) .$$

<u>Proof</u> Choose u_1 as in the previous lemma. For $u \le u_1$, let $l(u)$ be the solution near 1 of the equation,

$$(l(u)^2 - g(u))/l(u) = -(q-2) .$$

Then $l(u) = 1 + 0(e^{2u})$. We wish to show that $\alpha'(u) \le l(u) \sinh \alpha(u) \cosh \alpha(u)$ for $u \le u_1$.

Let $\beta(v)$ be the solution of the equation

$$\begin{cases} \beta'(v) = l(u) \sinh \beta(v) \cosh \beta(v) \\ \beta(u) = \alpha(u) . \end{cases}$$

Then β is asymptotic to 0 as $v \to -\infty$.

For $v \le u$,

$$\beta''(v) = l(u)^2 (\cosh^2 \beta(v) + \sinh^2 \beta(v)) \sinh \beta(v) \cosh (v)$$
$$\ge l(u)^2 \sinh \beta(v) \cosh \beta(v)$$
$$= \frac{(l(u)^2 - g(u))}{l(u)} \beta'(v) + g(u) \sinh \beta(v) \cosh \beta(v)$$
$$= -(q-2)\, \beta'(v) + g(u) \sinh \beta(v) \cosh \beta(v)$$
$$> h(v)\, \beta'(v) + g(u) \sinh \beta(v) \cosh \beta(v) . \qquad (6.2.9)$$

Now suppose $\alpha'(u) > \beta'(u)$, then equation (6.2.9) shows that $\alpha''(u) < \beta''(u)$. This yields a contradiction as in the previous proof. □

Lemma 6.2.4 There exist positive constants b_1 and b_2 such that for u sufficiently close to $-\infty$.

$$b_1 e^{lu} \le \sinh\alpha(u) \le b_2 e^{lu}$$

Proof We know eventually that

$$\alpha'(u) \ge 1^-(u)\sinh\alpha(u) ,$$

where $1^-(u) = 1 - 0(e^{2u})$. Then for $v \le u$,

$$\begin{aligned}\alpha'(v) &\ge 1^-(v)\sinh\alpha(v) \\ &\ge 1^-(u)\sinh\alpha(v) ,\end{aligned}$$

so α lies below the solution of

$$\begin{cases}\beta'(v) = 1^-(u)\sinh\beta(v) \\ \beta(u) = \alpha(u) ,\end{cases} \tag{6.2.10}$$

giving $\sinh\alpha(v) \le \sinh\beta(v)$ for $v \le u$.

The explicit solution of equation (6.2.10) is given by

$$\operatorname{cosech}\beta(v) + \coth\beta(v) = (\operatorname{cosech}\alpha(u) + \coth\alpha(u))\exp(1^-(u)(u - v)) .$$

Now the left-hand side is equal to $\sinh\beta(v)/(1 + \cosh\beta(v))$. We can assume u is sufficiently small that $\cosh\alpha(u) < 2$ say.

Then

$$\begin{aligned}\sinh\beta(v) &= (1 + \cosh\beta(v))\, c_u \exp(1^-(u)v) \\ &< 3c_u \exp(1^-(u)v) ,\end{aligned}$$

where c_u is a constant depending on u.

Hence

$$\sinh\alpha(v) \le d_u \exp(1^-(u)v) , \tag{6.1.11}$$

for some constant d_u depending on u.

Let $f(u) = \sinh \alpha(u)$.

Then

$$\begin{aligned} f'(u) &= \sinh \alpha(u)\ \alpha'(u) \\ &\leq l^+(u) \sinh \alpha(u) \cosh^2 \alpha(u) \\ &= (1 + \sinh^2 \alpha(u))\, l^+ f \\ &\leq (1 + \sinh \alpha(u))\, l^+ f \\ &= l^+ f + l^+ f^2 , \end{aligned}$$

where $l^+(u) = l + 0(e^{2u})$, and we assume that u is sufficiently small that $\sinh \alpha(u) < 1$,

From equation (6.2.11) we have that

$$\sinh^2 \alpha(u) \leq b_0 e^{\mu l u} ,$$

where b_0 and μ are constants with $\mu > 1$. Hence

$$f'(u) \leq l^+ f + b_0 e^{\mu l u}$$

We estimate a solution of the equation

$$\begin{cases} g'(v) = l^+ g(v) + b_0 e^{\mu l v} \\ g(u) = f(u) , \end{cases}$$

so that $g(v) \leq f(v)$, for $v \leq u$. The solution of the homogeneous part is given by

$$\begin{aligned} g_H(v) &= \tilde{c} \exp\left(\int_0^v l^+(s)\, ds\right) \\ &= \tilde{c} \exp lv \exp\left(\int_0^v 0(e^{2s})\, ds\right) \\ &\geq \hat{c}\, e^{lv} , \end{aligned}$$

since the integral is uniformly bounded, where we assume $\hat{c}$ is a negative constant.

A particular solution of the inhomogeneous equation is

$$g_I(v) = g_H(v) \int_0^v \exp\left(- \int_0^s l^+(x)\, dx\right) \exp(\mu l s)\, ds .$$

(Note that in order that $g_I(u) = f(u) > 0$, we must choose $\hat{c}$ above to be negative).

Then

$$g_I(v) \geq g_H(v) \int_0^v e^{(\mu-1)ls}\,ds$$

$$\geq - \hat{c}\, e^{lv} .$$

Hence $g_I(v) \geq de^{lv}$, for some constant d. So $f(v) \geq b_1 e^{lv}$, for $v \leq u$.

For the other half of the inequality, we have that

$$f'(u) = \cosh \alpha(u)\, \alpha'(u)$$

$$\geq \cosh \alpha(u)\, l^-(u) \sinh \alpha(u)$$

$$\geq l^-(u) f(u) .$$

If g is a solution of

$$\begin{cases} g'(v) = l^-(v) g(v) \\ g(u) = f(u) , \end{cases}$$

then

$$g(v) = c \exp \int_0^v l^-(x)\,dx$$

$$= c \exp \int_0^v (l - 0(e^{2x}))\,dx$$

$$= c\, e^{lv} \exp\left(\int_0^v - 0(e^{2x})\,dx\right)$$

$$= d\, e^{lv} ,$$

where c and d are constants. Hence $f(v) \leq g(v) \leq b_2 e^{lv}$, for $v \leq u$. □

The equations of cases (iii) and (iv) are very similar. We prove the results for equation (6.1.9) and similar considerations apply to equations (6.1.5), (6.1.6) and (6.1.8).

<u>Lemma 6.2.5</u> <u>If</u> α <u>is a solution of equation (6.1.9), then eventually</u> (u <u>close to</u> $-\infty$)

$$\alpha'(u) \leq (l - 0(e^{2u})) \sin \alpha(u) ,$$

<u>where</u> l <u>is defined as for Theorem 6.2.1.</u>

<u>Proof</u> Choose $u_1 \leq u_0$ such that $h(u_1)$ and $g(u_1)$ are close to their asymptotic values of $-b$ and $(m - 3)$ respectively. For $u \leq u_1$, let $l(u)$ be the solution near l of the equation,

$$(l(u)^2 - g(u))/l(u) = -b\ .$$

Then $l(u) = 1 - 0(e^{2u})$. We wish to show that $\alpha'(u) \leq l(u)\sin\alpha(u)$.

Let $\beta(v)$ be the solution of the equation

$$\begin{cases} \beta'(v) = l(u)\sin\beta(v) \\ \beta(u) = \alpha(u)\ . \end{cases}$$

Then β is asymptotic to 0 as $v \to -\infty$. For $v \leq u$,

$$\begin{aligned} \beta''(v) &= l(u)^2 \sin\beta(v)\cos\beta(v) \\ &= \frac{l(u)^2 - g(u)}{l(u)} \cdot \cos\beta(v)\,\beta'(v) + g(u)\sin\beta(v)\cos\beta(v) \\ &= -b\cos\beta(v)\,\beta'(v) + g(u)\sin\beta(v)\cos\beta(v) \\ &> h(v)\,\beta'(v) + g(u)\sin\beta(v)\cos\beta(v)\ . \end{aligned} \qquad (6.1.12)$$

Now suppose $\alpha'(u) > \beta'(u)$, then equation (6.1.12) shows that $\alpha''(u) < \beta''(u)$. The proof now concludes as for Lemma 6.2.2. □

Lemma 6.2.6 Eventually (u close to $-\infty$),

$$\alpha'(u) \geq (1 - 0(e^{2u}))\sin\alpha(u)\cos\alpha(u)\ .$$

Proof Choose u_1 as before. For $u \leq u_1$, let $l(u)$ be the solution near 1 of the equation

$$(l(u)^2 - g(u))/l(u) = h(u)\ .$$

Then $l(u) = 1 - 0(e^{2u})$ (there being two contributions to $-0(e^{2u})$, one from $g(u)$ and the other from $h(u)$). We wish to show that $\alpha'(u) \geq l(u)\sin\alpha(u)\cos\alpha(u)$.

Let β be the solution of the equation

$$\begin{cases} \beta'(v) = l(u)\cos\beta(v)\sin\beta(v) \\ \beta(u) = \alpha(u)\ . \end{cases}$$

Then β is asymptotic to 0 as $v \to -\infty$. For $v \leq u$,

$$\begin{aligned}
\beta''(v) &= l(u)^2(\cos^2\beta(v) - \sin^2\beta(v))\cos\beta(v)\sin\beta(v) \\
&\leq l(u)^2\cos\beta(v)\sin\beta(v) \\
&= \frac{(l(u)^2 - g(u))}{l(u)}\beta'(v) + g(u)\cos\beta(v)\sin\beta(v) \\
&= h(u)\beta'(v) + g(u)\cos\beta(v)\sin\beta(v) \\
&\leq h(v)\beta'(v) + g(u)\cos\beta(v)\sin\beta(v)\ . \qquad (6.1.13)
\end{aligned}$$

Now suppose $\alpha'(u) < \beta'(u)$, then equation (6.1.13) shows that $\alpha'' > \beta''(u)$. The proof concludes as before. □

Lemma 6.2.7 There exist positive constants b_1 and b_2 such that for u sufficiently close to $-\infty$;

$$b_1 e^{lu} \leq \sin\alpha(u) \leq b_2 e^{lu}\ .$$

Proof The proof is similar to that for Lemma 6.2.4, using the estimates for α' obtained in Lemma 6.2.5 and Lemma 6.2.6. □

This concludes the proof of Theorem 6.2.1.

6.3. Smoothness of certain equivariant harmonic maps

Recall that a solution to the reduction equation gives a smooth harmonic map from M^* to N^*. In the case when this map extends to a map from M to N, we need to know this extension is smooth at points of $M\setminus M^*$.

Theorem 6.3.1 The harmonic maps which arise from solutions of the equations of cases (i) - (v) all extend to smooth harmonic maps across $M\setminus M^*$, where M is the appropriate domain depending on which case is being considered.

In cases (i) and (iv), this theorem can most readily be derived from the following.

Suppose M and N are compact, and let $i : N \rightarrow V$ be an isometric immersion of N into a Euclidean space V. The completion of the space of smooth maps $\Phi : M \rightarrow V$ with norm

$$|\Phi| = \left(\int_M (|\Phi(x)|^2 + |\nabla\Phi(x)|^2)\,dx\right)^{\frac{1}{2}}\ ,$$

is a Banach space, denoted by $L_1^2(M,V)$.

The subset $L_1^2(M,N) = \{\Phi \in L_1^2(M,V);\ \Phi(M) \subset N\}$ is a smooth manifold. The energy integral (c.f. Remark 1.1.4) defines a function $E: L_1^2(M,N) \to \mathbb{R}$. There are extrema of E which are not C^0 [10]. However, Hildebrandt has shown that any extremal of E which is in $L_1^2 \cap C^0$ is smooth. Since the Laplacian is the Euler-Lagrange operator associated to the energy integral E, it is clear that the maps constructed in cases (i) and (v) are extrema of E. Also these maps lie in $L_1^2 \cap C^0$, and hence they are smooth.

In cases (ii), (iii) and (iv) the domain is no longer compact, and in general the energy is no longer finite. However, we can now appeal to the regularity theorem of Eells and Sampson [12], which states that any C^2-map between smooth manifolds which is harmonic is smooth. It suffices therefore to show the maps are C^2 at points of $M \setminus M^*$. We demonstrate this by proving a series of lemmas, using the estimates of the last section. We remark that we are again adapting the programme of Smith [36] which he used for the maps of case (i) (Hildebrandt's result was not known at that time).

Using the notations of Chapter 4, let $V = M \setminus M^*$ be the focal variety of the isoparametric function $s: M \to \mathbb{R}$ corresponding to $s = 0$ (or $u = -\infty$), and let $W \subset N$ be the focal variety or isoparametric hypersurface of N onto which V is mapped under ϕ. For a point $x \in V$ we pick suitable C^2 charts about x and $\phi(x)$. These charts can be thought of as generalized Fermi coordinates (see for example [1]), and they will be chosen in such a way as to induce an equivariant map between subsets of Euclidean spaces.

Since V is a smooth submanifold (Theorem 2.2.2), there is a smooth chart (β_1, B_1) about x in V, with $\beta_1: B_1 \to \mathbb{R}^p$ ($p = \dim V$). We can form a tube $T_\epsilon(B_1)$ about B_1, which through each point of B_1 consists of the union of geodesics subtended by a certain small sphere. More precisely, recalling the notation of Chapter 4, let $M_{s_0} = s^{-1}(s_0)$, so $V = M_0$. Let $\pi: M^* \to V$ be the projection down normal geodesics. Then $T_\epsilon(B_1) = B_1 \cup (\pi^{-1}(B_1) \cap (\bigcup_{s=0}^{\epsilon} M_s))$. We can choose coordinates for $T_\epsilon(B_1)$ by defining a map ψ_1 from $T_\epsilon(B_1)$ to the set $E_1 = \{(x, sy);\ x \in \beta_1(B_1),\ y \in S^{q-1},\ s \in [0,\epsilon)\} \subset \mathbb{R}^m$. Indeed M_s is a sphere bundle over V ($s \neq 0$), and we choose B_1 sufficiently small such that the induced bundle over B_1 is a product bundle:

$M_s = B_1 \times sS^{q-1}$. This defines the diffeomorphism $\psi_1 : T_\epsilon(B_1) \to E_1$, giving the desired chart (ψ_1, E_1). Note that ψ_1 is equivariant with respect to isoparametric functions. Similarly we can construct a chart (ψ_2, E_2) about $\phi(x)$, where $E_2 = \{(u, tv);\ u \in \mathbb{R}^r,\ v \in S^{s-1},\ t \in [0, \delta)\}$. With respect to these charts the equivariant map ϕ induces an equivariant map $\psi : E_1 \to E_2$ of the form

$$\psi(x, sy) = (g_2(x),\ \alpha(s)\, g_1(y)),$$

where $s \in [0, \epsilon)$ and $\alpha(0) = 0$. Also g_2 is a smooth map and g_1 is a homogeneous polynomial map of degree l. We must check that the map ψ between Euclidean spaces is C^2 when α satisfies the estimates of Section 6.2 .

Rewrite ψ as

$$\psi(x, y) = (g_2(x),\ \alpha(\log|y|)\, g_1(y/|y|)),$$

where $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q (m = p + q)$, and we have changed the parametrization from s to u where $e^u = s$. Then it is sufficient to check that the map H defined by

$$H(y) = \alpha(\log|y|)\, g_1(y/|y|),$$

extends over $|y| = 0$ to a C^2 map.

Write H as

$$H(y) = R^{\frac{1}{2}}(y)\, g_1(y),$$

where $R(y) = \alpha(\log|y|)^2 / |y|^{2l}$. Furthermore write

$$R = r \circ \rho,$$

where $\rho(y) = |y|^2 = v$ say, and $r(v) = \alpha(\tfrac{1}{2}\log v)^2 / v^l$.

<u>Lemma 6.3.2</u> <u>For</u> v <u>near</u> 0 , <u>the first derivative of</u> r <u>is uniformly bounded whilst</u> d^2r/dv^2 <u>is at worst of</u> $O(1/v)$.

<u>Proof</u> The derivative

$$dr/dv = \alpha(\alpha' - l\alpha)/v^{l+1} .$$

From Theorem 6.2.1 we see that $\alpha' - l\alpha = O(e^{(l+2)u})$ and $\alpha = O(e^{lu})$, hence $\alpha(\alpha' - l\alpha) = O(e^{2(l+1)u}) = O(v^{l+1})$ as required. The second derivative

$$d^2r/dv^2 = (\alpha(\alpha'' - l\alpha') + (\alpha' - 2(l+1)\alpha)(\alpha' - l\alpha))/2v^{l+2} .$$

Now $\alpha' - 2(l+1)\alpha = O(e^{lu})$ and $\alpha' - l\alpha = O(e^{(l+2)u})$, hence $(\alpha' - 2(l+1)\alpha)(\alpha' - l\alpha) = O(e^{2(l+1)u}) = O(v^{l+1})$, as required. For the first term on the right hand side we note that

$$\alpha'' = -b\alpha' + l(l+b)\alpha + O(e^{(l+2)u}) .$$

Hence

$$\begin{aligned}\alpha(\alpha'' - l\alpha') &= \alpha(-b\alpha' - l(l+b)\alpha) + O(e^{2(l+1)u}) \\ &= \alpha(l+b)(-\alpha' + l\alpha) + O(e^{2(l+1)u}) \\ &= O(e^{2(l+1)u}) \\ &= O(v^{l+1}),\end{aligned}$$

as required. □

<u>Lemma 6.3.3</u> (a) <u>As</u> $y \to 0$, $\partial R/\partial y_i \to 0$ $(y = (y_1, \ldots, y_q))$, <u>whilst all second derivatives remain bounded.</u>

(b) <u>Similarly for</u> $R^{\frac{1}{2}}$.

<u>Proof</u> (a) $R(y) = r \circ \rho(y)$, where $\rho(y) = |y|^2$, hence

$$\partial R/\partial y_i = 2y_i \, \partial r/\partial v ,$$

$$\frac{\partial^2 R}{\partial y_i \, \partial y_j} = \frac{\partial^2 r}{\partial v^2} \cdot 4 y_i y_j + 2\delta_{ij} \frac{\partial r}{\partial v} .$$

The conclusion now follows from Lemma 6.3.2 .

(b) It suffices to show that $R^{\frac{1}{2}}$ is bounded away from 0 as $y \to 0$. But by Theorem 6.2.1 (i) $R^{\frac{1}{2}} = \alpha(u)/|y|^l \geq b_1 e^{lu}/|y|^l = b_1$. □

<u>Lemma 6.3.4</u> H <u>extends to a</u> C^2 <u>function</u> .

<u>Proof</u> $H(y) = R^{\frac{1}{2}}(y) g_1(y)$, so that

$$\frac{\partial H}{\partial y_i} = \frac{\partial R^{\frac{1}{2}}}{\partial y_i} \cdot g_1 + R^{\frac{1}{2}} \frac{\partial g_1}{\partial y_i}$$

is continuous at $y = 0$ by Lemma 6.3.3 . Also

$$\frac{\partial^2 H}{\partial y_i \partial y_j} = \frac{\partial^2 R^{\frac{1}{2}}}{\partial y_i \partial y_j} \cdot g_1 + \frac{\partial R^{\frac{1}{2}}}{\partial y_i} \cdot \frac{\partial g_1}{\partial y_j} + \frac{\partial R^{\frac{1}{2}}}{\partial y_j} \cdot \frac{\partial g_1}{\partial y_i} + R^{\frac{1}{2}} \frac{\partial^2 g_1}{\partial y_i \partial y_j} ,$$

is again continuous at $y = 0$ by Lemma 6.3.3 and by the fact that g_1 is homogeneous and hence $g_1(0) = 0$. Indeed the first three terms tend to 0 as $y \to 0$. □

7 The general theory of harmonic morphisms

7.1 General theory

We were briefly introduced to harmonic morphisms in Section 1.3. Now we will give a much more detailed treatment.

Harmonic morphisms have been extensively studied by Fuglede and Ishihara [19,27], and we will now describe some of their results.

Let $\phi : (M,g) \to (N,h)$ be a map of Riemannian manifolds. If $x \in M$ is such that $d\phi_x \neq 0$, then T_xM can be decomposed into $\mathscr{V}_x = \ker d\phi_x$ and $\mathscr{H}_x$ the orthogonal complement with respect to g. The map ϕ is called horizontally conformal if, for all $x \in M$ where $d\phi_x \neq 0$; $d\phi_x|_{\mathscr{H}_x} : \mathscr{H}_x \to T_{\phi(x)}N$ is conformal and surjective. That is, for all $X, Y \in \mathscr{H}_x$, $h(d\phi_x(X), d\phi_x(Y)) = \lambda^2(x).g(X,Y)$, for some function $\lambda : M \to \mathbb{R}$, called the dilation of ϕ.

Theorem 7.1.1 [19,27] A map $\phi : (M,g) \to (N,h)$ is a harmonic morphism if and only if ϕ is harmonic and horizontally conformal.

Thus, for each $x \in M$, $d\phi_x$ either has maximal rank or $d\phi_x = 0$. Denote the critical set of ϕ by $C_\phi = \{x \in M;\ d\phi_x = 0\}$. The function $\lambda^2 : M \to \mathbb{R}$ is smooth and $C_\phi = (\lambda^2)^{-1}(0)$. The critical set C_ϕ is a polar set; see [19]. Roughly speaking this means that codim $C_\phi \geq 2$ in M. We remark also that ϕ is an open mapping.

Remark 7.1.2 If C_ϕ is non-empty; one can easily see from the condition that $d\phi_x = 0$ on C_ϕ, that each connected component of C_ϕ is mapped under ϕ to a single point of N.

For maps into $\mathbb{R}^n$, where $\mathbb{R}^n$ has the standard metric $\langle\ ,\ \rangle$, the condition of harmonicity and horizontal conformality become the following.

Theorem 7.1.3 [19] If $\phi : (M,g) \to (\mathbb{R}^n, \langle\ ,\ \rangle)$, then ϕ is a harmonic morphism if and only if the components $\phi_1, \ldots, \phi_n$ of ϕ are harmonic on M, and their gradients are mutually orthogonal and of equal length $\lambda(x)$ at each point $x \in M$:

$$g(\nabla\phi_k, \nabla\phi_l) = \lambda^2 \delta_{kl}, \text{ for all } k,l = 1, \dots, n. \quad (7.1.1)$$

Example 7.1.4 Any Riemannian submersion is horizontally conformal with $\lambda^2 = 1$. By a result of Eells and Sampson [12], a Riemannian submersion is harmonic if and only if the fibres are minimal submanifolds.

Example 7.1.5 Define $\phi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ by $\phi(x) = x/|x|$, for all $x \in \mathbb{R}^n \setminus \{0\}$. Then ϕ is a harmonic morphism with dilation λ given by $\lambda^2(x) = 1/|x|^2$.

Example 7.1.6 Let ϕ be multiplication of real, complex, quaternionic or Cayley numbers; $\phi : \mathbb{R}^{2n} \to \mathbb{R}^n$, $\phi(u,v) = uv$, where $n = 1,2,4$ or 8 respectively. Then ϕ is a harmonic morphism from Theorem 7.1.3 with dilation λ given by $\lambda^2(x) = |x|^2$. The critical set C_ϕ consists of the point 0 in $\mathbb{R}^n$.

Example 7.1.7 Let $\phi_1 : M \to N$ and $\phi_2 : N \to P$ be harmonic morphisms with dilations λ_1 and λ_2 respectively. Then, from equation (1.1.3), $\phi_2 \circ \phi_1 : M \to P$ is a harmonic morphism with dilation λ given by $\lambda^2(x) = \lambda_2^2(\phi_1(x)) . \lambda_1^2(x)$, for all $x \in M$.

In view of Example 7.1.4 we might expect to find conditions when the fibres of a harmonic morphism are minimal submanifolds. This is what we consider in the next theorem.

Theorem 7.1.8 Let $\phi : (M,g) \to (N,h)$ be a submersion which is a harmonic morphism. Then (setting $n = \dim N$)

(a) If $n = 2$, the fibres are minimal submanifolds

(b) if $n \neq 2$, the following properties are equivalent:

(i) the fibres are minimal submanifolds ;

(ii) $\nabla \lambda^2$ is vertical, where λ is the dilation of ϕ ;

(iii) the horizontal distribution has mean curvature $\nabla\lambda^2 / 2\lambda^2$.

Proof First of all we have $e(\phi) = n\lambda^2/2$, consequently the stress-energy tensor S_ϕ is given by

$$S_\phi = \tfrac{1}{2} n \lambda^2 . g - \phi^* h . \quad (7.1.2)$$

Take a point $x \in M$ and an orthonormal frame field $(X_a)_{a=1,\dots,m}$ near x with $X_1, \dots, X_n$ horizontal and $X_{n+1}, \dots, X_m$ vertical. Use the following ranges of indices:

$1 \le a,b, \ldots \le m;\ 1 \le i,j, \ldots \le n;\ n+1 \le r,s,\ldots \le m$.

The map $\emptyset$ is harmonic so $\nabla^* S_\emptyset = 0$; therefore, summing over repeated indices,

$$0 = (\nabla_{X_a} S_\emptyset)(X_b, X_a)$$

$$= \tfrac{1}{2} n X_b(\lambda^2) - (X_a(\emptyset^* h(X_b, X_a)) - \emptyset^* h(\nabla_{X_a} X_b, X_a)$$

$$- \emptyset^* h(X_b, \nabla_{X_a} X_a) . \qquad (7.1.3)$$

Since the frame is orthonormal

$$0 = X_i g(X_j, X_i)$$

$$= g(\nabla_{X_i} X_j, X_i) + g(X_j, \nabla_{X_i} X_i)$$

$$= g(\mathscr{H} \nabla_{X_i} X_j, X_i) + g(X_j, \mathscr{H} \nabla_{X_i} X_i)$$

$$= (1/\lambda^2)(\emptyset^* h(\nabla_{X_i} X_j, X_i) + \emptyset^* h(X_j, \nabla_{X_i} X_i)) , \qquad (7.1.4)$$

where $\mathscr{H}$ denotes horizontal projection.

Choose $X_b = X_j$, then $X_a(\emptyset^* h(X_j, X_a)) = X_j(\lambda^2)$, and using equation (7.1.4); equation (7.1.3) becomes

$$0 = \tfrac{1}{2}(n-2) X_j(\lambda^2) + \emptyset^* h(\nabla_{X_r} X_j, X_r) + \emptyset^* h(X_j, \nabla_{X_r} X_r)$$

$$= \tfrac{1}{2}(n-2) X_j(\lambda^2) + \lambda^2 g(X_j, \nabla_{X_r} X_r)$$

$$= \tfrac{1}{2}(n-2) X_j(\lambda^2) - \lambda^2 (m-n) \text{ (mean curvature of fibre in the } X_j \text{ direction)} .$$

$$(7.1.5)$$

Thus we have proved (a), and (i) if and only if (ii) in (b).

Now choose $X_b = X_r$. Equation (7.1.3) becomes

$$0 = \tfrac{1}{2} n X_r(\lambda^2) - \emptyset^* h(\nabla_{X_i} X_r, X_i)$$

$$= \tfrac{1}{2} n X_r(\lambda^2) - \lambda^2 g(\mathscr{H} \nabla_{X_i} X_r, X_i)$$

$$= \tfrac{1}{2} n X_r(\lambda^2) + \lambda^2 g(\nabla_{X_i} X_i, X_r) \qquad (7.1.6)$$

$$= \tfrac{1}{2} n X_r(\lambda^2) - \lambda^2 n \text{ (mean curvature of horizontal distribution in } X_r \text{ direction).}$$

We now choose X_r to be in the direction of the vertical projection of $\nabla\lambda^2$, and we obtain (ii) if and only if (iii) in (b). □

To what extent can we regard Theorem 7.1.8 as being valid for arbitrary harmonic morphisms (i.e. allowing $C_\emptyset$ to be non-empty)? In fact we can remove the "submersion" condition in the statement of Theorem 7.1.8 on account of the following recent result of Fuglede.

Theorem 7.1.9 [20] If $\nabla\lambda^2$ is vertical, where $\lambda : M \to \mathbb{R}$ is the dilation of a harmonic morphism $\emptyset : (M,g) \to (N,h)$, then $C_\emptyset$ is the empty set.

7.2 Examples and non-examples of harmonic morphisms

Example 7.2.1 The Hopf maps defined as follows are all harmonic morphisms. Let F denote either the real, complex, quaternionic or Cayley numbers, and let $n = \dim F$. Define $\emptyset : \mathbb{R}^{2n} \to \mathbb{R}^{n+1}$ by

$$\emptyset(x,y) = (|x|^2 - |y|^2, 2xy), \tag{7.2.1}$$

for all $x,y \in F$. Then $\emptyset$ is a harmonic morphism with dilation λ given by $\lambda^2((x,y)) = |(x,y)|^2$, for all $x,y \in F$. Also $|\emptyset(z)|^2 = |z|^4$, for $z \in R^{2n}$, so the fibres are all compact, and being in $\mathbb{R}^{2n}$ cannot be minimal. Thus the condition (b) of Theorem 7.1.8 is indeed restrictive.

Example 7.2.2 Isoparametric families of hypersurfaces with 2 distinct principal curvatures give rise to harmonic morphisms.

Let M be $\mathbb{R}^m$, and express $z \in \mathbb{R}^m$ in the form $z = (x, s.y)$, where $x \in \mathbb{R}^p$, $y \in S^q$, $p + q = m$ and $s \in [0,\infty)$. Then the level surfaces s = constant form an isoparametric family (c.f. Example 2.1.4). The map $\emptyset : \mathbb{R}^m \to \mathbb{R}^p$; $\emptyset(z) = x$ defines a harmonic Riemannian submersion.

Let M be S^{m-1}, and express $z \in S^{m-1}$ as $z = (\cos s.x, \sin s.y)$ where $x \in S^{p-1}$, $y \in S^{q-1}$, $p+q = m$ and $s \in [0, \pi/2]$. The level surfaces s = constant form the isoparametric family of Example 2.1.5. Define $\emptyset : S^m \setminus S^q \to S^p$ by $\emptyset(z) = x$. Then $\emptyset$ is a harmonic morphism with dilation λ given by $\lambda^2(z) = 1/\cos^2 s$. Here $\nabla\lambda^2$ is vertical and the fibres are minimal.

Let M be H^{m-1}, and express each $z \in H^{m-1}$ as $z = (\cosh s.x, \sinh s.y)$, where $x \in H^{p-1}$, $y \in S^{q-1}$, $p+q = m$ and $s \in [0,\infty)$. The level surfaces s = constant form the isoparametric family of Example 2.1.6 . Define $\phi : H^{m-1} \to H^{p-1}$ by $\phi(z) = x$. Then ϕ is a harmonic morphism with dilation given by $\lambda^2(z) = 1/\cosh^2 s$. Also the map $\psi : H^{m-1} \setminus H^{p-1} \to S^{q-1}$ defined by $\psi(z) = y$ is a harmonic morphism with dilation μ given by $\mu^2(z) = 1/\sinh^2 s$.

All the above harmonic morphisms are applications of Proposition 2.2.5 .

Example 7.2.3 Isoparametric families of hypersurfaces on a space form M with 1 distinct principal curvature give rise to the following harmonic morphisms.

Let M_c be a particular hypersurface in the family, and let V be the focal variety; then $M \setminus V$ is connected. Define $\phi : M \setminus V \to M_c$ to be projection down the normal geodesics. Then ϕ is a harmonic morphism with geodesics as fibres. Conversely we have the following.

Example 7.2.4 Suppose $\phi : (M,g) \to (N,h)$, $\dim M = \dim N + 1$, is a harmonic morphism, and $\nabla\lambda^2$ is vertical, where λ is the dilation of ϕ. Then from Theorem 7.1.8, the integral curves of $\nabla\lambda^2$ are geodesics, and so $|\nabla\lambda^2|^2$ is a function of λ^2 (one can see this by reversing the proof of Lemma 2.1.1). The horizontal distribution is integrable in this case, the integral submanifolds being the level surfaces of λ^2. Thus property (b) (iii) of Theorem 7.1.8 implies the mean curvature of these level surfaces is a function of λ^2 - otherwise said, λ^2 is a generalized isoparametric function.

Example 7.2.5 Consider the tangent bundle (TM,G) of some Riemannian manifold (M,g), where G is the Sasaki metric of Example 2.4.7 . Let T^1M be the unit sphere bundle:

$$T^1M = \{(x,v) \in TM;\ |v|^2 = 1\},$$

endowed with the metric induced from G. Then, as a consequence of Example 2.4.7 and Example 7.2.3, $\phi : TM \setminus (\text{zero section}) \to T^1M$; $\phi(x,v) = (x, v/|v|)$ is a harmonic morphism. The dilation λ of ϕ is given by $\lambda^2(x,v) = 1/|v|^2$.

Example 7.2.6 In Example 7.1.6 we saw that real, complex, quaternionic and Cayley multiplication are all examples of harmonic morphisms. Such multiplications are all orthogonal multiplications; i.e. a bilinear map $\phi : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ such that $|\phi(x,y)| = |x|.|y|$, for all $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$. Which orthogonal multiplications are harmonic morphisms? It turns out that the multiplications of Example 7.1.6 are the only ones:

Theorem 7.2.7 If $\phi : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n$ is an orthogonal multiplication, then ϕ is a harmonic morphism if and only if $p = q = n$ and $n = 1,2,4$ or 8.

Proof Suppose ϕ is a harmonic morphism. Since ϕ is an orthogonal multiplication

$$|\phi(x,y)| = |x|.|y| , \tag{7.2.2}$$

for all $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. Fix $x_0 \in \mathbb{R}^p$ and $y_0 \in \mathbb{R}^q$, then $\phi_{y_0} : \mathbb{R}^p \to \mathbb{R}^n$ given by $\phi_{y_0}(x) = \phi(x,y_0)$ and $\phi_{x_0} : \mathbb{R}^q \to \mathbb{R}^n$ given by $\phi_{x_0}(y) = \phi(x_0,y)$ are both linear. Therefore

$$d(\phi_{y_0})_{x_0}(u_1) = \phi_{y_0} u_1 , \tag{7.2.3}$$

for all $u_1 \in \mathbb{R}^p$, and has norm

$$|d(\phi_{y_0})_{x_0}(u_1)| = |u_1|\ |y_0| . \tag{7.2.4}$$

Similarly

$$|d(\phi_{x_0})_{y_0}(u_2)| = |u_2|\ |x_0| , \tag{7.2.5}$$

for all $u_2 \in \mathbb{R}^p$.

Now

$$d\phi_{(x_0,y_0)}(v_1,v_2) = d(\phi_{y_0})_{x_0}(v_1) + d(\phi_{x_0})_{y_0}(v_2) , \tag{7.2.6}$$

for all $(v_1,v_2) \in \mathbb{R}^p \times \mathbb{R}^q$. Thus

$$\ker d\phi_{(x_0,y_0)} = \ker(d\phi_{y_0})_{x_0} \oplus \ker(d\phi_{x_0})_{y_0} \oplus \{(v_1,v_2) \in \mathbb{R}^p \times \mathbb{R}^q ;$$
$$d(\phi_{y_0})_{x_0}(v_1) = d(\phi_{x_0})_{y_0}(v_2)\} . \tag{7.2.7}$$

From equations (7.2.4) and (7.2.5) we see that $\ker(d\phi_{y_0})_{x_0} = \{0\}$ and $\ker(d\phi_{x_0})_{y_0} = \{0\}$ whenever x_0 and y_0 are non-zero, and so $p,q \geq n$. Henceforth assume $x_0, y_0 \neq 0$.

From equation (7.2.7)

$$\ker d\phi_{(x_0,y_0)} = \{(v_1,v_2) \in \mathbb{R}^p \times \mathbb{R}^q ;\ d(\phi_{y_0})_{x_0} v_1 = -d(\phi_{x_0})_{y_0} v_2\} . \quad (7.2.8)$$

Let $L = d(\phi_{y_0})_{x_0} \mathbb{R}^p \cap d(\phi_{x_0})_{y_0} \mathbb{R}^q$. Since $d\phi_{(x_0,y_0)}$ is surjective; $\dim L = \dim \ker(d\phi)_{(x_0,y_0)} = p + q - n$. Let $\Pi : \mathbb{R}^n \to L$ be the projection map, and define

$$L_{y_0} = (\ker(\Pi \circ \phi_{y_0}))^{\perp} .$$

$$L_{x_0} = (\ker(\Pi \circ \phi_{x_0}))^{\perp} .$$

If both $L_{y_0}^{\perp} = L_{x_0}^{\perp} = \{0\}$, then $p = q$ and L has maximum dimension p. In which case, $L_{y_0} = \mathbb{R}^p$ and $L_{x_0} = \mathbb{R}^q$. But then $d\phi_{(x_0,y_0)}$ cannot be onto $\mathbb{R}^n$ unless $p = q = n$, which is the required result. So without loss of generality, assume $L_{y_0}^{\perp} = \{0\}$.

From equation (7.2.8), $\ker d\phi_{(x_0,y_0)} \subset L_{y_0} \times L_{x_0}$. Thus the horizontal subspace $\mathscr{H}(x_0,y_0)$ consists of

(i) the orthogonal complement of $\ker d\phi_{(x_0,y_0)}$ in $L_{y_0} \times L_{x_0}$, and

(ii) $L_{y_0}^{\perp} \times L_{x_0}^{\perp}$.

Choose $(u_1,0) \in L_{y_0}^{\perp} \times L_{x_0}^{\perp}$, then

$$d\phi_{(x_0,y_0)}(u_1,0) = d(\phi_{y_0})_{x_0} u_1 ,$$

and

$$|d(\phi_{y_0})_{x_0} u_1|^2 = |\phi_{y_0} u_1|^2$$
$$= |u_1|^2 \, |y_0|^2 ,$$

and since $|(u_1,0)|^2 = |u_1|^2$; we conclude that the dilation λ is given by

$$\lambda^2(x_0,y_0) = |y_0|^2 . \quad (7.2.9)$$

On the other hand, choose (u_1,u_2) as lying in that part of the horizontal space given by (i). For $u \in L_{y_0}$, $\phi_{y_0} u \in L$, so that $\Pi \circ \phi_{y_0}(u) = \phi_{y_0}(u)$, and

$$|\Pi \circ \phi_{y_0}(u)| = |\phi_{y_0}(u)|$$

$$= |u|\ |y_0| .$$

Thus $\Pi \circ \phi_{y_0}|L_{y_0}$ is homothetic, and so

$$d(\phi_{y_0})|L_{x_0} = \lambda_{x_0} A_{x_0} , \qquad (7.2.10)$$

for some $\lambda_{x_0} \in \mathbb{R}$, and some matrix A_{x_0} with $A_{x_0}{}^* A_{x_0}$ = identity. Similarly

$$d(\phi_{x_0})_{L_{y_0}} = \lambda_{y_0} B_{y_0} ,$$

for some $\lambda_{y_0} \in \mathbb{R}$, and some matrix B_{y_0} with $B_{y_0}{}^* B_{y_0}$ = identity. From equation (7.2.8)

$$\ker d\phi_{(x_0,y_0)} = \{(v_1,v_2) \in \mathbb{R}^p \times \mathbb{R}^q ;\ \lambda_{x_0} A_{x_0}(v_1) = -\lambda_{y_0} B_{y_0}(v_2)\}$$

$$= \{((-\lambda_{y_0}/\lambda_{x_0}).A_{x_0}{}^* B_{y_0} v_2, v_2);\ v_2 \in L_{y_0}\} .$$

Thus, if $(u_1,u_2) \in L_{y_0} \times L_{x_0}$, then $(u_1,u_2) \in (\ker d\phi_{(x_0,y_0)})^{\perp}$ if and only if, for all $v_2 \in L_{y_0}$,

$$0 = \langle(-\lambda_{y_0}/\lambda_{x_0}).A_{x_0}{}^* B_{y_0} v_2, u_1\rangle + \langle v_2,u_2\rangle$$

$$= (-\lambda_{y_0}/\lambda_{x_0}).\langle v_2, B_{y_0}{}^* A_{x_0} u_1\rangle + \langle v_2,u_2\rangle$$

$$= \langle v_2, (-\lambda_{y_0}/\lambda_{x_0}).B_{y_0}{}^* A_{x_0} u_1 + u_2\rangle ,$$

if and only if

$$u_2 = (\lambda_{y_0}/\lambda_{x_0}).B_{y_0}{}^* A_{x_0} u_1 . \qquad (7.2.11)$$

Thus, that part of the horizontal space given by (i) is spanned by

$$\{(\lambda_{x_0} u_1 ,\ \lambda_{y_0} B_{y_0}{}^* A_{x_0} u_1);\ u_1 \in L_{x_0}\} .$$

Now

$$|(\lambda_{x_0} u_1, \lambda_{y_0} B_{y_0}{}^* A_{x_0} u_1)|^2 = (\lambda_{x_0}{}^2 + \lambda_{y_0}{}^2)\ |u_1|^2 .$$

Also, for all $u_1 \in L_{x_0}$,

$$d\emptyset_{(x_0,y_0)}(\lambda_{x_0} u_1, \lambda_{y_0} B_{y_0}{}^* A_{x_0} u_1) = \lambda_{x_0}{}^2 A_{x_0} u_1 + \lambda_{y_0}{}^2 B_{y_0} B_{y_0}{}^* A_{x_0} u_1$$

$$= (\lambda_{x_0}{}^2 + \lambda_{y_0}{}^2)\, A_{x_0} u_1 ,$$

and the square of the norm of this vector is equal to $(\lambda_{x_0}{}^2 + \lambda_{y_0}{}^2)$. Thus the dilation λ is given by

$$\lambda^2(x_0,y_0) = (\lambda_{x_0}{}^2 + \lambda_{y_0}{}^2) .$$

But

$$d(\emptyset_{y_0})_{x_0}(\lambda_{x_0} u_1) = \lambda_{x_0}{}^2 A_{x_0} u_1 ,$$

and equation (7.2.4) implies

$$\lambda_{x_0}{}^4 |u_1|^2 = \lambda_{x_0}{}^2 |u_1|^2 |y_0|^2 ,$$

that is

$$\lambda_{x_0}(y_0)^2 = |y_0|^2 .$$

Similarly

$$\lambda_{y_0}(x_0)^2 = |x_0|^2 ,$$

and so

$$\lambda^2(x_0,y_0) = |x_0|^2 + |y_0|^2 . \tag{7.2.12}$$

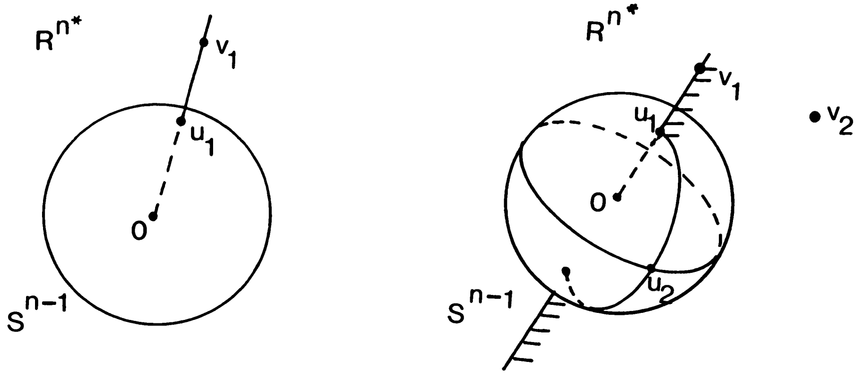
On account of equation (7.2.9) we see that $\emptyset$ is horizontally conformal only when $L_{y_0}{}^\perp = L_{x_0}{}^\perp = \{0\}$, i.e. only when $p = q = n$. It is well known [10] that in this case $\emptyset$ must be one of the standard multiplications of Example 7.1.6 . □

Example 7.2.8 Example 7.1.5 is a special case of the following.

Let $V_{n,k}$ be the space of k-frames in $\mathbb{R}^n$ at 0, and $0_{n,k}$ the space of orthonormal k-frames in $\mathbb{R}^n$ at 0. The Gram-Schmidt orthonormalization process defines a natural deformation retract $\emptyset : V_{n,k} \to 0_{n,k}$. We might ask whether this retraction is a harmonic morphism.

Let $\mathbb{R}^{n^*}$ denote $\mathbb{R}^n \backslash 0$. Then $V_{n,k}$ can be regarded as an open subset of the Euclidean space $\mathbb{R}^{n^*} \times \ldots \times \mathbb{R}^{n^*}$ (k times), from where it inherits its metric. Similarly $0_{n,k}$ can be regarded as an open subset of $S^{n-1} \times \ldots \times S^{n-1}$ (k times) with the induced metric. Let $\Pi: \mathbb{R}^{n^*} \to S^{n-1}$ be the natural retraction $\Pi(x) = x/|x|$. The Gram-Schmidt orthonormalization process can be described as follows.

Let $(v_1, \ldots, v_k) \in V_{n,k}$, then $u_1 = \Pi(v_1) \in S^{n-1}$. There exists a natural projection $\Pi_{v_1}: S^{n-1} \backslash \{\pm \Pi(v_1)\} \to S^{n-2}_{u_1}$ which maps down normal geodesics to the equator $S^{n-2}_{u_1}$ subtended by the two poles $\pm \Pi(v_1)$. Let $u_2 = \Pi_{v_1}(\Pi(v_2))$. Define the projection $\Pi_{v_2}: S^{n-2}_{u_1} \backslash \{\pm u_2\} \to S^{n-3}_{u_2}$, which maps down normal geodesics to the equator $S^{n-3}_{u_2}$ subtended by the poles $\pm u_2$. Let $u_3 = \Pi_{v_2}(\Pi_{v_1}(\Pi(v_3)))$, and so on. In this way we obtain the desired orthonormal basis $(u_1, \ldots, u_k) \in 0_{n,k}$.



The fibre over a point $(u_1, \ldots, u_k) \in 0_{n,k}$ consists of products of various open half-spaces, and hence is minimal in $V_{n,k}$. The tangent space to $V_{n,k}$ at each point can be identified with the Euclidean space $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$ (k times), and the horizontal space to the fibre over $(u_1, \ldots, u_k) \in 0_{n,k}$ consists of k-tuples $(w_1, \ldots, w_k) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ such that

$$
\begin{aligned}
&\langle u_1, w_1 \rangle = 0 \\
&\langle u_1, w_2 \rangle = \langle u_2, w_2 \rangle = 0 \\
&\quad\vdots \\
&\langle u_1, w_k \rangle = \langle u_2, w_k \rangle = \ldots\ldots\ldots = \langle u_k, w_k \rangle = 0 .
\end{aligned} \tag{7.2.13}
$$

We can identify the tangent space to $0_{n,k}$ at $(u_1, \ldots, u_k)$ with a subspace of Euclidean space, and the map ϕ_* sends $(w_1, \ldots, w_k)_{(v_1, \ldots, v_k)} \in T_{(v_1,\ldots,v_k)}V_{n,k}$ to

$$\left(\frac{w_1}{|v_1|}, \frac{w_2}{|v_2|} \Big/ \frac{v_2 . u_2}{|v_2|}, \ldots, \frac{w_k}{|v_k|} \Big/ \frac{v_k . u_k}{|v_k|}\right)$$

$$= \left(\frac{w_1}{v_1 . u_1}, \frac{w_2}{v_2 . u_2}, \ldots, \frac{w_k}{v_k . u_k}\right) .$$

Thus

$$|\phi_*(w_1, \ldots, w_k)_{(v_1, \ldots, v_k)}|^2 = \frac{|w_1|^2}{|v_1 . u_1|^2} + \ldots + \frac{|w_k|^2}{|v_k . u_k|^2}$$

and the dilation λ of ϕ is given by

$$\lambda^2(v_1, \ldots, v_k) = \left(\frac{|w_1|^2}{|v_1 . u_1|^2} + \ldots + \frac{|w_k|^2}{|v_k . u_k|^2}\right) \Big/ (w_1^2 + \ldots + w_k^2) . \tag{7.2.14}$$

This is independent of the choice of $(w_1, \ldots, w_k)$ only in the following cases

(i) $k = 1$; in which case we have $\phi : \mathbb{R}^{n*} \to S^{n-1}$ as in Example 6.1.5 .

(ii) $k = n = 2$; for then equation (7.2.13) implies that $w_2 = 0$, and equation (7.2.14) shows that ϕ is horizontally conformal.

In case (ii); $\phi : GL(2) \to 0(2)$. By choosing appropriate geodesics through $(v_1, \ldots, v_k)$ one can see that ϕ is harmonic, and so is a harmonic morphism (since dim $0(2) = 1$ the map ϕ is obviously horizontally conformal).

Of course the above retraction may be a harmonic morphism with respect to some more natural metric on $0_{n,k}$, such as the left invariant metric which arises from regarding $0_{n,k}$ as a homogeneous space.

<u>Remark 7.2.9</u> Although the map $\phi : V_{n,k} \to 0_{n,k}$ of Example 7.2.8 is not in general a harmonic morphism, it is harmonic and has minimal fibres. In fact it is an example of a more general kind of map than a harmonic morphism, where $\phi^* h(\phi : (M,g) \to (N,h))$ has several distinct eigenvalues instead of just one (c.f. Example 3.3.2) (in Example

7.2.8 $\phi^* h$ has min (k,n - 1) distinct eigenvalues). In the next section we shall briefly consider maps with $\phi^* h$ having more than one distinct eigenvalue.

7.3 Maps $\phi : (M,g) \to (N,h)$ where $\phi^* h$ has two distinct non-zero eigenvalues.

Suppose $\phi : (M,g) \to (N,h)$ is a submersion almost everywhere, with $\phi^* h$ having at most two distinct eigenvalues. Denote these by λ_1 and λ_2, and let U be an open set in M on which λ_1 and λ_2 are non-zero.

Let S_i denote the eigenspace of λ_i, with $r_i = \dim S_i$, $i = 1,2$. Choose a frame field $X_1, \ldots, X_m$ on U such that $X_1, \ldots, X_p$ span S_1, $X_{p+1}, \ldots, X_n$ span S_2 and $X_{n+1}, \ldots, X_m$ are vertical. Choose the following ranges of indices:

$$1 \le a,b, .. \le m;\ 1 \le i,j, .. \le p;\ p+1 \le r,s,.. \le n;\ n+1 \le \alpha, \beta, \ldots \le m .$$

The energy density of ϕ is given by

$$e(\phi) = \tfrac{1}{2} \operatorname{trace} \phi^* h = (r_1\lambda_1 + r_2\lambda_2)/2 \ , \qquad (7.3.1)$$

and the stress-energy tensor S_ϕ is given by

$$S_\phi = \tfrac{1}{2}(r_1\lambda_1 + r_2\lambda_2).g - \phi^* h . \qquad (7.3.2)$$

Suppose ϕ is harmonic; then calculations similar to those in the proof of Theorem 7.1.8 establish the following two equations (summing over repeated indices)

$$\tfrac{1}{2}((r_1 - 2)\, d\lambda_1 + r_2 d\lambda_2)(X_j) + (\lambda_1 - \lambda_2)\, g(X_j, \nabla_{X_r} X_r) + \lambda_1 g(X_j, \nabla_{X_\alpha} X_\alpha) = 0 , \qquad (7.3.3)$$

$$\tfrac{1}{2}(r_1 d\lambda_1 + (r_2 - 2) d\lambda_2)(X_s) + (\lambda_2 - \lambda_1)\, g(X_s, \nabla_{X_i} X_i) + \lambda_2 g(X_s, \nabla_{X_\alpha} X_\alpha) = 0 . \qquad (7.3.4)$$

We therefore have

<u>Theorem 7.3.1</u> <u>Let $\phi : (M,g) \to (N,h)$ be defined as above. If ϕ is harmonic and the mean curvature of each eigenspace S_i is vertical, $i = 1,2$, then the fibre is minimal if and only if</u>

$$((r_1 - 2)\, d\lambda_1 + r_2 d\lambda_2)(X_j) = 0 \qquad (7.3.5)$$

<u>and</u>

$$(r_1 d\lambda_1 + (r_2 - 2) d\lambda_2)(X_s) = 0 \ , \qquad (7.3.6)$$

for each $i = 1, \ldots, p$, $s = p+1, \ldots, n$. In particular if $\dim N = 2$ and $r_1 = r_2 = 1$, then equations (7.3.5) and (7.3.6) imply $\nabla(\lambda_1 - \lambda_2)$ is vertical.

Example 7.3.2 Let $M = S^{m-1}$, and $f: S^{m-1} \to \mathbb{R}$ be an isoparametric function of degree 3, with focal varieties V_1 and V_2. Let $\Pi: M \setminus V_2 \to V_1$ be defined as in Proposition 2.2.5, then from that proposition Π is harmonic. Also $\phi^* h$ has two distinct eigenvalues (c.f. Example 3.3.3), where h is the induced metric on V_1; the corresponding eigenspaces are precisely the principal curvature spaces. The mean curvature of these spaces is vertical, and so Theorem 7.3.1 applies. Since λ_1 and λ_2 are both vertical; equations (7.3.5) and (7.3.6) are satisfied and so the fibres are minimal.

8 Harmonic morphisms defined by homogeneous polynomials

8.1 Properties of harmonic polynomial morphisms

Suppose $\emptyset : \mathbb{R}^m \to \mathbb{R}^n$ is a harmonic morphism defined by harmonic homogeneous polynomials $\emptyset_1, \ldots, \emptyset_n$, each of degree p. Normalize $\emptyset$ such that $\sup_{|x|=1} |\emptyset(x)|^2 = 1$. Following Fuglede [19], we define the set

$$\Gamma = \{ x \in S^{m-1} ; \ |\emptyset(x)|^2 = 1 \} .$$

Then $|x|^{2p} - |\emptyset(x)|^2 \geq 0$ in $\mathbb{R}^m \setminus \{0\}$ with equality on the cone $\mathbb{R}^+ \Gamma \setminus \{0\}$. Thus on $\mathbb{R}^+ \Gamma \setminus \{0\}$

$$\begin{aligned} 0 &= \tfrac{1}{2} \nabla(|x|^{2p} - |\emptyset(x)|^2) \\ &= p \ |x|^{2p-2} . x - \sum_{k=1}^{n} \emptyset_k(x) \nabla \emptyset_k(x) . \end{aligned} \tag{8.1.1}$$

From Theorem 7.1.3 , this implies

$$p^2 \ |x|^{4p-2} = \lambda(x)^2 \ |\emptyset(x)|^2 , \tag{8.1.2}$$

where $\lambda : \mathbb{R}^m \to \mathbb{R}$ is the dilation of $\emptyset$. Thus on $\mathbb{R}^+ \Gamma \setminus \{0\}$

$$\begin{aligned} \lambda^2(x) &= p^2 |x|^{4p-2} / |\emptyset(x)|^2 \\ &= p^2 \ |x|^{2p-2} . \end{aligned} \tag{8.1.3}$$

By the smoothness of λ^2, we see that $\lambda^2(0) = 0$, and $\lambda^2(x) = p^2 \ |x|^{2p-2}$ on $\mathbb{R}^+ \Gamma$.

Consider the Laplacian

$$\Delta(|x|^{2p} - |\emptyset(x)|^2) = p(2p - 2) \ |x|^{2p-2} + mp \ |x|^{2p-2} - n\lambda^2 . \tag{8.1.4}$$

This is ≥ 0 on $\mathbb{R}^+ \Gamma$, thus $p(2p - 2) + mp \geq np^2$, that is

$$(m - 2) \geq (n - 2).p .$$

Equality is obtained when $\Delta(|x|^{2p} - |\emptyset(x)|^2) = 0$, that is when $|x|^{2p} - |\emptyset(x)|^2 = 0$ on S^{m-1}. We therefore have

Proposition 8.1.1 *If* $\phi : \mathbb{R}^m \to \mathbb{R}^n$ *is a harmonic morphism defined by homogeneous polynomials of degree* p , *then*

$$m - 2 \geq (n - 2).p \ , \qquad (8.1.5)$$

with equality if and only if $|\phi(x)|^2$ = *constant, for all* $x \in S^{m-1}$.

Example 8.1.2 Let $\phi : \mathbb{R}^4 \to \mathbb{R}^3$ be the Hopf map

$$\phi((x,y)) = (|x|^2 - |y|^2 , 2xy) ,$$

where we regard x,y as being complex numbers. Then ϕ is a harmonic morphism defined by homogeneous polynomials of degree 2. The map $\phi|_{S^{m-1}} : S^{m-1} \to S^{n-1}$ is a harmonic Riemannian fibration.

From now on we will assume that the dilation $\lambda : \mathbb{R}^m \to \mathbb{R}$ *is given by*

$$\lambda^2(x) = p^2 |x|^{2p-2} \ , \qquad (8.1.6)$$

for all $x \in \mathbb{R}^m$. Let $F : \mathbb{R}^m \to \mathbb{R}$ be defined by $F(x) = |\phi(x)|^2$, for all $x \in \mathbb{R}^m$, then F is homogeneous of degree $2p$. Let $f = F|_{S^{m-1}} : S^{m-1} \to \mathbb{R}$.

Lemma 8.1.3 *If* f *is as above then*

$$|df|^2 = \psi_1(f) \ , \quad \Delta f = \psi_2(f) \ ,$$

for some smooth functions ψ_1 *and* ψ_2. *In particular* f *is an isoparametric function (c.f. Chapter 2).*

Proof First of all

$$\begin{aligned} |df|^2 &= |dF|^2 - (\partial F/\partial r)^2 \\ &= 4 < \sum_{k=1}^{n} \phi_k \nabla\phi_k, \sum_{k=1}^{n} \phi_k \nabla\phi_k > - (2pF)^2 \\ &= 4\lambda^2 |\phi|^2 - (2pF)^2 \\ &= 4p^2(1 - F).F \ , \text{ using equation } (8.1.6) \ . \end{aligned} \qquad (8.1.7)$$

Write F as $F = \sum_{k=1}^{n} \phi_k^{\ 2}$, then

$$\begin{aligned} \Delta^{\mathbb{R}^m} F &= 2 \sum_k \nabla\phi_k . \nabla\phi_k \\ &= 2n \ \lambda^2 \ . \end{aligned}$$

Also

$$\frac{\partial F}{\partial r} = (1/r).2pF ,$$

and

$$\frac{\partial^2 F}{\partial r^2} = \frac{-2pF}{r^2} + \frac{4p^2F^2}{r^2} .$$

where $r^2 = |x|^2$. Therefore from Lemma 1.1.5

$$\Delta^{S^{m-1}} f = 2n\lambda^2 - 4p^2f^2 - (m-2)2pf . \qquad \square \qquad (8.1.8)$$

<u>Theorem 8.1.4</u> <u>If</u> $\emptyset : \mathbf{R}^m \to \mathbf{R}^n$ <u>is a harmonic morphism defined by homogeneous polynomials of degree</u> p, <u>with dilation</u> λ <u>given by</u> $\lambda^2(x) = p^2|x|^{2p-2}$, <u>then</u> Γ <u>is a smooth submanifold of</u> S^{m-1}, <u>and both</u> $\mathbf{R}^+\Gamma$ <u>and the fibre over the origin in</u> $\mathbf{R}^n$ <u>are minimal cones through the origin in</u> $\mathbf{R}^m$.

<u>Proof</u> Let $\Sigma = \{x \in \mathbf{R}^m ; \emptyset(x) = 0\}$ denote the fibre over the origin. Then $F|_\Sigma = 0$, where $F = |\emptyset|^2$, and so Γ and $\Sigma \cap S^{m-1}$ are both critical sets of $f = F|_{S^{m-1}}$. From Lemma 8.1.3 f is isoparametric, and hence from Theorem 2.2.2 Γ and $\Sigma \cap S^{m-1}$ are minimal submanifolds of S^{m-1}.

Consider the map $\emptyset|_\Gamma : \Gamma \to S^{n-1}$, then by a result of Fuglede [19] this map is surjective. We prove the following theorem.

<u>Theorem 8.1.5</u> <u>The map</u> $\emptyset|_\Gamma : \Gamma \to S^{n-1}$ <u>defined above is a harmonic Riemannian submersion.</u>

<u>Proof</u> <u>Claim 1</u>: The set Γ is precisely the set of $x \in S^{m-1}$ such that T_x(fibre of $\emptyset$ through x) $\subset T_xS^{m-1}$.

<u>Proof of Claim 1</u>: Let $y \in S^{n-1}$ be such that $\emptyset(x) = y$, $x \in \Gamma$, and write $\emptyset^{-1}(y)$ for the fibre over y. Let $\gamma(u) \subset \emptyset^{-1}(y)$ be a curve with $\gamma(0) = x$. Then $\emptyset(\gamma(u)) = y$. Let $\mu(u) = \gamma(u)/|\gamma(u)| \in S^{m-1}$. Then

$$\emptyset(\mu(u)) = y/|\gamma(u)|^p ,$$

by the homogeneity of $\emptyset$. Thus

$$d\emptyset(\mu'(0)) = (d/du)(y/|\gamma(u)|^p)\Big|_{u=0}$$

$$= -p<\gamma'(0),\gamma(0)>.y\,; \qquad (8.1.9)$$

so that $\gamma(u)$ is tangent to S^{m-1} at $u = 0$ if and only if $<\gamma'(0),\gamma(0)> = 0$ if and only if $d\emptyset(\mu'(0) = 0$. But we know that $|\emptyset(x)|^2 = 1$ is maximum on the sphere, i.e. $|\emptyset(\mu(u))|^2 \leq 1$ and $= 1$ at $s = 0$. Therefore

$$0 = (d/du)\,|\emptyset(\mu(u))|^2\Big|_{u=0}$$

$$= (d/du)\sum_{k=1}^{n} \emptyset_k(\mu(u))^2\Big|_{u=0}$$

$$= 2\sum_{k=1}^{n} \emptyset_k(\mu(0))\,d\emptyset_k(\mu'(0))$$

$$= 2 < \emptyset(\mu(0)),\ d\emptyset(\mu'(0)) >$$

$$= 2<y,-p<\gamma'(0),\gamma(0)>y>$$

$$= -2p<\gamma'(0),\ \gamma(0)> \ . \qquad (8.1.10)$$

Thus γ is tangent to S^{m-1} at $u = 0$ and Claim 1 is proved. In particular $\nabla\lambda^2$ is horizontal over Γ, and equation (7.1.5) of Theorem 7.1.8 implies that the mean curvature of the fibre is proportional to the horizontal projection of $\nabla\lambda^2$, i.e. to $\nabla\lambda^2$ (in particular if the fibre were contained in S^{m-1} then it would be minimal in S^{m-1}).

Claim 2: The set $\emptyset^{-1}(y) \cap \Gamma$ is minimal in Γ.

Proof of Claim 2 : Suppose we are given a curve $\alpha(u)$ on S^{m-1} such that

$$\emptyset(\alpha(u)) = a(u).y \ . \qquad (8.1.11)$$

Then, since $\emptyset(\rho(u).\alpha(u)) = \rho(u)^p.a(u).y$, if we let $\rho(u) = a(u)^{-1/p}$ whenever $a(u) \neq 0$; the curve $\gamma(u) = \alpha(u)/a(u)^{1/p}$ is a curve in the fibre $\emptyset^{-1}(y)$ whenever $a(u) \neq 0$.

Consider the particular case when $\alpha(u)$ is an integral curve of ∇f through a point $x \in \Gamma$. Then we find that

(a) $\emptyset(\alpha(0)) = y$

(b) $d\emptyset(\nabla f) = 2p^2\emptyset.(1-f)$.

Both the conditions (a) and (b) imply that either (i) $\alpha(u)$ satisfies equation (8.1.11), or (ii) $\phi(\alpha(u)) \subset S^{n-1}$.

Consider case (i). Since $\alpha(u)$ is a geodesic in S^{m-1}; $\nabla_{\alpha'(0)}\alpha'(0)$ is perpendicular to S^{m-1}, which implies that $\nabla_{\gamma'(0)}\gamma'(0)$ is perpendicular to S^{m-1}.

Consider case (ii). Then $\alpha(u)$ is horizontal, and so $\alpha'(u)$ is perpendicular to $\phi^{-1}(y)$.

We can now perform the following construction.

Consider the set $\phi^{-1}(y) \cap \Gamma$. Locally we can choose an orthonormal basis of $T(\phi^{-1}(y); X_1, \ldots, X_r$, such that $X_1, \ldots, X_s$ span $T(\phi^{-1}(y) \cap \Gamma)$ and $X_{s+1}, \ldots, X_r$ arise as the tangent vectors to integral curves of ∇f (this is always possible since Γ is the focal variety of an isoparametric function) as above, so that $\nabla_{X_j} X_j$ is perpendicular to S^{m-1}, for $j = s+1, \ldots, r$. We therefore find that the mean curvature vector of $\phi^{-1}(y) \cap \Gamma$ is perpendicular to S^{m-1}, and in particular that $\phi^{-1}(y) \cap \Gamma$ is minimal in Γ.

Furthermore, since $\lambda^2|_\Gamma$ = constant and $\nabla\lambda^2$ (which is horizontal) is perpendicular to Γ, then the map ϕ is a Riemannian submersion. The minimality of the fibres now imply that $\phi|_\Gamma$ is a harmonic Riemannian submersion. □

8.2. Some examples

✓Example 8.2.1 Let $\phi : \mathbb{R}^8 \to \mathbb{R}^5$ be the Hopf map defined by

$$\begin{aligned}\phi(x,y) = (\, |x|^2 - |y|^2,\ & 2(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4), \\ & 2(x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3), \\ & 2(x_1y_3 + x_3y_1 + x_4y_2 - x_2y_4), \\ & 2(x_1y_4 + x_4y_1 + x_2y_3 - x_3y_2)) .\end{aligned}$$

Then ϕ is a harmonic morphism defined by homogeneous polynomials of degree 2. Write $\phi = (\phi_1, \ldots, \phi_5)$. Then

$$\phi_1 = (x_1, x_2, x_3, x_4, -y_1, -y_2, -y_3, -y_4)$$

$$\phi_2 = (y_1, -y_2, -y_3, -y_4, x_1, -x_2, -x_3, -x_4)$$

$$\nabla \phi_3 = 2(y_2, y_1, y_4, -y_3, x_2, x_1, -x_4, x_3)$$

$$\nabla \phi_4 = 2(y_3, -y_4, y_1, y_2, x_3, x_4, x_1, -x_2)$$

$$\nabla \phi_5 = 2(y_4, y_3, -y_2, y_1, x_4, -x_3, x_2, x_1) ,$$

from which we see that $\lambda^2(z) = 4|z|^2$ for all $z \in \mathbb{R}^8$. We also note that there exists an $A \in O_8(\mathbb{R})$, such that $\phi_{i+1} = \phi_i \circ A$;

$$A = \frac{1}{2^{\frac{1}{2}}} \begin{bmatrix} 1&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&-1 \\ 0&0&-1&0&0&0&1&0 \\ 0&1&0&0&0&-1&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \\ 1&0&0&0&-1&0&0&0 \end{bmatrix} .$$

Let $\chi : \mathbb{R}^8 \to \mathbb{R}^n$, $n \leq 5$, be defined by a subset of $\{\phi_1, \ldots, \phi_5\}$. Let $\Gamma = \{x \in S^7 ;\ |\chi(x)|^2 = 1\}$, $F = |\chi|^2$ the associated isoparametric function and Σ the fibre over the origin in $\mathbb{R}^n$. We consider the different cases in turn.

Case 0 $\chi(z) = 0$, $z = (x,y) \in \mathbb{R}^8$. Then Γ is empty, $\Sigma \cap S^7 = S^7$ and $F = 0$.

Case 1 $\chi(z) = \phi_1(z) = |x|^2 - |y|^2$. Then $F = (|x|^2 - |y|^2)^2$, $\Gamma = S^3 \cup S^3$, $\chi|_\Gamma : \Gamma \to S^0$ is the obvious map, and $\Sigma \cap S^7 = S^3 \times S^3/2^{\frac{1}{2}}$.

Case 2 $\chi = (\phi_1, \phi_2)$. Let $B \in O_8(\mathbb{R})$ be the matrix

$$B = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&-1 \end{bmatrix} ,$$

then B preserves ϕ_1, while taking ϕ_2 into the function $2\langle x,y\rangle$. Then $F = (|x|^2 - |y|^2)^2 + 4\langle x,y\rangle^2$, which is the isoparametric function of Example 2.3.5. The set Γ is given by $\Gamma = S^1 \times S^2/S^0 = \{e^{i\theta}.x;\ x \in S^3\} = \{\cos\theta.x, \sin\theta.x)$;

$x \in S^3\}$, and $\chi|_\Gamma : (\cos\theta.x, \sin\theta.x) \to (\cos 2\theta, \sin 2\theta)$, i.e. $\chi|_\Gamma : (\theta, x) \to 2\theta$, which is clearly a Riemannian submersion. Also $\Sigma \cap S^7 = S^{\pi/4}_{4,2}$, the Stiefel manifold of orthonormal 2-frames in 4-space (defined in Example 5.3.3).

<u>Case 3</u> $\chi = (\phi_1, \phi_2, \phi_3)$. Let $\psi = (\phi_2, \phi_3, \phi_4, \phi_5)$. Then the isoparametric function corresponding to ψ, F_ψ, is given by $F_\psi(x,y) = 4\,|x|^2\,|y|^2 = (|x|^2 + |y|^2)^2 - (|x|^2 - |y|^2)^2 = 1 - (|x|^2 - |y|^2)^2$. Thus the level sets of χ coincide, (i.e. up to orthogonal transformation) with the level sets of $\chi \circ A^2$, which coincide with the level sets of $|\psi|^2 - \phi_2^2$, i.e. with the level sets of $1 - \phi_1^2 - \phi_2^2$. Using the matrix B, we can transform ϕ_2 into $2 < x,y >$ while preserving ϕ_1. Thus

$$F(x,y) = 1 - (|x|^2 - |y|^2)^2 - 4 < x,y >^2 ,$$

and $\Gamma = S^{\pi/4}_{4,2}$ and $\Sigma = S^1 \times S^3/S^0$. We now perform a computation of $\chi|_\Gamma : \Gamma \to S^2$.
The map $\chi \circ B \circ A^2$ is given by

$$\chi \circ B \circ A^2(x,y) = 2(-x_1y_2 + x_2y_1 - x_3y_4 + x_4y_3, -x_1y_3 + x_3y_1 - x_4y_2 + x_2y_4, -x_1y_4 + x_4y_1 - x_2y_3 + x_3y_2).$$

Then $|\chi \circ B \circ A^2(x,y)|^2 = 4(|x|^2|y|^2 - < x,y >^2)$, which is equal to 1 when $|x|^2 = |y|^2 = \frac{1}{2}$ and $< x,y > = 0$, i.e. on the focal variety $S^{\pi/4}_{4,2} = \{(x+iy)/2^{\frac{1}{2}} ; (x,y) \in S_{4,2}\}$. We are thus precisely in the situation of Example 5.3.3, and the map $\chi|_\Gamma : \Gamma \to S^2$ is given by the map $\phi^{\pi/4} : S^{\pi/4}_{4,2} \to S^2$ of that example.

Similarly, if $\chi = (\phi_1, \phi_2, \phi_4)$ say, then $\chi \circ A = (\phi_2, \phi_3, \phi_5)$. The level sets of $|\chi \circ A|^2$ coincide with the level sets of $|\psi|^2 - \phi_4^2$, which coincide with the level sets of $1 - \phi_1^2 - \phi_4^2$. If $C \in 0_8(\mathbb{R})$ is the matrix

$$C = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&-1&0&0 \end{bmatrix} ,$$

then C takes the function ϕ_4 into the function $2 < x,y >$, while preserving ϕ_1, i.e. we have the same case as above.

Case 4 $\chi = (\phi_1, \phi_2, \phi_3, \phi_4)$. Then $\chi \circ A = \psi$, so that $F = 1 - (|x|^2 - |y|^2)^2$. The set Γ is isometric to $S^3 \times S^3 / 2^{\frac{1}{2}}$, and $\chi|_\Gamma : \Gamma \to S^3$; $\chi|_\Gamma((x,y)/2^{\frac{1}{2}}) = xy \in S^3$ multiplication of unit quarternions). The set $\Sigma \cap S^7$ is isometric to $S^3 \cup S^3$.

Case 5 $\chi = \phi$. Then $F = 1$, $\Gamma = S^7$, $\Sigma \cap S^7$ is the empty set and $\chi|_\Gamma : \Gamma \to S^4$ is the Hopf fibration.

We remark on the duality between Case i and Case $(5-i)$, $i = 1, \ldots, 5$ - that is, the set Γ of case i is identical with the set $\Sigma \cap S^7$ of case $(5-i)$.

Example 8.2.2 Consider the example of an isoparametric function given by Ozeki and Takeuchi in [33], and defined as follows. Let H denote the space of quaternions, and write $u = (u_0, u_1)$, $v = (v_0, v_1)$, $u_i, v_i \in H$; let $u \to \bar{u}$ denote the canonical involution. Define a function $F : \mathbb{R}^{16} \to \mathbb{R}$ by

$$F((u,v)) = 4({}^t u \bar{v} - \langle u, v\rangle^2) + (|u_1|^2 - |v_1|^2 + 2\langle u_0, v_0\rangle)^2 .$$

Then $f = F|_{S^{15}}$ is isoparametric.

Let $\phi : \mathbb{R}^{16} \to \mathbb{R}^4$ be defined in the following way. Write $u_i = (u_i^1, u_i^2, u_i^3, u_i^4)$, $i = 1,2$, and let $\phi = (\phi_1, \ldots, \phi_4)$ be given by

$$\phi_1 = |u_1|^2 - |v_1|^2 + 2\langle u_0, v_0\rangle$$

$$\phi_2 = u_0^2 v_0^1 - u_0^1 v_0^2 - u_0^3 v_0^4 + u_0^4 v_0^3 + u_1^2 v_1^1 - u_1^1 v_1^2 - u_1^3 v_1^4 + u_1^4 v_1^3$$

$$\phi_3 = u_0^3 v_0^1 - u_0^1 v_0^3 - u_0^4 v_0^2 + u_0^2 v_0^4 + u_1^3 v_1^1 - u_1^1 v_1^3 - u_1^4 v_1^2 + u_1^2 v_1^4$$

$$\phi_4 = u_0^4 v_0^1 - u_0^1 v_0^4 - u_0^2 v_0^3 + u_0^3 v_0^2 + u_1^4 v_1^1 - u_1^1 v_1^4 - u_1^2 v_1^3 + u_1^3 v_1^2 .$$

Then the square of the norm of the gradients of the ϕ_i, $i = 1, \ldots, 4$, are all equal and mutually perpendicular. Since each ϕ_i is harmonic we see that ϕ is a harmonic morphism. Furthermore $F = |\phi|^2$. Thus from Theorem 8.1.5 there exists an interesting harmonic Riemannian submersion from the set Γ onto S^3.

8.3. Harmonic morphisms defined by homogeneous polynomials of degree bigger than two.

We can easily construct harmonic morphisms defined by homogeneous polynomials of degree bigger than 2, simply by composing two harmonic polynomial morphisms of degree 2. Since we recall from Example 7.1.7 that the composition of two harmonic morphisms is again a harmonic morphism.

Example 8.3.1 Let $\phi : \mathbb{R}^8 \to \mathbb{R}^4$ be defined by $\phi(x,y) = 2xy$, where x and y are quaternions, and let $\psi : \mathbb{R}^4 \to \mathbb{R}^3$ by the Hopf map; $\psi(u,v) = (|u|^2 - |v|^2, 2uv)$, where u and v are complex numbers. Then $\chi = \psi \circ \phi : \mathbb{R}^8 \to \mathbb{R}^3$ is a harmonic morphism defined by homogeneous polynomials of degree 4. The critical set $C_\chi = \phi^{-1}(0)$ is a minimal cone in $\mathbb{R}^8$.

However, we ask whether there exist harmonic polynomial morphisms defined by homogeneous polynomials of degree bigger than 2, which do not arise in this way as the composition of two harmonic morphisms. In particular, do there exist harmonic morphisms defined by homogeneous polynomials of degree 3?

Recall the theorem of Munzner; Theorem 2.2.5. This states that an isoparametric function $f: S^{m-1} \to \mathbb{R}$ arises as the restriction of a homogeneous polynomial $F : \mathbb{R}^m \to \mathbb{R}$ of degree p with

(i) $|\nabla F|^2 = p^2 |x|^{2p-2}$

(ii) $\Delta F = c\,|x|^{p-2}$,

where the constant c is zero when the multiplicities of the distinct principal curvatures are equal.

Suppose we are given such a polynomial $F : \mathbb{R}^m \to \mathbb{R}$ with $c = 0$. Given any matrix $A \in O_m(\mathbb{R})$; define $G: \mathbb{R}^m \to \mathbb{R}$ by $G = F \circ A$. Then, for all $x \in \mathbb{R}^m$ and for all vectors $v \in T_x\mathbb{R}^m$,

$$\begin{aligned} \langle v, \nabla G_x \rangle &= dG_x(v) \\ &= dF_{A(x)} \circ dA_x(v) \\ &= dF_{A(x)}(Av) \\ &= \langle Av, \nabla F_{A(x)} \rangle \\ &= \langle v, A^* \nabla F_{A(x)} \rangle . \end{aligned}$$

Thus

$$\nabla G_x = A^* \nabla F_{A(x)} . \tag{8.3.1}$$

Therefore, from (i), $|\nabla G_x|^2 = p^2 |x|^{2p-2}$. Since $A: \mathbb{R}^m \to \mathbb{R}^m$ is an isometry; G is also harmonic, so that G satisfies condition (ii) also. In particular, if there exists an $A \in 0_m(\mathbb{R})$ such that $< \nabla G_x, \nabla F_x > = 0$, for all $x \in \mathbb{R}^m$, then $\emptyset = (F,G)$ gives a non-trivial harmonic morphism $\emptyset: \mathbb{R}^m \to \mathbb{R}^2$, defined by homogeneous polynomials of degree p. This would give a method of constructing harmonic morphisms of various degrees.

Remark 8.3.2 All known harmonic morphisms defined by homogeneous polynomials of degree 2 arise from a single polynomial in this way. We now demonstrate that this procedure will now always work.

Let F,G be as above, and write $f = F|_{S^{m-1}}$ and $g = G|_{S^{m-1}}$.

Lemma 8.3.3 *Let* V(f) *denote the focal varieties of* f, *and* $M_0(f)$ *the minimal hypersurface* $f^{-1}(0)$. *Then*

(a) $V(f) = \{ x \in S^{m-1} ; \nabla F_x \perp T_x S^{m-1} \}$

(b) $M_0(f) = \{ x \in S^{m-1} ; \nabla F_x \in T_x S^{m-1} \}$.

Proof The statement follows immediately from the equation

$$\nabla F_x = \nabla f_x + pf.(\partial/\partial r)(x) , \tag{8.3.2}$$

for all $x \in S^{m-1}$, where $r^2(x) = |x|^2$. □

Proposition 8.3.4 *If*

$$< \nabla F_x, \nabla G_x > = 0 ,$$

for all $x \in \mathbb{R}^m$, *then*

$$V(f) \subset M_0(g)$$

and

$$V(g) \subset M_0(f) .$$

Proof From Lemma 8.3.3. (a) we see that

$$x \in V(f) \quad \text{implies} \quad x \in M_0(g) ,$$

and similarly

$x \in V(g)$ implies $x \in M_0(f)$. □

Theorem 8.3.5 Let $F: \mathbb{R}^6 \to \mathbb{R}$ be the standard homogeneous polynomial of degree 4 of Theorem 2.2.5, defining the family of isoparametric hypersurfaces of Example 2.3.5 with $n = 2$ (so the multiplicities of the principal curvatures are equal and F is harmonic). Then there exists no $A \in 0_6(\mathbb{R})$ such that $A(V(f)) \subset M_0(f)$, where $f = F|_{S^{m-1}}$ (hence from equation (8.3.1) the above method of constructing a harmonic morphism fails in this case).

Proof The polynomial F is given by

$$F(x,y) = 2((|x|^2 - |y|^2)^2 + 4 <x,y>^2) - |(x,y)|^4 ,$$

where $x,y \in \mathbb{R}^3$. As in Example 2.3.5 the level surfaces of f can be parametrized by the sets

$$\{ e^{i\theta}(\cos s.x + i\sin s.y);\ \theta \in [0,2\pi] \text{ and } (x,y) \in S_{3,2}\} ,$$

for each $s \in [0,\pi/4]$.

Consider the focal variety V_1 corresponding to $s = \pi/4$. Then $V_1 = \{(x+iy)/2^{\frac{1}{2}};$ $(x,y) \in S_{3,2}\}$. The minimal hypersurface of f is given by

$$M_0(f) = \{ e^{i\theta}(\cos(\pi/8).u + i\sin(\pi/8).v);\ \theta \in [0,2\pi], (u,v) \in S_{3,2}\} .$$

Suppose the point $z = (x + iy)/2^{\frac{1}{2}}$ of V_1 is mapped under A to the point $w = e^{i\psi}(\cos(\pi/8).u + i\sin(\pi/8).v) \in M_0(f)$. Then

$$w = (\cos\psi\cos(\pi/8).u - \sin\psi\sin(\pi/8).v)$$
$$+ i(\sin\psi\cos(\pi/8).u + \cos\psi\sin(\pi/8).v) .$$

Now, since $0_3(\mathbb{R})$ acts transitively on $S_{3,2}$, there exists a $B \in 0_3(\mathbb{R})$ with $u = Bx$ and $v = By$. Thus the matrix A has the form

$$A = \left(\begin{array}{c|c} \cos\psi\cos(\pi/8).B & -\sin\psi\sin(\pi/8).B \\ \hline \sin\psi\cos(\pi/8).B & \cos\psi\sin(\pi/8).B \end{array}\right).$$

We must check that this is indeed orthogonal.

We compute AA^* and obtain

$$AA^* = \left(\begin{array}{c|c} \cos^2\psi\cos^2\frac{\pi}{8}\,BB^* + \sin^2\psi\sin^2\frac{\pi}{8}BB^* & \frac{1}{2}\sin 2\psi\cos\frac{\pi}{4}.BB^* \\ \hline \frac{1}{2}\sin 2\psi\cos\frac{\pi}{4}.BB^* & \sin^2\psi\cos^2\frac{\pi}{8}BB^* + \cos^2\psi\sin^2\frac{\pi}{8}BB^* \end{array}\right).$$

This can only be the identity matrix if $\sin 2\psi = 0$, i.e. $\psi = 0, \pi/2, \pi$ or $3\pi/2$. But in this case the non-zero diagonal entries are not all equal, and certainly do not equal 1, e.g. if $\psi = 0$, then

$$A.A^* = \left[\begin{array}{ccc|ccc} \cos^2(\pi/8) & & & & & \\ & \ddots & & & 0 & \\ & & \cos^2(\pi/8) & & & \\ \hline & & & \sin^2(\pi/8) & & \\ & & & & \ddots & \\ 0 & & & & & \sin^2(\pi/8) \end{array}\right]. \qquad \square$$

8.4 Harmonic polynomial morphisms and equivariant maps between spheres

Recall Example 2.3.8 and Theorem 2.3.9 of Chapter 2. In that example we were given an n-tuple $(P_1, \ldots, P_n)$ of symmetric endomorphisms of $\mathbb{R}^{21}$ with

$$P_iP_j + P_jP_i = 2\,\delta_{ij}.I, \quad i,j = 1, \ldots, n \;,$$

called a Clifford system. To such a Clifford system, with $m_1 = n$ and $m_2 = 1-n$ both positive, is associated the isoparametric function $f = F|_{S^{21-1}}$, where $F : \mathbb{R}^{21} \to \mathbb{R}$ is defined by

$$F(x) = |x|^4 - 2\sum_i < P_ix,x >^2$$

We could equally well write F as

$$F(x) = \sum_i < P_i\, x,x >^2, \qquad (8.4.1)$$

and then, in view of Lemma 8.1.3 it is natural to ask whether such an isoparametric function arises from a harmonic polynomial morphism.

Theorem 8.4.1 Given a Clifford system $(P_1, \ldots, P_n)$ on $\mathbb{R}^{21}$; define $f_i : \mathbb{R}^{21} \to \mathbb{R}$ by $f_i(x) = < P_ix,x >$, for all $x \in \mathbb{R}^{21}$, and for each $i = 1, \ldots, n$.

Then $\phi : \mathbb{R}^{21} \to \mathbb{R}^n$ *given by* $\phi = (f_1, \ldots, f_n)$, *is a harmonic morphism defined by homogeneous polynomials of degree* 2, *with dilation* λ *given by* $\lambda^2(x) = 4|x|^2$, *for each* $x \in \mathbb{R}^{21}$.

Proof Let $(x_i)_{i=1,\ldots,m=21}$ be the standard coordinates on $\mathbb{R}^m$. Then, for each $x \in \mathbb{R}^m$

$$\begin{aligned}
\langle \nabla f_i(x), \partial/\partial x_1 \rangle &= df_i(x)(\partial/\partial x_1) \\
&= (d/du)(f_i(x+u(1,0,\ldots,0)))|_{u=0} \\
&= (d/du)\langle P_i(x+u(1,0,\ldots,0)), x+u(1,0,\ldots,0)\rangle |_{u=0} \\
&= (d/du)(\langle P_i(x),x\rangle + \langle P_i(x), u(1,0,\ldots,0)\rangle \\
&\qquad + \langle uP_i(1,0,\ldots,0), x\rangle + 0(u^2))|_{u=0} \\
&= \langle P_i(x),(1,0,\ldots,0)\rangle + \langle P_i(1,0,\ldots,0),x\rangle .
\end{aligned}$$

Thus

$$\nabla f_i(x) = P_i(x) + \sum_{k=1}^{m} \langle P_i(\partial/\partial x_k), x\rangle . \partial/\partial x_k . \tag{8.4.2}$$

But P_i is symmetric, i.e. $\langle P_i(x),y\rangle = \langle P_i(y),x\rangle$. Thus, from equation (8.4.2),

$$\nabla f_i(x) = 2P_i(x) , \tag{8.4.3}$$

for each $i = 1, \ldots, n$. Then

$$\begin{aligned}
\langle \nabla f_i(x), \nabla f_j(x)\rangle &= 4\langle P_i(x), P_j(x)\rangle \\
&= 2\langle P_jP_i(x),x\rangle + 2\langle x, P_iP_j(x)\rangle \\
&= 4\langle x,x\rangle \delta_{ij} .
\end{aligned}$$

Thus ϕ is horizontally conformal with dilation λ given by $\lambda^2(x) = 4|x|^2$, for all $x \in \mathbb{R}^m$.

Write P_i in matrix form as $(P_i^{ab})_{a,b=1,\ldots,m}$, for each $i = 1, \ldots, n$. Then

$$f_i(x) = \sum_{a,b} P_i^{ab} x_a x_b .$$

Thus

$$\Delta f_i = \sum_a P_i^{aa} .$$

Since $P_i^2 = 1$, the eigenvalues of P_i are +1 and -1. If the eigenspaces of +1,-1 are $E_+(P_i)$ and $E_-(P_i)$ respectively, then since $P_iP_j + P_jP_i = 0$, $i \neq j$, we see that P_j interchanges the eigenspaces $E_+(P_i)$ and $E_-(P_i)$ of P_i. Thus $\dim E_+(P_i) = \dim E_-(P_i) = 1$, whence

$$\sum_a P_i^{aa} = 0 ,$$

and f_i is harmonic for each $i = 1, \ldots, n$. □

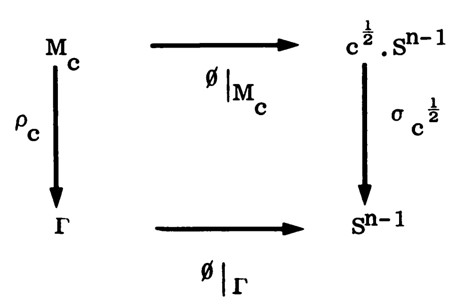
Examples of harmonic morphisms defined by Clifford systems, include as special cases Examples 8.2.1. and 8.2.2. From Theorem 8.1.5, to each such example we can associate a harmonic Riemannian submersion onto a sphere.

Amongst the family of isoparametric functions (8.4.1), some are inhomogeneous ; furthermore in some cases the focal varieties are inhomogeneous [17]. Thus we can associate harmonic Riemannian submersions from compact inhomogeneous spaces onto spheres.

We can modify the proof of Theorem 8.1.5 to obtain the following.

Theorem 8.4.2 Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a harmonic morphism defined by homogeneous polynomials of degree p, with dilation λ given by $\lambda^2(x) = p^2 |x|^{2p-2}$. Let $F: \mathbb{R}^m \to \mathbb{R}$ be defined by $F(x) = |\phi(x)|^2$, and write $f = F|_{S^{m-1}}$, then $\phi|_{M_c} : M_c \to c^{\frac{1}{2}} S^{n-1}$ is a harmonic homothetic submersion, where $M_c = f^{-1}(c)$ and $c \neq 0$.

Proof Since $c \neq 0$, the projection $\rho_c : M_c \to \Gamma$, which maps down normal geodesics, is well-defined. Therefore $\phi|_{M_c} : M_c \to c^{\frac{1}{2}}. S^{n-1}$ is onto, since it factors through ρ_c, $\phi|_\Gamma : \Gamma \to S^{n-1}$ and the projection map $\sigma_{c^{\frac{1}{2}}} : c^{\frac{1}{2}} S^{n-1} \to S^{n-1}$; i.e. the following diagram commutes:



In order to prove the theorem, the crucial point to observe is that the proof of Theorem 8.1.5 will go through provided that the following claim is true.

Claim 8.4.3 For $x \in M_c$, $\mathscr{H}(\nabla\lambda^2(x))$ lies in the plane spanned by $\nabla\lambda^2(x)$ and ξ_x, where ξ is the unit normal vector field to M_c.

For then the mean curvature of the fibre of ϕ over $\phi(x)$ would be perpendicular to M_c at x, and the rest of the proof goes through as before.

Proof of Claim 8.4.3 Write $\phi = (\phi_1, \ldots, \phi_n)$, then the horizontal space through $x \in M_c$ is spanned by $\{\nabla\phi_i(x)\}_{i=1,\ldots,n}$. Now

$$F = \sum_{k=1}^{n} \phi_k^2 \quad ,$$

so

$$\nabla F = 2 \sum_{k=1}^{n} \phi_k \nabla \phi_k \quad ,$$

therefore

$$< \nabla F, \nabla\phi_i > = 2\phi_i \lambda^2 \quad .$$

Also $\lambda^2(x) = p^2 |x|^{2p-2}$, therefore

$$\nabla\lambda^2 = 2p^2(p-1)|x|^{2p-4} \cdot \sum_k x_k \nabla x_k \, .$$

Now

$$\sum_k < x_k \nabla x_k, \nabla\phi_i > = p.\phi_i \, ,$$

by the homogeneity of ϕ_i. Thus

$$< \nabla\lambda^2, \nabla\phi_i > = 2p^3(p-1)\phi_i$$

on S^{m-1}. Hence the horizontal projection of $\nabla\lambda^2$ is proportional to the horizontal projection of ∇F. Since ξ is proportional to the projection of ∇F onto TS^{m-1}; the claim is proved. □

Example 8.4.4 Consider Example 8.2.1, Case (2). Here $\phi: \mathbb{R}^8 \to \mathbb{R}^2$, $\phi = (\phi_1, \phi_2)$ is given by

$$\phi_1(x,y) = |x|^2 - |y|^2$$

$$\phi_2(x,y) = 2 < x,y > \, ,$$

where x,y are quaternions. The level hypersurfaces are parametrized by the sets $M_s = \{ e^{i\theta}(\cos s . x + i \sin s . y) ; \theta \in [0,2\Pi], (x,y) \in S_{3,2} \}$, where $s \in [0, \Pi/4]$.

If $z \in M_s$, then z can be expressed in the form

$$z = e^{i\theta}(\cos s . x + i \sin s . y)$$
$$= (\cos\theta \cos s . x - \sin\theta \sin s . y) + i(\sin\theta \cos s . x + \cos\theta \sin s . y) .$$

Then

$$\phi_1(z) = |x|^2 - |y|^2$$
$$= \cos^2\theta \cos^2 s + \sin^2\theta \sin^2 s - \sin^2\theta \cos^2 s - \cos^2\theta \sin^2 s$$
$$= \cos 2\theta . \cos 2s ,$$

and

$$\phi_2(z) = 2(\cos\theta \cos s \sin\theta \cos s - \sin\theta \sin s \cos\theta \sin s)$$
$$= \sin 2\theta . \cos 2s .$$

Thus $\phi|_{M_s} : M_s \to \cos 2s . S^1 ; \phi(z) = \cos 2s . e^{2i\theta}$.

Theorem 8.4.2 allows us to construct many equivariant maps $\chi : S^{m-1} \to S^n$ associated to a harmonic polynomial morphism $\phi : \mathbb{R}^m \to \mathbb{R}^n$ with dilation λ given by $\lambda^2(x) = p^2 |x|^{2p-2}$. We simply define χ by

$$\chi(z) = (\cos 2\alpha(s)(\phi|_{\Gamma} \circ \rho(z)), \sin 2\alpha(s)) ,$$

where $F = |\phi|^2$ is parametrized such that $F = \cos ps$, $\Gamma = s^{-1}(0)$, ρ is the projection down normal geodesics onto Γ and $\alpha(0) = 0$, $\alpha(\Pi/p) = \Pi/2$.

In particular all the harmonic morphisms associated to a Clifford system give such equivariant maps, and hence we have examples of equivariant maps between spheres with respect to isoparametric functions, where one of the isoparametric functions has non-homogeneous hypersurfaces. This then justifies our use of isoparametric hypersurfaces as opposed to homogeneous hypersurfaces.

Furthermore, when the same isoparametric function gives rise to two distinct harmonic Riemannian submersions via Theorem 8.1.5, we expect to be able to construct examples of equivariant maps similar to Example 5.3.4. Indeed the two Riemannian submersions of Example 8.2.1 given by Cases (2) and (3) give rise to

Example 5.3.4. Similarly Cases (1) and (4) of Example 8.2.1 generate the Hopf map from S^7 to S^4.

In fact, suppose $\emptyset : \mathbb{R}^m \to \mathbb{R}^n$ is a harmonic polynomial morphism, satisfying

$$m - 2 = p(n-2)$$

where p is the degree of the homogeneous polynomials defining $\emptyset$. Then from Proposition 8.1.1 we see that

$$|\emptyset(x)|^2 = |x|^{2p}$$

for all $x \in \mathbb{R}^m$. Let $\emptyset = (\emptyset_1, \ldots, \emptyset_n)$, and define harmonic polynomial morphisms ρ and σ by

$$\rho = (\emptyset_1, \ldots, \emptyset_p)$$

$$\sigma = (\emptyset_{p+1}, \ldots, \emptyset_n) .$$

Call such ρ and σ complementary.

Let M_s be a level hypersurface of the isoparametric function f defined by $f(x) = |\rho(x)|^2 |_{S^{m-1}}$, for each $x \in S^{m-1}$. Then since

$$1 = |\rho|^2 + |\sigma|^2 ,$$

M_s must also be a level hypersurface of the isoparametric function g defined by $g(x) = |\sigma(x)|^2 |_{S^{m-1}}$, for all $x \in S^{m-1}$. From Theorem 8.4.2, we obtain harmonic homothetic submersions $\rho|_{M_s} : M_s \to a(s)S^{p-1}$, $\sigma|_{M_s} : M_s \to b(s)S^{q-1}$, $p + q = n$, for some functions a(s) and b(s). Since $1 = |\rho|^2 + |\sigma|^2$, we have $a(s)^2 + b(s)^2 = 1$, so we can choose $a(s) = \cos ps$ and $b(s) = \sin ps$ (by writing $M_s = f^{-1}(\cos^2 ps)$).

Given two harmonic polynomial maps $g_1 : S^{p-1} \to S^{r-1}$ and $g_2 : S^{q-1} \to S^{s-1}$, we can now define an equivariant map from S^{m-1} to S^{r+s-1} as follows. Let $x \in S^{m-1}$, then $(\rho(x), \sigma(x)) \in S^{p-1} * S^{q-1}$; we then compose with $g_1 * g_2$, to obtain the point $g_1 * g_2(\rho(x), \sigma(x)) \in S^{r-1} * S^{s-1} = S^{r+s-1}$. The map so defined is clearly equivariant since the map of level hypersurfaces is harmonic of constant energy density by Theorem 8.4.2.

The Smith maps of Section 1.3 can be seen to arise in this way as follows. Consider the harmonic polynomial morphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\phi(x_1, \ldots, x_m) = (x_1, \ldots, x_m) .$$

Define the complementary maps ρ and σ by

$$\rho(x_1, \ldots, x_m) = (x_1, \ldots, x_p)$$

and

$$\sigma(x_1, \ldots, x_m) = (x_{p+1}, \ldots, x_m).$$

Then $f(x) = |\rho(x)|^2 |_{S^{m-1}}$ is isoparametric of degree 2. Let M_s be the level hypersurface given by $f(x) = \cos^2 s$. Then $g(x) = |\sigma(x)|^2 |_{S^{m-1}}$ is given by $g(x) = \sin^2 s$ on M_s. For $x \in S^{m-1}$; $(\rho(x), \sigma(x) \in S^{p-1} * S^{q-1}$, $p + q = m$. We now compose with harmonic polynomial maps $g_1 : S^{p-1} \to S^{r-1}$ and $g_2 : S^{q-1} \to S^{s-1}$ as above, to obtain the Smith map from S^{m-1} to S^{n-1}, $r + s = n$.

9 Deformations of metrics

9.1 Deformations of the metric for harmonic morphisms

Let $\phi : (M,g) \to (N,h)$ be a horizontally conformal map between Riemannian manifolds with dilation λ. Let $U \subset M$ be an open set upon which $d\phi$ is non-zero. Let $x \in U$, then for horizontal vectors $X, Y \in \mathscr{H}_x$, we have

$$\lambda^2 g(X,Y) = \phi^* h(X,Y) .$$

Thus we can decompose g as

$$g = (1/\lambda^2).\phi^* h + k \quad , \tag{9.1.1}$$

over U, where $(1/\lambda^2)\phi^* h$ represents the horizontal part of the metric, and k the vertical part.

The stress-energy tensor of ϕ is given by

$$\begin{aligned} S_\phi &= e(\phi) g - \phi^* h \\ &= \tfrac{1}{2} n \lambda^2 . g - \lambda^2 (g - k) \\ &= \tfrac{1}{2}(n-2)\lambda^2 . g + \lambda^2 . k \quad , \end{aligned} \tag{9.1.2}$$

where n = dim N. Thus

$$\nabla^* S_\phi = \tfrac{1}{2}(n-2).d(\lambda^2) + \nabla^*(\lambda^2 k) \quad . \tag{9.1.3}$$

We therefore have

<u>Lemma 9.1.1</u> <u>If</u> $\phi : (M,g) \to (N.h)$ <u>is horizontally conformal with dilation</u> λ, <u>and</u> ϕ <u>is a submersion almost everywhere, then</u> ϕ <u>is harmonic (and so a harmonic morphism) if and only if</u>

$$\tfrac{1}{2}(2-n).d\lambda^2 = \nabla^*(\lambda^2 k) \quad , \tag{9.1.4}$$

<u>where</u> k <u>represents the fibre metric (where defined) (assume both sides of (9.1.4) are zero when</u> $\lambda = 0$).

Define a new metric $\bar{g}$ on M by

$$\bar{g} = (1/\lambda^2\sigma^2).\phi^*h + (1/\rho^2).k \ , \tag{9.1.5}$$

where $\sigma^2, \rho^2 : M \to \mathbb{R}$ are smooth functions. The new metric $\bar{g}$ may not be well-defined everywhere if σ^2, ρ^2 have zeros, and we remove such points to obtain the Riemannian manifold $(\bar{M}, \bar{g})$. Denote by $\bar{\nabla}$ the associated Levi-Civita connection. The map $\phi : (M,g) \to (N,h)$ induces a new map $\bar{\phi} : (\bar{M},\bar{g}) \to (N,h)$. If $\bar{\phi}$ is a submersion almost everywhere then, from Lemma 9.1.1, $\bar{\phi}$ is a harmonic morphism if and only if

$$\tfrac{1}{2}(2 - n).d(\lambda^2\sigma^2) = \bar{\nabla}^*(\lambda^2\sigma^2.k/\rho^2) \ . \tag{9.1.6}$$

Our aim is to reformulate and solve equation (9.1.6), given that equation (9.1.4) is satisfied.

<u>Proposition 9.1.2</u> <u>The connection coefficients of</u> $\bar{\nabla}$ <u>are described in terms of those of</u> ∇ <u>by the following formulae. Use the following ranges of indices:</u>

$$1 \le i,j, \ldots \le n = \dim N;\ n+1 \le r,s, \ldots \le m = \dim M;\ 1 \le a,b, \ldots \le m\ ;$$

<u>and let</u> $(e_a)_{1\le a\le m} = (e_i, e_r)$ <u>denote a local orthonormal basis with respect to</u> g <u>over a subset</u> U <u>of</u> M <u>where</u> $d\phi \neq 0$, <u>and</u> e_i, e_r are horizontal, vertical, for each i,r respectively. <u>Let</u> $(\bar{e}_a) = (\bar{e}_i, \bar{e}_r) = (\sigma e_i, \rho e_r)$ <u>denote the corresponding orthonormal basis with respect to</u> $\bar{g}$ <u>over</u> U. <u>Then</u>

$$g(e_t, \bar{\nabla}_{\bar{e}_r}\bar{e}_s) = \rho^2 g(e_t, \nabla_{e_r}e_s) + \tfrac{1}{2}\rho(g(e_t, d\rho(e_r)e_s) + g(e_s, d\rho(e_t)e_r) - g(e_r, d\rho(e_s)e_t))\ ; \tag{9.1.7}$$

$$g(e_i, \bar{\nabla}_{\bar{e}_r}\bar{e}_s) = \sigma^2 g(e_i, \nabla_{e_r}e_s) + (\sigma^2/2\rho)g(e_s, d\rho(e_i)e_r)\ ; \tag{9.1.8}$$

$$g(e_k, \bar{\nabla}_{\bar{e}_i}\bar{e}_j) = \sigma^2 g(e_k, \nabla_{e_i}e_j) + \tfrac{1}{2}\sigma(g(e_k, d\sigma(e_i)e_j) + g(e_j, d\sigma(e_k)e_i) - g(e_i, d\sigma(e_j)e_k))\ ; \tag{9.1.9}$$

$$g(e_r, \bar{\nabla}_{\bar{e}_i}\bar{e}_j) = \tfrac{1}{2}(g(e_r, (\sigma^2 + \rho^2)\nabla_{e_i}e_j) - g(e_r, (\sigma^2 - \rho^2)\nabla_{e_j}e_i) + (\rho^2/\sigma)g(e_j, d\sigma(e_r)e_i))\ ; \tag{9.1.10}$$

$$g(e_r, \bar\nabla_{\bar e_i} \bar e_s) = \rho\sigma g(e_r, \nabla_{e_i} e_s) + \tfrac{1}{2}\sigma g(e_r, d\rho(e_i) e_s) ; \tag{9.1.11}$$

$$g(e_j, \bar\nabla_{\bar e_r} \bar e_i) = \rho\sigma g(e_j, \nabla_{e_r} e_i) + \tfrac{1}{2}\sigma(1-(\sigma^2/\rho)) g(e_r, [e_i, e_j]) + (\rho/\sigma) g(e_j, d\sigma(e_r) e_i)) ; \tag{9.1.12}$$

$$g(e_s, \bar\nabla_{\bar e_r} \bar e_i) = -(\rho/\sigma) g(e_i, \bar\nabla_{\bar e_r} \bar e_s) . \tag{9.1.13}$$

Proof We use the fundamental formula for an orthonormal frame $X, Y, Z \in \mathscr{C}TM$ (see for example [14]):

$$g(Z, \nabla_X Y) = \tfrac{1}{2}(g(Z, [X,Y]) + g(Y, [Z,X]) - g(X, [Y,Z])) . \tag{9.1.14}$$

Similarly

$$\bar g(\bar Z, \bar\nabla_{\bar X} \bar Y) = \tfrac{1}{2}(\bar g(\bar Z, [\bar X, \bar Y]) + \bar g(\bar X, [\bar Z, \bar X]) - \bar g(\bar X, [\bar Y, \bar Z])) . \tag{9.1.15}$$

Let $\mathscr{L}$ denote Lie derivation. First of all we prove equation (9.1.7).

From equation (9.1.15), letting k, $\bar k$ denote the vertical part of the metric $g, \bar g$ respectively,

$$\begin{aligned}\bar g(\bar e_t, \bar\nabla_{\bar e_r} \bar e_s) &= \tfrac{1}{2}(\bar k(\bar e_t, \mathscr{L}_{\bar e_r} \bar e_s) + \bar k(\bar e_s, \mathscr{L}_{\bar e_t} \bar e_r) - \bar k(\bar e_r, \mathscr{L}_{\bar e_s} \bar e_t)) \\ &= \tfrac{1}{2}(\rho k(e_t, \mathscr{L}_{e_r} e_s) + k(e_t, d\rho(e_r) e_s) + \dots) \\ &= \tfrac{1}{2}(\rho g(e_t, \mathscr{L}_{e_r} e_s) + g(e_t, d\rho(e_r) e_s + \dots) .\end{aligned}$$

On the other hand

$$\begin{aligned}\bar g(\bar e_t, \bar\nabla_{\bar e_r} \bar e_s) &= \bar k(\bar e_t, \bar\nabla_{\bar e_r} \bar e_s) \\ &= (1/\rho) k(e_t, \bar\nabla_{\bar e_r} \bar e_s) \\ &= (1/\rho) g(e_t, \bar\nabla_{\bar e_r} \bar e_s) .\end{aligned}$$

Therefore

$$g(e_t, \bar\nabla_{\bar e_r} \bar e_s) = \rho^2 g(e_t, \nabla_{e_r} e_s) + \tfrac{1}{2}\rho(g(e_t, d\rho(e_r) e_s) + \dots) .$$

The proof of equation (9.1.9) is similar. We prove one more, say equation (9.1.10);

the others use similar arguments.

From equation (9.1.15), and writing $H, \bar{H}$ for the horizontal part of the metrics $g, \bar{g}$ respectively

$$\bar{g}(\bar{e}_r, \bar{\nabla}_{\bar{e}_i} \bar{e}_j) = \tfrac{1}{2}(\bar{k}(\bar{e}_r, \mathcal{L}_{\bar{e}_i} \bar{e}_j) + \bar{H}(\bar{e}_j, \mathcal{L}_{\bar{e}_r} \bar{e}_i) - \bar{H}(\bar{e}_i, \mathcal{L}_{\bar{e}_j} \bar{e}_r))$$

$$= \tfrac{1}{2}((1/\rho)k(e_r, \mathcal{L}_{\sigma e_i} \sigma e_j) + (\rho/\sigma) H(e_j, \mathcal{L}_{e_r} (\sigma e_i) - H(e_i, \mathcal{L}_{e_j} \rho e_r))$$

$$= \tfrac{1}{2}((\sigma^2/\rho) k(e_r, \mathcal{L}_{e_i} e_j) + \rho H(e_j, \mathcal{L}_{e_r} e_i) - \rho H(e_i, \mathcal{L}_{e_j} e_r) + (\rho/\sigma) H(e_j, d\sigma(e_r) e_i)) .$$

Now

$$H(e_j, \mathcal{L}_{e_r} e_i) - H(e_i, \mathcal{L}_{e_j} e_r) = g(e_j, \nabla_{e_r} e_i - \nabla_{e_i} e_r) - g(e_i, \nabla_{e_j} e_r - \nabla_{e_r} e_j)$$

$$= g(e_j, \nabla_{e_i} e_r) - g(e_i, \nabla_{e_j} e_r)$$

$$= g(e_r, \nabla_{e_i} e_j + \nabla_{e_j} e_i).$$

Thus

$$\bar{g}(\bar{e}_r, \bar{\nabla}_{\bar{e}_i} \bar{e}_j) = \tfrac{1}{2}(g(e_r, (\sigma^2/\rho)(\nabla_{e_i} e_j - \nabla_{e_j} e_i)) + g(e_r, \rho(\nabla_{e_i} e_j + \nabla_{e_j} e_i))$$

$$g(e_i, d\sigma(e_r) e_i)) .$$

But

$$\bar{g}(\bar{e}_r, \bar{\nabla}_{\bar{e}_i} \bar{e}_j) = \bar{k}(\bar{e}_r, \bar{\nabla}_{\bar{e}_i} \bar{e}_j)$$

$$= (1/\rho) k(e_r, \bar{\nabla}_{\bar{e}_i} \bar{e}_j)$$

$$= (1/\rho) g(e_r, \bar{\nabla}_{\bar{e}_i} \bar{e}_j) ,$$

giving equation (9.1.10) . □

We now compare $\bar{\nabla}^* k$ with $\nabla^* k$.

<u>Lemma 9.1.3</u> <u>Using the notations of Proposition 9.1.2</u> ;

$$\bar{\nabla}^* k(e_j) = \rho^2 \nabla^* k(e_j) + \tfrac{1}{2}\rho(m-n)d\rho(e_j) , \tag{9.1.16}$$

for all $j = 1, \ldots, n$.

Proof The divergence with respect to $\bar{g}$ is given by (summing over repeated indices)

$$\bar{\nabla}^* k(e_j) = \bar{e}_a k(\bar{e}_a, e_j) - k(\bar{\nabla}_{\bar{e}_a} \bar{e}_a, e_j) - k(\bar{e}_a, \bar{\nabla}_{\bar{e}_a} e_j)$$

$$= - k(\bar{e}_a, \bar{\nabla}_{\bar{e}_a} e_j)$$

$$= - (1/\sigma)(k(\bar{e}_a, \bar{\nabla}_{\bar{e}_a} \bar{e}_j) - \sigma k(\bar{e}_a, d\sigma(\bar{e}_a) e_j))$$

$$= - (\rho/\sigma) k(e_a, \bar{\nabla}_{\bar{e}_a} \bar{e}_j)$$

$$= - (\rho/\sigma) g(e_r, \bar{\nabla}_{\bar{e}_r} \bar{e}_j)$$

$$= (\rho^2/\sigma^2)(\sigma^2 g(e_j, \nabla_{e_r} e_r) + (\sigma^2/2\rho) g(e_r, d\rho(e_j) e_r)) ,$$

from equations (9.1.11) and (9.1.13),

$$= \rho^2 g(e_j, \nabla_{e_r} e_r) + \tfrac{1}{2}\rho(m-n) d\rho(e_j) .$$

On the other hand

$$\nabla^* k(e_j) = - k(e_a, \nabla_{e_a} e_j)$$

$$= - g(e_r, \nabla_{e_r} e_j)$$

$$= g(e_j, \nabla_{e_r} e_r) . \qquad \square$$

Lemma 9.1.4 Using the notations of Proposition 9.1.2;

$$\bar{\nabla}^* k(e_s) = \rho^2 d\rho(e_s) - \tfrac{1}{2}n(\rho/\sigma)(d\sigma(e_s) + \rho^2 \nabla^* k(e_s) , \tag{9.1.17}$$

for all $s = n+1, \ldots, m$.

Proof The divergence with respect to $\bar{g}$ is given by (summing over repeated indices)

$$\bar{\nabla}^* k(e_s) = \bar{e}_a k(\bar{e}_a, e_s) - k(\bar{\nabla}_{\bar{e}_a} \bar{e}_a, e_s) - k(\bar{e}_a, \bar{\nabla}_{\bar{e}_a} e_s)$$

$$= \rho e_r(\rho k(e_r, e_s)) - g(\bar{\nabla}_{\bar{e}_i} \bar{e}_i, e_s) - g(\bar{\nabla}_{\bar{e}_r} \bar{e}_r, e_s)$$

$$- \rho g(e_r, \bar{\nabla}_{\bar{e}_r} e_s)$$

$$= \rho d\rho(e_s) - \rho^2 g(e_s, \nabla_{e_i} e_i) - (\rho^2/2\sigma) g(e_i, d\sigma(e_s) e_i)$$

$$- \rho^2 g(e_s, \nabla_{e_r} e_r) - \tfrac{1}{2}\rho g(e_r, d\rho(e_s) e_r)$$

$$- \rho((1/\rho) g(e_r, \bar{\nabla}_{\bar{e}_r} \bar{e}_s) - g(e_r, d\rho(\bar{e}_r) e_s)) ,$$

where we have used equations (9.1.7) and (9.1.10);

$$= \rho d\rho(e_s) - \rho^2 g(e_s, \nabla_{e_i} e_i) - n(\rho^2/2\sigma) d\sigma(e_s) - \rho^2 g(e_s, \nabla_{e_r} e_r)$$

$$- \tfrac{1}{2}(m - n)\rho d\rho(e_s) - \rho^2 g(e_r, \nabla_{e_r} e_s) - \rho d\rho(e_s)$$

$$+ \tfrac{1}{2}(m - n)\rho d\rho(e_s) + \rho^2 d\rho(e_s) ,$$

using equation (9.1.7) again;

$$= \rho^2 d\rho(e_s) - \tfrac{1}{2} n(\rho^2/\sigma) d\sigma(e_s) - \rho^2 (g(e_s, \nabla_{e_i} e_i)$$

$$+ g(e_s, \nabla_{e_r} e_r) + g(e_r, \nabla_{e_r} e_s)) . \square$$

<u>Theorem 9.1.5</u> <u>Assume equation (9.1.4) is satisfied, i.e.</u> $\emptyset$ <u>is a harmonic morphism. Then</u> $\bar{\emptyset} : (\bar{M}, \bar{g}) \to (N, h)$ <u>is a harmonic morphism if and only if, using the notations of Proposition 9.1.2</u>,

$$\begin{cases} (2 - n) d(\sigma^2)(e_j) = \tfrac{1}{2}(\sigma^2/\rho^2)(m - n) d(\rho^2)(e_j) & (9.1.18) \\ (2 - n) d(\lambda^2 \sigma^2)(e_s) = 2\rho^2 d(\sigma^2 \lambda^2/\rho^2)(e_s) + \sigma^2 \lambda^2 (2 d\rho(e_s) & \\ \qquad - \tfrac{1}{2} n(1/\sigma^2) d(\sigma^2)(e_s) - n(1/\lambda^2) d\lambda^2(e_s)) & (9.1.19) \end{cases}$$

<u>if and only if</u>

$$\begin{cases} (2-n)d(\log\sigma^2)(e_j) = \tfrac{1}{2}(m-n)d(\log\rho^2)(e_j) & (9.1.20) \\ (2-n)d(\log\lambda^2\sigma^2)(e_s) = 2d(\log(\sigma^2\lambda^2/\rho^2)(e_s) + 2d\rho(e_s) & \\ \qquad - \tfrac{1}{2}nd(\log\sigma^2)(e_s) - nd(\log\lambda^2)(e_s), & (9.1.21) \end{cases}$$

<u>for all</u> $j = 1, \ldots, n$ <u>and all</u> $s = n+1, \ldots, m$.

<u>Remark 9.1.6</u> It may be more natural to write $\mu^2 = \sigma^2\lambda^2$, whence equations (9.1.18) and (9.1.19) become

$$\begin{cases} (2-n)(d(\log\mu^2) - d(\log\lambda^2))(e_j) = (m-n)d\rho(e_j) & (9.1.22) \\ (2-n)d(\log\mu^2)(e_s) = 2d(\log(\mu^2/\rho^2))(e_s) + 2d\rho(e_s) & \\ \qquad - nd(\log(\mu/\lambda))(e_s) - nd(\log\lambda^2)(e_s), & (9.1.23) \end{cases}$$

for all $j = 1, \ldots, n$ and $s = n+1, \ldots, m$.

<u>Proof of Theorem 9.1.5</u> First note that (summing over repeated indices)

$$\nabla^*(\lambda^2 k) = d\lambda^2(e_r)k(e_r) + \lambda^2\nabla^* k, \tag{9.1.24}$$

$$\bar{\nabla}^*((\lambda^2\sigma^2/\rho^2)k) = d(\sigma^2\lambda^2/\rho^2)(\bar{e}_r)k(\bar{e}_r) + (\sigma^2\lambda^2/\rho^2)\bar{\nabla}^* k. \tag{9.1.25}$$

Let equation (9.1.6) act on e_j;

$$\begin{aligned} \tfrac{1}{2}(2-n)d(\lambda^2\sigma^2)(e_j) &= (\sigma^2\lambda^2/\rho^2)\bar{\nabla}^* k(e_j) \\ &= (\sigma^2\lambda^2/\rho^2)(\rho^2\nabla^* k(e_j) + \tfrac{1}{2}\rho(m-n)d\rho(e_j)) \\ &\qquad \text{from Lemma 9.1.3,} \\ &= \sigma^2\tfrac{1}{2}(2-n)d\lambda^2(e_j) + (\sigma^2\lambda^2/\rho)\tfrac{1}{2}(m-n)d\rho(e_j) \\ &\qquad \text{from equation (9.1.4),} \end{aligned}$$

giving equation (9.1.18).

Now let equation (9.1.6) act on e_s, to give

$$\begin{aligned} \tfrac{1}{2}(2-n)d(\lambda^2\sigma^2)(e_s) &= d(\sigma^2\lambda^2/\rho^2)(\bar{e}_r)k(\bar{e}_r, e_s) + (\sigma^2\lambda^2/\rho^2)\bar{\nabla}^* k(e_s) \\ &= \rho^2 d(\sigma^2\lambda^2/\rho^2)(e_s) + (\sigma^2\lambda^2/\rho^2)(\rho^2 d\rho(e_s) \\ &\qquad \tfrac{1}{2}n(\rho^2/\sigma)d\sigma(e_s) + \rho^2\nabla^* k(e_s)). \end{aligned}$$

Now

$$\begin{aligned}\nabla^* k(e_s) &= (1/\lambda^2)\nabla^*(\lambda^2 k)(e_s) - (1/\lambda^2)d\lambda^2(e_s)\\ &= \tfrac{1}{2}(2-n)(1/\lambda^2)d\lambda^2(e_s) - (1/\lambda^2)d\lambda^2(e_s)\\ &= \tfrac{1}{2}n(1/\lambda^2)d\lambda^2(e_s)\ ,\end{aligned}$$ □

giving equation (9.1.19) .

Remark 9.1.7 If we put $\sigma^2 = \rho^2 = 1$, then equations (9.1.18) and (9.1.19) are satisfied - similarly, if σ^2 and ρ^2 are both constant then the equations are also satisfied.

Remark 9.1.18 There is a striking analogy of the above methods with the classical notion of a Backlund transformation. There one has a hyperbolic surface M in $\mathbb{R}^3$, which is parametrized by a function $\alpha: M \to \mathbb{R}$. This function is in fact the angle between the asymptotic coordinates, and the Codazzi equation for M is equivalent to α satisfying a certain second order equation called the Sine-Gordon equation. Conversely to each solution of the Sine-Gordon equation one can construct a hyperbolic surface. The idea of Backlund was to write down a first order equation in two variables α and $\tilde{\alpha}$, such that if α is a solution of the Sine-Gordon equation, and $\tilde{\alpha}$ satisfies the first order equation, then $\tilde{\alpha}$ is also a solution of the Sine-Gordon equation.

We view Backlund's idea in a more general context as the following fundamental principal:

(i) we are given a second order problem parametrized by a set of functions $(\alpha, \beta, \ldots)$,

(ii) there is a set of first order equations in two sets of parameters; $(\alpha, \beta, \ldots)$ and $(\tilde{\alpha}, \tilde{\beta}, \ldots)$, which associates to a solution $(\alpha, \beta, \ldots)$ of (i) another solution $(\tilde{\alpha}, \tilde{\beta}, \ldots)$ of (i) .

In the context of harmonic morphisms, the parameter is the dilation λ .

9.2 Examples

Given a particular harmonic morphism $\phi: (M,g) \to (N,h)$, we attempt to find non-trivial solutions to equations (9.1.18) and (9.1.19). We also consider instances when there are no non-trivial solutions.

Example 9.2.1 If $n > 2$ and ρ is constant, then equation (9.1.18) implies that $\nabla\sigma^2$ is vertical.

Example 9.2.2 If ρ is constant and $\nabla\lambda^2$, $\nabla\sigma^2$ are both vertical, then equation (9.1.18) is satisfied. Equation (9.1.19) now becomes

$$d(\lambda^2\sigma^2)(e_s) = \sigma\lambda^2 d\sigma(e_s) + \sigma^2 d\lambda^2(e_s) ,$$

for all $s = n+1, \ldots, m$. This is satisfied if and only if

$$2\,d\sigma(e_s) = d\sigma(e_s) .$$

for all $s = n+1, \ldots, m$; if and only if

$$d\sigma = 0 ,$$

i.e. the function σ is constant.

Example 9.2.3 If $\rho = 1$, $n = 2$, $\nabla\sigma^2$ is horizontal and $\nabla\lambda^2$ is vertical, then equations (9.1.18) and (9.1.19) are both satisfied. For example, let $\phi : \mathbb{R}^3\backslash\{0\} \to S^2$ be defined by $\phi(x) = x/|x|$, for all $x \in \mathbb{R}^3\backslash\{0\}$. Then if $\bar{\sigma}: S^2 \to \mathbb{R}$ is any smooth function which does not take on the value $0 \in \mathbb{R}$; $\sigma: \mathbb{R}^3\backslash\{0\} \to \mathbb{R}$, $\sigma(x) = \bar{\sigma}(\phi(x))$ has the property that $\nabla\sigma^2$ is horizontal. We thus obtain a new harmonic morphism $\bar{\phi} : \overline{\mathbb{R}^3\backslash\{0\}} \to S^2$.

Example 9.2.4 Let $\phi : S^3 \to S^2$ be the Hopf fibration; so $\lambda^2 = 1$. Put $\rho = 1$, then equation (9.1.19) is satisfied if $\nabla\sigma^2$ is horizontal. For example, if $\bar{\sigma}: S^2 \to \mathbb{R}$ is smooth and does not take on the value zero; define $\sigma: S^3 \to \mathbb{R}$ by $\sigma(x) = \bar{\sigma}(\phi(x))$, for all $x \in S^3$, then $\nabla\sigma^2$ is horizontal.

Example 9.2.5 Let $\phi: \mathbb{R}^4 \to \mathbb{R}^3$ be the Hopf map of Example 7.2.1. Then ϕ is a harmonic morphism with dilation λ given by $\lambda^2(x) = 4|x|^2$, for all $x \in \mathbb{R}^4$. The fibres of ϕ consist of great circles of Euclidean spheres of $\mathbb{R}^4$, and hence $\nabla\lambda^2$ is perpendicular to the fibres and hence horizontal. Equation (9.1.20) becomes

$$-d(\log\sigma^2)(e_j) = \tfrac{1}{2}d(\log\rho^2)(e_j) ,$$

for all $j = 1, \ldots, n$. This is solved if σ^{-2} = constant $\times\ \rho$. If we choose ρ^2 such that $\nabla\rho^2$ is horizontal - for example $\rho^2(x) = \psi(|x|^2)$ for some function ψ, then equation (9.1.21) is also satisfied. More generally we have the following.

Example 9.2.6 Let $\phi : \mathbb{R}^4 \to \mathbb{R}^3$ be the Hopf map of Example 9.2.5, and let $\Pi^p_4 : \mathbb{R}^p \to \mathbb{R}^4$, $p \geq 4$, be the projection map. Define $\chi^p : \mathbb{R}^p \to \mathbb{R}^3$, by $\chi^p = \phi \circ \Pi^p_4$. Then χ^p is a harmonic morphism with dilation λ given by $\lambda^2(x) = 4\,|\Pi^p_4(x)|^2$, for all $x \in \mathbb{R}^p$. Thus $\nabla\lambda^2$ is horizontal. Equation (9.1.20) becomes

$$- d(\log \sigma^2)(e_j) = \tfrac{1}{2}(p-3)\, d(\log \rho^2)(e_j) \quad ,$$

for all $j = 1, \ldots, n$. This is satisfied if

$$\sigma^{-2} = \text{constant} \times \rho^{(p-3)} \quad .$$

If we choose σ^2 such that $\nabla\sigma^2$ is horizontal, for instance $\sigma^2(x)$ is a function of $|\Pi^p_4(x)|^2$, then equation (9.1.21) is also satisfied.

Remark 9.2.7 When $\nabla\sigma^2$ is horizontal, then $\sigma^2|_{\phi^{-1}(y)}$ = constant, for all $y \in N$. Thus σ^2 can be thought of as a function on N, and we can view the change from the map ϕ to the map $\bar{\phi}$ as changing the metric h on N to $\bar{h} = h/\sigma^2$. That is, equation (9.1.5) can be seen as

$$\bar{g} = (1/\lambda^2)\phi^*\bar{h} + (1/\rho^2)k = \phi^* h(\sigma^2\lambda^2) + (1/\rho^2)k \; .$$

In this way, by suitable choices of the functions ρ^2 and σ^2, it may be possible, by removing and adding certain points, to change the topology of both M and N.

9.3. Deformations of metrics for equivariant maps

Suppose $\phi : M \to N$ is equivariant with respect to the isoparametric functions $s: M \to \mathbb{R}$ and $t: N \to \mathbb{R}$, where we suppose M and N are space forms. Using the notations of Chapter 4, we consider the map

$$\phi_{s,t} : M_s \to N_t$$

between level hypersurfaces of s and t. Away from the focal varieties we can express the metric g of M as

$$g = ds^2 + g_s \quad ,$$

where g_s is the induced metric on M_s. Similarly on N, we can write h as

$$h = dt^2 + h_t \quad ,$$

where h_t is the induced metric on N_t.

There exist four reasonable kinds of deformations of the metrics g and h which we can consider.

(i) Define the new metric $\overline{g}$ on M^* by

$$\overline{g} = (1/\mu(s)^2)\, ds^2 + g_s , \tag{9.3.1}$$

where $\mu(s)$ is smooth and positive and such that the resulting metric $\overline{g}$ extends smoothly across the focal varieties.

(ii) Define the new metric $\overline{h}$ on N^* by

$$\overline{h} = (1/\nu(t)^2)\, dt^2 + h_t , \tag{9.3.2}$$

where $\nu(t)$ is smooth and positive and such that the resulting metric $\overline{h}$ extends smoothly across the focal varieties.

(iii) If M_s has principal curvatures $\lambda_1(s), \ldots, \lambda_p(s)$ with corresponding eigenspaces $S_1, \ldots, S_p$, then we can express g_s as

$$g_s = g_1(s) + g_2(s) + \ldots + g_p(s) ,$$

where, for each $x \in M_s$, and for all $X, Y \in T_x M_s$, then

$$g_i(s)_x(X,Y) = \begin{cases} g_s(X,Y), & \text{if } X, Y \in S_i(x) \\ 0 , & \text{if either } X \in S_j(x),\ j \neq i \\ & \quad \text{or } \ Y \in S_j(x),\ j \neq i \end{cases}$$

Define a new metric $\overline{g}$ on M^* by $\overline{g} = ds^2 + \overline{g}_s$, where

$$\overline{g}_s = (1/\sigma_1(s)^2)g_1(s) + \ldots + (1/\sigma_p(s)^2)g_p(s) , \tag{9.3.3}$$

where $\sigma_1, \ldots, \sigma_p$ are smooth functions, which are chosen such that the resulting metric $\overline{g}$ extends smoothly across the focal varieties. Essentially, if M is a sphere, we are changing the sizes of the principal curvature small spheres of M_s by a factor depending on s.

(iv) We perform a similar construction to case (iii) for the level hypersurface N_t, by expressing h_t in its principal curvature components as

$$h_t = h_1 + \ldots + h_q ,$$

and defining

$$\bar{h}_t = (1/\nu_1(t)^2)h_1 + \dots + (1/\nu_q(t)^2)h_q , \tag{9.3.4}$$

for some suitably chosen functions $\nu_1(t) , \dots , \nu_q(t)$.

(v) We can perform various combinations of (i) ... (iv) always in such a way that the resulting metric extends smoothly over all of $M(N)$.

We consider each case in turn.

(i) We first of all work out the connection coefficients for the new metric. Let $x \in M^*$, and locally about x choose an orthonormal basis (X_a, ξ) with respect to g, such that if $y \in M_s$ for some s, then $(X_a(y))$ is an orthonormal basis for $T_y M_s$. Now $\bar{g} = (1/\mu(s)^2) ds^2 + g_s$, and so $(\bar{X}_a, \bar{\xi}) = (X_a, \mu\xi)$ is an orthonormal basis with respect to $\bar{g}$.

<u>Proposition 9.3.1</u> <u>If</u> $\bar{\nabla}$ <u>is the Levi-Civita connection with respect to</u> $\bar{g}$, <u>then</u>

$$\bar{\nabla}_{\bar{X}} \bar{\xi} = \mu \nabla_X \xi , \tag{9.3.5}$$

<u>for all</u> $X \in TM_s$, <u>for</u> $s \in \operatorname{int} I_s$.

<u>Proof</u> We can adapt Proposition 9.1.2 to apply to our present situation, and if we write $\sigma^2 = 1$ and $\rho^2 = \mu^2$, then equation (9.1.10) tells us that

$$g(\xi, \nabla_X X) = \frac{1}{\mu^2} g(\xi, \bar{\nabla}_{\bar{X}} \bar{X}) ;$$

therefore

$$\begin{aligned}
- g(\nabla_X \xi, X) &= g(\xi, \nabla_X X) \\
&= \frac{1}{\mu^2} g(\xi, \bar{\nabla}_{\bar{X}} \bar{X}) \\
&= \bar{g}(\xi, \bar{\nabla}_{\bar{X}} \bar{X}) \\
&= \frac{1}{\mu} \bar{g}(\bar{\xi}, \bar{\nabla}_{\bar{X}} \bar{X}) \\
&= -\frac{1}{\mu} \bar{g}(\bar{\nabla}_{\bar{X}} \bar{\xi}, \bar{X}) \\
&= -\frac{1}{\mu} g(\bar{\nabla}_{\bar{X}} \bar{\xi}, X) .
\end{aligned}$$

In the new metric $\bar{g}$, $\bar{\nabla} s$ no longer has unit length. We therefore reparametrize s into $\bar{s} = \bar{s}(s)$, such that $\bar{g}(\bar{\nabla}\bar{s}, \bar{\nabla}\bar{s}) = 1$. Clearly the new map $\phi: (M,\bar{g}) \to (N,h)$ (we assume that $\bar{g}$ extends smoothly to M) is harmonically equivariant with respect to $\bar{s}$ and t. We must therefore compute the equations (4.1.4) and (4.1.5) in the new variable $\bar{s}$. Note we are using Theorem 4.2.1, since $\bar{s}$ is now a generalized isoparametric function.

Suppose $\lambda_1(s_0), \ldots, \lambda_p(s_0)$ are the principal curvatures of M_{s_0} with respect to the isoparametric function s, for some $s_0 \in \operatorname{int} I_s$. Then from Proposition 9.3.1, the principal curvatures of M_{s_0} with respect to the generalized isoparametric function $\bar{s}$ are given by $\bar{\lambda}_1(s_0), \ldots, \bar{\lambda}_p(s_0)$, where

$$\bar{\lambda}_k(s_0) = \mu(s_0)\lambda_k(s_0) , \tag{9.3.6}$$

for $k = 1, \ldots, p$. Therefore

$$\bar{\Delta}\bar{s} = \mu \Delta s , \tag{9.3.7}$$

where $\bar{\Delta}$ is the Laplacian with respect to $\bar{g}$.

Since $\mu_{j_k}(t)$ and $\gamma_k(s,t)$ remain unchanged, equation (4.1.4) becomes

$$\alpha''(\bar{s}) + \bar{\Delta}\bar{s}\,\alpha'(\bar{s}) + \sum_{k=1}^{p} \mu_{j_k}\gamma_k = 0; \tag{9.3.8}$$

this is the reduction equation in the new metric $\bar{g}$. Alternatively this equation can be written in terms of the variable s (from Lemma 5.1.1) as

$$\alpha''(s)\,|ds|^2_{\bar{g}} + \bar{\Delta} s\,\alpha'(s) + \sum_{k=1}^{p} \mu_{j_k}\gamma_k = 0 . \tag{9.3.9}$$

Now $|ds|^2_{\bar{g}} = \mu(s)^2$. Whilst

$$\bar{\Delta} s = s''(\bar{s})\,|d\bar{s}|^2_{\bar{g}} + s'(\bar{s})\,\bar{\Delta}\bar{s} . \tag{9.3.10}$$

But without loss of generality assume $\inf I_s = 0$, then $\bar{s}$ is given by

$$\bar{s}(s) = \int_0^s (1/\mu(u))\,du.$$

Therefore $s'(\bar{s}) = \mu(s)$ and $s''(\bar{s}) = \mu'(s)\mu(s)$. Thus equation (9.3.12) becomes comes

$$\mu(s)^2\alpha''(s) + (\mu'(s)\mu(s) + \mu(s)\bar{\Delta}\bar{s})\alpha'(s) + \sum_{k=1}^{p} \mu_{j_k}\gamma_k = 0. \qquad (9.3.11)$$

We therefore have, substituting equation (9.3.7) into equation (9.3.11);

Theorem 9.3.2 If $\phi:(M,g) \to (N.h)$ is equivariant with respect to isoparametric functions s and t. Then ϕ is harmonic with respect to the metric $\bar{g}$ of (i) if and only if

$$\alpha''(s) + \left(\frac{\mu'(s)}{\mu(s)} + \Delta s\right)\alpha'(s) + (1/\mu(s)^2)\sum_{k=1}^{p} \mu_{j_k}\gamma_k = 0\,, \qquad (9.3.12)$$

for all $s \in \text{int } I_s$.

(ii) Such a change in the metric h still preserves the fact that ϕ is equivariant, for we are simply reparametrizing the isoparametric function t. We therefore have

Theorem 9.3.3 If $\phi:(M,g) \to (N,h)$ is equivariant with respect to the isoparametric functions s and t. Then ϕ is harmonic with respect to the metric $\bar{h}$ of (ii) on N, if and only if

$$\alpha''(s) + \Delta s\,\alpha'(s) + \nu\sum_{k=1}^{p} \mu_{j_k}\gamma_k = 0\,. \qquad (9.3.13)$$

for all $s \in \text{int } I_s$.

Proof We have simply changed the principal curvatures $\mu_1, \ldots, \mu_q$ on the level hypersurfaces N_t by an amount given by equation (9.3.9), that is

$$\bar{\mu}_j(t) = \nu(t)\mu_j(t)\,.$$

Since t is a generalized isoparametric function on $(N,\bar{h})$, the result now follows from equation (4.2.1). □

(iii) First of all assume $p = 2$. Let $m_i = \dim S_i$, $i = 1,2$, and consider a local frame field adapted to the principal curvature spaces; $(X_a,\xi)_{a=1,\ldots,m-1} =$

$(X_i,X_r,\xi)_{i=1,\ldots,m_1,\ r=m_1+1,\ldots,m_1+m_2=m-1}$.

Proposition 9.3.4 If (X_a,ξ) is the frame field defined above, then $(\bar{X}_a,\bar{\xi}) =$

$(\bar{X}_i, \bar{X}_r, \bar{\xi}) = (\sigma_1 X_i, \sigma_2 X_r, \xi)$ is an orthonormal frame field with respect to $\bar{g}$. If $\bar{\nabla}$ is the Levi-Civita connection for $\bar{g}$ then the following formulae hold:

$$g(X_i, \bar{\nabla}_{\bar{\xi}} \bar{\xi}) = \sigma_1^2 g(X_i, \nabla_\xi \xi) = 0 \qquad (9.3.14)$$

$$g(X_r, \bar{\nabla}_{\bar{\xi}} \bar{\xi}) = \sigma_2^2 g(X_r, \nabla_\xi \xi) = 0 \qquad (9.3.15)$$

$$g(\xi, \bar{\nabla}_{\bar{\xi}} \bar{\xi}) = g(\xi, \nabla_\xi \xi) = 0 \qquad (9.3.16)$$

$$g(X_t, \bar{\nabla}_{\bar{X}_r} \bar{X}_s) = \sigma_2^2 g(X_t, \nabla_{X_r} X_s) \qquad (9.3.17)$$

$$g(X_k, \bar{\nabla}_{\bar{X}_i} \bar{X}_j) = \sigma_1^2 g(X_k, \nabla_{X_i} X_j) \qquad (9.3.18)$$

$$g(X_i, \bar{\nabla}_{\bar{X}_r} \bar{X}_s) = \sigma_1^2 g(X_i, \nabla_{X_r} X_s) \qquad (9.3.19)$$

$$g(X_r, \bar{\nabla}_{\bar{X}_i} \bar{X}_j) = \sigma_2^2 (X_r, \nabla_{X_i} X_j) \qquad (9.3.20)$$

$$g(X_r, \bar{\nabla}_{\bar{X}_i} \bar{X}_s) = \sigma_1 \sigma_2 g(X_r, \nabla_{X_i} X_s) \qquad (9.3.21)$$

$$g(X_j, \bar{\nabla}_{\bar{X}_r} \bar{X}_i) = \sigma_1 \sigma_2 g(X_j, \nabla_{X_r} X_i) \qquad (9.3.22)$$

$$g(X_i, \bar{\nabla}_{\bar{X}_j} \bar{\xi}) = g(X_i, \nabla_{X_j} \xi) - d\sigma_1(\xi) g(X_j, X_i)/2\sigma_1 \qquad (9.3.23)$$

$$g(X_r, \bar{\nabla}_{\bar{X}_s} \bar{\xi}) = g(X_r, \nabla_{X_s} \xi) - d\sigma(\xi) g(X_r, X_s)/2\sigma_2 \qquad (9.3.24)$$

$$g(X_r, \bar{\nabla}_{\bar{X}_i} \bar{\xi}) = 0 , \qquad (9.3.25)$$

etc., for all $i,j,k = 1, \ldots, m_1$ and $r,s,t = m_1 + 1, \ldots, m_1 + m_2$.

Similar formulae hold for any p, and we now assume p is arbitrary, and the principal curvatures are $\lambda_1, \ldots, \lambda_p$ with multiplicities $m_1, \ldots, m_p$ respectively.

Corollary 9.3.5 The integral curves of $\bar{\xi}$ are affinely parametrized geodesics with respect to $\bar{g}$.

Proof This follows from equations (9.3.14), (9.3.15) and (9.3.16). □

Corollary 9.3.6 If X_i is a principal curvature vector of M_s, with principal curvature λ_{k_i}, then in the metric $\bar{g}$, X_i is still a principal curvature vector of M_s with principal curvature

$$\bar{\lambda}_{k_i} = \lambda_{k_i} + \sigma_{k_i}'(s)/2\sigma_{k_i} ,$$

where $X_i \in S_{k_i}$.

Proof This follows from equations (9.3.23), (9.3.24) and (9.3.25). □

Corollary 9.3.7 With respect to the metric $\bar{g}$, s is a generalized isoparametric function.

Corollary 9.3.8 If $\bar{\Delta}$ is the Laplacian with respect to $\bar{g}$, then

$$\bar{\Delta} s = \Delta s - \sum_{k=1}^{p} m_k \sigma_k'(s)/2\sigma_k .$$

Proof This follows from Corollary 9.3.6 and Lemma 2.2.9 . □

Let $\bar{S}_\phi$ denote the stress-energy tensor of ϕ in the metric $\bar{g}$. Assume $p = 2$ again, and use the ranges of indices $1 \le i,j, \ldots \le m_1$; $m_1 + 1 \le r,s, \ldots \le m_1 + m_2$; $1 \le a,b, \ldots \le m_1 + m_2 = m-1$.

We make the following further assumption on the equivariant map ϕ . One of the conditions for equivariance is that $d\phi(S_k) \subset T_{j_k}$, for each $k = 1, \ldots, p$ and some $j_k = 1, \ldots, q$. If whenever $k \ne 1$, then $j_k \ne j_1$; call ϕ p- equivariant. For example, the map $\phi: S^{m-1} \to S^{n-1}$; $\phi(\cos s.x, \sin s.y)) = (\cos\alpha(s) g_1(x), \sin\alpha(s) g_2(y))$, $x \in S^{p-1}$, $y \in S^{q-1}$, $p+q = m$, $g_1: S^{p-1} \to S^{r-1}$, $g_2: S^{q-1} \to S^{s-1}$, harmonic of constant energy, is p- equivariant. On the other hand the map $\phi: S^3 \to S^2$; $\phi(\cos s.x, \sin s\, y)) = (\cos\alpha(s), \sin\alpha(s).x^k y^1)$, $x,y \in S^1$, is not p- equivariant.

Proposition 9.3.9 In the above notation, and provided ϕ is p- equivariant, then

$$\bar{\nabla}^* \bar{S}_\phi(\bar{X}_k) = 0 , \quad \bar{\nabla}^* \bar{S}_\phi(\bar{X}_r) = 0 ,$$

for all $k = 1, \ldots, m_1$ and all $r = m_1 + 1, \ldots, m_1 + m_2$.

Proof The divergence is given by (summing over repeated indices)

$$\bar{\nabla}^*\bar{S}_\phi(\bar{X}_k) = -\bar{\nabla}^*(\phi^* h_t)(\bar{X}_k)$$

$$= -\bar{X}_a(\phi^* h_t(\bar{X}_a,\bar{X}_k)) + \phi^* h_t(\bar{\nabla}_{\bar{X}_a}\bar{X}_a,\bar{X}_k) + \phi^* h_t(\bar{X}_a,\bar{\nabla}_{\bar{X}_a}\bar{X}_k)$$

$$= -\bar{X}_i(\phi^* h_t(\bar{X}_i,\bar{X}_k)) - \bar{X}_r(\phi^* h_t(\bar{X}_r,\bar{X}_k))$$

$$+ \phi^* h_t(\bar{\nabla}_{\bar{X}_i}\bar{X}_i,\bar{X}_k) + \phi^* h_t(\bar{\nabla}_{\bar{X}_r}\bar{X}_r,\bar{X}_k)$$

$$+ \phi^* h_t(\bar{X}_i,\bar{\nabla}_{\bar{X}_i}\bar{X}_k) + \phi^* h_t(\bar{X}_r,\bar{\nabla}_{\bar{X}_r}\bar{X}_k) \,.$$

Now the $2^{\underline{nd}}$ term vanishes since ϕ preserves the orthogonality of the principal curvature spaces. The $4^{\underline{th}}$ term vanishes from equation (9.3.19) and the fact that the mean curvature of the small sphere $\mathscr{S}_2$ is proportional to ξ (Proposition 2.2.1). The $6^{\underline{th}}$ term is equal to $\sigma_2^2\,\sigma_1\,\phi^* h_t(X_r,\nabla_{X_r}X_k)$. Now

$$g(X_s,\nabla_{X_r}X_k) = -g(\nabla_{X_r}X_s,X_k)\,.$$

But $\nabla_{X_r}X_s$ points in the ξ direction, since by Proposition 2.2.1, $\mathscr{S}_2$ is totally geodesic in M_s, and so $g(\nabla_{X_r}X_s,X_k) = 0$. Thus $\nabla_{X_r}X_k$ is either in S_1 space or proportional to ξ. Both are perpendicular to S_2 space, and since ϕ preserves this orthogonality; the $6^{\underline{th}}$ term vanishes also.

We have now established that

$$\bar{\nabla}^*\bar{S}_\phi(\bar{X}_k) = \sigma_1^3(-X_i(\phi^* h_t(X_i,X_k)) + \phi^* h_t(\nabla_{X_i}X_i,X_k) + \phi^* h_t(X_i,\nabla_{X_i}X_k))\,.$$

But this is zero since the right hand side of equation (4.1.11) is zero.

Similarly

$$\bar{\nabla}^*\bar{S}_\phi(\bar{X}_r) = 0\,.$$

Similarly for arbitrary p, and provided ϕ is p-equivariant, one can show $\bar{\nabla}^*\bar{S}_\phi$ is proportional to ξ. □

Theorem 9.3.10 *If $\phi:(M,g) \to (N,h)$ is p-equivariant with respect to isoparametric functions s and t. Then ϕ is harmonic with respect to the metric $\bar{g}$ of (iii) if and only if*

$$\alpha''(s) + (\Delta s - \sum_{k=1}^{p} m_k \frac{\sigma_k'(s)}{2\sigma_k(s)})\alpha'(s) - \tfrac{1}{2} d_t (\sum_{k=1}^{p} \sigma_k^2 \gamma_k)$$

$$- (1/2\,\alpha') \sum_{k=1}^{p} \sigma_k \sigma_k' \gamma_k = 0 ,$$

for $s \in \operatorname{int} I_s$.

Proof As before

$$\bar\nabla^* \bar S_\phi(\xi) = \tfrac{1}{2}(d_s \bar\gamma + d_t \bar\gamma . \alpha'(s)) - \alpha''(s)\alpha'(s)$$

$$- \bar\Delta s (\alpha'(s))^2 - \bar\nabla^*(\phi^* h_t)(\bar\xi) ,$$

where $\bar\gamma = \sum_{k=1}^{p} \bar\gamma_k$, with $\bar\gamma_k = \sigma_k^2 \gamma_k$. Then

$$(d/ds)\bar\gamma = 2\sum_{k=1}^{p} \sigma_k \sigma_k' \gamma_k + \sum_{k=1}^{p} \sigma_k^2 d_s \gamma_k .$$

Also

$$\bar\nabla^*(\phi^* h_t)(\bar\xi) = - \sum_a \phi^* h_t(\bar X_a, \bar\nabla_{\bar X_a} \bar\xi)$$

$$= - \sum_k \sum_{i_k} \sigma_k \phi^* h_t(X_{i_k}, -\bar\lambda_k X_{i_k}), \text{ where } (X_{i_k})_{i_k} \text{ span } S_k;$$

$$= \sum_k \sum_{i_k} \sigma_k^2 \bar\lambda_k \phi^* h_t(X_{i_k}, X_{i_k})$$

$$= \sum_k \sigma_k^2 \lambda_k \gamma_k + \sum_k \sigma_k^2 \sigma_k'(s)\gamma_k / 2\sigma_k .$$

Thus

$$\tfrac{1}{2} d_s \bar\gamma - \bar\nabla^*(\phi^* h_t)(\bar\xi) = \sum_{k=1}^{p} \sigma_k^2 (\tfrac{1}{2} d_s \gamma_k - \lambda_k \gamma_k) + \tfrac{1}{2}\sum_k \sigma_k \sigma_k' \gamma_k$$

$$= \tfrac{1}{2}\sum_k \sigma_k \sigma_k' \gamma_k , \text{ from Proposition 4.1.12 (i)} .$$

□

The result now follows from Corollary 9.3.8 .

Remark 9.3.11 We have defined the notion of ϕ being p-equivariant in order to carry out the deformation. We can replace this condition by another condition.

If $\phi : (M,g) \to (N,h)$ is equivariant (M,N are space forms) and $\phi|_{\mathscr{S}_k} : \mathscr{S}_k \to \mathscr{T}_{j_k}$

is harmonic of constant energy density, for each $k = 1, \dots, p$, then call ϕ <u>$\mathscr{S}$-equivariant with respect to</u> s <u>and</u> t. We remark that all the examples of equivariant maps which we have considered are also $\mathscr{S}$- equivariant.

If ϕ is $\mathscr{S}$- equivariant and $i_k: \mathscr{S}_k \to M$ is the inclusion map, then

$$\Delta(\phi \circ i_k) = d\phi(\Delta i_k) + \text{trace } \nabla d\phi(di_k, di_k)$$

is proportional to η, and since Δi_k is proportional to ξ; trace $\nabla d\phi(di_k, di_k)$ is proportional to η.

The new metric $\bar{g}$ has simply changed the sizes of the small spheres $\mathscr{S}_k$, and we conclude that $\phi_{s,t}: M_s \to N_t$ is still harmonic and of constant energy density in the metric $\bar{g}$. Thus ϕ is equivariant with respect to the generalized isoparametric functions s and t, and Theorem 9.3.10 is true with "p-equivariant" replaced by "$\mathscr{S}$-equivariant" as a consequence of Theorem 4.2.1 .

(iv) As in case (i), the integral curves of $\eta = \nabla t$ are geodesics in the metric $\bar{h}$, and the function $t: N \to \mathbb{R}$ is a generalized isoparametric function.

If we assume that $\phi: (M,g) \to (N,h)$ is $\mathscr{S}$-harmonically equivariant, then in the metric $\bar{h}$, $\phi_{s,t}: M_s \to N_t$ is harmonic of constant energy density. Thus $\phi:(M,g) \to (N,\bar{h})$ is equivariant with respect to the generalized isoparametric functions s and t. Theorem 4.2.1 applies, and we have

<u>Theorem 9.3.12</u> <u>If</u> $\phi: (M,g) \to (N,h)$ <u>is $\mathscr{S}$-equivariant with respect to the isoparametric functions</u> s <u>and</u> t. <u>Then</u> $\phi: (M,g) \to (N,\bar{h})$ <u>is harmonic, where</u> $\bar{h}$ <u>is defined in (iv), if and only if</u>

$$\alpha''(s) + \Delta s\, \alpha'(s) + \sum_k (\mu_{j_k} + \nu_{j_k}'/2\nu_{j_k})\gamma_k/\nu_{j_k}^2 = 0 ,$$

<u>for all</u> $s \in \text{int } I_s$.

<u>Proof</u> For each $k = 1, \dots, p$, we have

$$\bar{\gamma}_k(s,t) = \sum_{i_k} \phi^* \bar{h}_{j_k}(X_{i_k}, X_{i_k}) = (1/\nu_{j_k}(t)^2)\gamma_k(s,t) ,$$

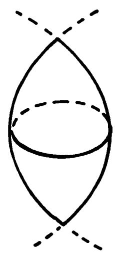
where $\bar{h}_j = h_j/\nu_j(t)^2$). In the metric $\bar{h}$ the principal curvatures become

$$\bar{\mu}_j = \mu_j + \nu_j'/2\nu_j ,$$

for each $j = 1, \ldots, q$. The result now follows from equation (4.2.1). □

9.4 Examples

Example 9.4.1 Consider a deformation of the kind given by case (i) of Section 9.3. The problem here is to define $\mu(s)$ in such a way that $\bar{g}$ is a smooth metric. For example, suppose $\mu(s) = K$, where K is some constant not equal to 1. Consider the join $S^2 = S^1 * S^0$. Then, with respect to the metric $\bar{g}$, S^2 still has constant positive curvature [38, vol.2, chapter 7, addendum 1] equal to K^2, and so has the appearance of a rugby ball [13].



The metric $\bar{g}$ is no longer C^1. However, we can use bump functions as follows.

Define

$$\mu(s) = 1 + \xi(s), \quad \xi > -1,$$

where $\xi : I_s \to \mathbb{R}$ is smooth, and $\overline{\text{supp}\ \xi} \subset \text{int}\ I_s$. Then $(M, \bar{g})$ is a smooth manifold, since the question of smoothness only arises across the focal varieties of s, and $g = \bar{g}$ in a neighbourhood of these focal varieties.

Suppose we are given one of Smith's maps between two spheres. Consider the reduction equation (equation (1.3.7)) - we use the reparametrized equation to avoid singularities. Use the same reparametrization for equation (9.3.12), so that time varies between $-\infty$ and $+\infty$. Then the asymptotic form of the equation as time $u \to \pm\infty$ will be the same in the deformed case as in the undeformed case. Theorem 6.1.9 and Lemma 6.1.11 will still apply to yield a non-trivial solution. Therefore for the maps of Example 5.3.1, the existence of solutions will be unaffected provided $|\xi|^2$ is small enough. For maps from Euclidean space to sphere and from hyperbolic space to sphere, the existence of solutions will again be unaffected. However, we would expect the asymptotic behaviour as time $u \to +\infty$ to change substantially with such deformations of the metric.

Example 9.4.2 Consider the join of two harmonic polynomial maps as described in Section 1.3 and Example 5.3.1. We thus consider a map $\phi: S^{m-1} \to S^{n-1}$ of the form

$$\phi(\cos s\, x, \sin s\, y) = (\cos\alpha(s) g_1(x), \sin\alpha(s) g_2(y)),$$

where $x \in S^{p-1}$, $y \in S^{q-1}$, $g_1: S^{p-1} \to S^{r-1}$ and $g_2: S^{q-1} \to S^{s-1}$ are harmonic polynomial maps with $|dg_i|^2 = \lambda_i$ constant, $p + q = m$ and $r + s = n$. Express the Euclidean metric on S^{m-1} in the form

$$g = ds^2 + \cos^2 s\, dx^2 + \sin^2 s.\, dy^2,$$

where dx^2 is the Euclidean metric for S^{p-1} and dy^2 that for S^{q-1}. We now perform a deformation of the metric incorporating deformations of types (i) and (iv) in order to make S^{m-1} ellipsoidal. The deformed metric has the form

$$\bar{g} = (a^2\sin^2 s + b^2\cos^2 s)\, ds^2 + a^2\cos^2 s\, dx^2 + b^2\sin^2 s\, dy^2.$$

The sphere has now become ellipsoidal with one set of axes of length a having multiplicity (p - 1), one set of length b with multiplicity (q - 1), and the other axis retaining its original length of 1.

Using the notations of equation (9.3.1) and equation (9.3.3) we have

$$\mu(s)^2 = 1/(a^2\sin^2 s + b^2\cos^2 s), \quad \sigma_1^2 = 1/a^2, \quad \sigma_2^2 = 1/b^2. \tag{9.4.1}$$

The reduction equation before the deformation has the form

$$\alpha''(s) + \Delta s\, \alpha'(s) + \sum_{k=1}^{2} \mu_{j_k}\gamma_k,$$

where $\Delta s = (q - 1)\cot s - (p - 1)\tan s$, $\gamma_1 = \lambda_1\cos^2(s)/\cos^2 s$, $\gamma_2 = \lambda_2\sin^2(s)/\sin^2 s$, $\mu_{j_1} = \tan\alpha(s)$ and $\mu_{j_2} = -\cot\alpha(s)$. Using Theorem 9.3.2 and Theorem 9.3.10, after the deformation the reduction equation becomes

$$\alpha''(s) + \left(\frac{\mu'(s)}{\mu(s)} + \Delta s\right)\alpha'(s) + (1/\mu(s))^2 \sum_{k=1}^{2} \mu_{j_k}\sigma_k^2\gamma_k. \tag{9.4.2}$$

We reparametrize equation (9.4.2) as before, defining a new variable u by $e^u = \tan s$. Note that care must be taken since equation (9.4.2) no longer has the form of equation (5.1.1) and so Lemma 5.1.1 no longer applies.

A computation shows that

$$\mu'(s) = -(a^2 - b^2)\sin s \cos s/(a^2\sin^2 s + b^2\cos^2 s)^{3/2} ,$$

or

$$\mu'(s(u)) = -\frac{(a^2 - b^2)}{e^u + e^{-u}} \left(\frac{a^2 e^u + b^2 e^{-u}}{e^u + e^{-u}}\right)^{3/2} .$$

This tends to 0 as u tends to both $-\infty$ and $+\infty$. Also

$$\mu(s(u))^2 = \frac{e^u + e^{-u}}{a^2 e^u + b^2 e^{-u}} .$$

which tends to $1/b^2$, $1/a^2$ as u tends to $-\infty$, $+\infty$ respectively. Equation (9.4.2) becomes, in the variable u ,

$$\alpha''(u) + \frac{1}{e^u + e^{-u}} \left[\frac{\mu'}{\mu} + (q-2)e^{-u} - (p-2)e^u\right] \alpha'(u)$$

$$+ \frac{(1/\mu^2)}{e^u + e^{-u}} \sin\alpha(u)\cos\alpha(u) \left(\frac{\lambda_1 e^u}{a^2} - \frac{\lambda_2 e^{-u}}{b^2}\right) = 0. \qquad (9.4.3)$$

If we abbreviate this in the form

$$\alpha''(u) + h(u)\alpha'(u) + g(u)\sin\alpha(u)\cos\alpha(u) = 0 ,$$

then as $u \to -\infty$,

$$\begin{cases} h(u) \to q - 2 \\ g(u) \to -\lambda_2 \end{cases}$$

and as $u \to +\infty$,

$$\begin{cases} h(u) \to -(p-2) \\ g(u) \to \lambda_1. \end{cases}$$

The damping conditions become

Thus the damping conditions are as for the Smith maps, given by equations (1.4.1). The methods of Section 6.1 (in particular Theorem 6.1.9 and Lemma 6.1.10) show that, provided the damping conditions are satisfied, equation (9.4.3) has a solution yielding a smooth harmonic map $\emptyset : S^{m-1} \to S^{n-1}$, where S^{m-1} has the ellipsoidal metric described above. We therefore have

Theorem 9.4.3 The join of two harmonic polynomial maps between spheres for which the damping conditions are satisfied, can be deformed into a harmonic map from the ellipsoid described above into the sphere. The ellipsoid has distinct eccentricities given by a, b and 1, for any $a, b > 0$.

Theorem 9.4.4. The join of two harmonic polynomial maps between spheres for which the damping conditions are not satisfied, cannot be rendered harmonic by an ellipsoidal deformation of the above kind on the domain sphere.

In contrast, however, we see in the next example that such ellipsoidal deformations on the range sphere do yield harmonic maps.

Example 9.4.5 Again consider the join of two harmonic polynomial maps as in the last example. We can deform the range sphere into an ellipsoid, by using deformations of types (ii) and (iv). That is, we express the Euclidean metric on S^{n-1} in the form

$$h = dt^2 + \cos^2 t \, du^2 + \sin^2 t \, dv^2 ,$$

where du^2, dv^2 are the Euclidean metrics for S^{r-1}, S^{s-1} respectively, then the deformed metric has the form

$$\bar{h} = (a^2 \sin^2 t + b^2 \cos^2 t) dt^2 + a^2 \cos^2 t \, du^2 + b^2 \sin^2 t \, dv^2 .$$

In the notations of equation (9.3.2) and the equation (9.3.4), we have

$$\nu(t)^2 = 1/(a^2 \sin^2 t + b^2 \cos^2 t), \ \nu_1^2 = 1/a^2, \ \nu_2^2 = 1/b^2 . \qquad (9.4.5)$$

From Theorem 9.3.3 and Theorem 9.3.12, the reduction equation becomes

$$\alpha''(s) + \Delta s \, \alpha'(s) + \nu(\alpha(s)) \sum_{k=1}^{2} \mu_{j_k} \frac{\gamma k}{\nu_{j_k}^2} = 0 .$$

Using the parameter u defined by $e^u = \tan s$, we obtain

$$\alpha''(u) + \frac{1}{e^u + e^{-u}}\left((q-2)e^{-u} - (p-2)e^{u}\right)\alpha'(u)$$

$$+ \frac{\sin\alpha(u)\cos\alpha(u)}{(e^u + e^{-u})(a^2\sin^2\alpha(u) + b^2\cos^2\alpha(u))^{\frac{1}{2}}} \cdot (a^2\lambda_1 e^u - b^2\lambda_2 e^{-u}) = 0 .$$

(9.4.5)

This equation fits into the category of equations described in Theorem 6.1.9. The arguments of that section apply, and we are assured of a solution provided the damping conditions are satisfied. These damping conditions are that

$$\begin{cases} (q-2)^2 < 4b\,\lambda_2 \\ (p-2)^2 < 4a\,\lambda_1 . \end{cases} \qquad (9.4.6)$$

Given the join of two harmonic polynomial maps, we can always find a and b such that (9.4.6) are satisfied. We therefore have

Theorem 9.4.6 The join of any two harmonic polynomial maps can always be rendered harmonic by a suitable deformation on the range sphere.

In particular we can apply Theorem 9.4.6 to Example 1.4.1 to yield

Theorem 9.4.7 For each n = 1,2, ... , there exists a smooth metric on the range sphere S^n (depending on n), such that each class of $\Pi_n(S^n) = \mathbb{Z}$ contains a harmonic representative. The deformed spheres are familiar ellipsoids whose eccentricities depend only on n and the degree.

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Index of definitions

Clifford system 39
Critical set of a map 122
Damping conditions 16
Degree of an isoparametric function 33
Dilation 12, 122
Energy density 8
Equivariant with respect to isoparametric functions 65
Focal variety 31
Generalized family of isoparametric hypersurfaces 40
Generalized isoparametric function 40
Harmonic map 1, 7
Harmonic morphism 11
Hopf invariant 18
Hopf map 11
Horizontally conformal 12, 122
Isoparametric function 27
Isoparametric hypersurface 28
Join of a map 11
Join of two spheres 11
Laplacian 7
Orthogonal multiplication 126
p- equivariant with respect to isoparametric functions 168
Reduction 2, 54
Reduction equation 2
Rendering problem 2
S-equivariant with respect to isoparametric functions 53
S-wavefront preserving with respect to isoparametric functions 51

Second fundamental form of a map 1, 7

Stiefel manifold 36

Stress-energy tensor 43

Unit speed reparametrization of a generalized family of isoparametric hypersurfaces 66

Wavefront preserving with respect to isoparametric functions 50