Exercices d'entrainement – solutions 3

14 (c) $f(x,y,z,t) = x^2y^2 + z^2t^2 + xz + yt$

$$\frac{\partial f}{\partial x} = 2xy^2 + z = 0 \quad (1)$$
$$\frac{\partial f}{\partial y} = 2x^2y + t = 0 \quad (2)$$
$$\frac{\partial f}{\partial z} = 2zt^2 + x = 0 \quad (3)$$
$$\frac{\partial f}{\partial t} = 2z^2t + y = 0 \quad (4)$$

$(4) \Rightarrow y = -2z^2t \overset{(2)}{\Longrightarrow} -4x^2z^2t + t = 0$: soit $t=0$ soit $x^2z^2 = \frac{1}{4}$
$\Rightarrow xz = \pm\frac{1}{2}$

$t = 0 \overset{(4)}{\Longrightarrow} y = 0 \overset{(1)}{\Longrightarrow} z = 0 \overset{(3)}{\Longrightarrow} x = 0$: $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ pt. critique.

Si $xz = \frac{1}{2}$: $(1)\times x \Rightarrow 2x^2y^2 + xz = 0 \Rightarrow x^2y^2 = -\frac{1}{4}$ impossible

d'où $\boxed{xz = -\frac{1}{2}}$ $\quad (1)\times x \Rightarrow x^2y^2 = \frac{1}{4} \Rightarrow xy = \pm\frac{1}{2}$

$xy = \frac{1}{2}$: $(1) \Rightarrow y + z = 0$, $(2) \Rightarrow x + t = 0$

$(3) \Rightarrow 2zt^2 - t = 0 \overset{t\neq 0}{\Longrightarrow} 2zt = 1 \Rightarrow zt = \frac{1}{2}$

$\left((1) \Rightarrow 2xy^2 - y = 0 \Rightarrow 2xy = 1 \Rightarrow xy = \frac{1}{2} \right)$

$\left((1) \Rightarrow 2xty^2 + tz = 0 \Rightarrow 2xty^2 \mp \frac{1}{2} = 0 \Rightarrow xty^2 = -\frac{1}{4} \Rightarrow xtz^2 = -\frac{1}{4} \right)$
$\Rightarrow tz = -\frac{1}{4xz} = \frac{1}{2}$

Resumé : $xz = -1/2$, $xy = \frac{1}{2}$, $zt = \frac{1}{2}$, $ty = -\frac{1}{2}$, $y + z = 0$, $x + t = 0$

Famille de solutions critiques $x = \lambda$, $z = -\frac{1}{2\lambda}$, $y = \frac{1}{2\lambda}$, $t = -\lambda$

$\begin{pmatrix}\lambda\\ \frac{1}{2\lambda}\\ -\frac{1}{2\lambda}\\ -\lambda\end{pmatrix}$ vérifient (1), (2), (3) et (4)

Si ~~$xy = -1/2$~~ $xy = -1/2$ $\quad (1) \Rightarrow y - z = 0$, $(2) \Rightarrow x - t = 0$

$\Rightarrow y = -\frac{1}{2x}$, $z = y = -\frac{1}{2x}$, $t = x$ $= \begin{pmatrix}\lambda\\ -\frac{1}{2\lambda}\\ -1/(2\lambda)\\ \lambda\end{pmatrix}$

Hessienne

$f_{xx} = 2y^2$, $f_{xy} = 4xy$, $f_{xz} = 1$, $f_{yy} = 2x^2$, $f_{yz} = 0$, $f_{yt} = 1$

$f_{zz} = 2t^2$, $f_{zt} = 4zt$, $f_{tt} = 2z^2$

$$\begin{pmatrix} 2y^2 & 4xy & 1 & 0 \\ 4xy & 2x^2 & 0 & 1 \\ 1 & 0 & 2t^2 & 4zt \\ 0 & 1 & 4zt & 2z^2 \end{pmatrix}$$

~~Difficile de calculer la nature~~

En l'origine $\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{vmatrix} = \lambda(\lambda^3 - \lambda) \mp \lambda^2 + 1$

$\lambda = 1$

donc $\lambda^2(\lambda+1) \mp \lambda + 1$

$\lambda = -1$ $\quad$ pas point de selle

14 (c) autre : pts $\begin{pmatrix} a \\ 1/2a \\ -1/2a \\ -a \end{pmatrix}$

$$H(f) = \begin{pmatrix} 2y^2 & 2 & 1 & 0 \\ 2 & 2x^2 & 0 & 1 \\ 1 & 0 & 2t^2 & 2 \\ 0 & 1 & 2 & 2z^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2a^2} & 2 & 1 & 0 \\ 2 & 2a^2 & 0 & 1 \\ 1 & 0 & 2a^2 & 2 \\ 0 & 1 & 2 & \frac{1}{2a^2} \end{pmatrix}$$

etc !

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15. $f = x_1 x_2 \dots x_n$ $\nabla f = \begin{pmatrix} x_2 x_3 \dots x_n \\ x_1 x_3 \dots x_n \\ \vdots \end{pmatrix} = f \begin{pmatrix} 1/x_1 \\ 1/x_2 \\ \vdots \\ 1/x_n \end{pmatrix}$

(a) $\nabla g = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ $\nabla f = \lambda \nabla g$

$\Leftrightarrow \frac{1}{x_1} = \frac{1}{x_2} = \dots = \frac{1}{x_n}$

$\Leftrightarrow x_1 = x_2 = \dots = x_n = \mu$ disons

$\Rightarrow n\mu - S = 0 \Rightarrow \mu = \frac{S}{n}$

et $\left(\frac{S}{n}, \dots, \frac{S}{n}\right)$ pt maximal

(b) En $\left(\frac{S}{n}, \dots, \frac{S}{n}\right)$, $f = \frac{S^n}{n^n} = \frac{(x_1 + \dots + x_n)^n}{n^n}$

d'où en générale : $x_1 x_2 \dots x_n \leq \frac{(x_1 + \dots + x_n)^n}{n^n}$

$\Rightarrow (x_1 x_2 \dots x_n)^{1/n} \leq \frac{x_1 + \dots + x_n}{n}$ //

16. $f(x,y,z) = x^2+y^2+z^2$, $\underbrace{x^2+y^2-1}_{g_1} = 0$, $\underbrace{x-2y+3z}_{g_2} = 0$

$\nabla f = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\nabla g_1 = 2\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, $\nabla g_2 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\lambda_1 \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$\Rightarrow \begin{array}{l} 2x = 2\lambda_1 x + \lambda_2 \\ 2y = 2\lambda_1 y - 2\lambda_2 \\ 2z = 3\lambda_2 \end{array} \Rightarrow \left.\begin{array}{l} 4x = 4\lambda_1 x + 2\lambda_2 \\ 2y = 2\lambda_1 y - 2\lambda_2 \end{array}\right\} \Rightarrow 4x + 2y = 4\lambda_1 x + 2\lambda_1 y$

$\Rightarrow (2x+y)(1-\lambda_1) = 0$

$\lambda_1 = 1 \Rightarrow \lambda_2 = 0 \Rightarrow z = 0 \Rightarrow x = \pm 2y$

$\Rightarrow 5y^2 - 1 = 0 \Rightarrow y = \pm\frac{1}{\sqrt{5}}$

$x = \pm\frac{2}{\sqrt{5}}$, $z = 0$

$y = -2x \Rightarrow 5x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{5}}$

$y = \mp\frac{2}{\sqrt{5}}$, $3z = 2y - x = \mp\frac{4}{\sqrt{5}} \mp \frac{1}{\sqrt{5}} = \mp\frac{5}{\sqrt{5}} = \mp\sqrt{5}$

Deux pnts : on calcule f pour savoir lequel est max, lequel est min. etc

1r(b) $\quad f = \text{distance}^2 = x^2 + y^2 + (z-1)^2$

Contraintes: $g_1 = x^2 + y^2 - z^2 = 0$, $\quad g_2 = x - 2y + z - 1 = 0$

$$\nabla f = 2\begin{pmatrix} x \\ y \\ z-1 \end{pmatrix}, \quad \nabla g_1 = 2\begin{pmatrix} x \\ y \\ -z \end{pmatrix}, \quad \nabla g_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$2\begin{pmatrix} x \\ y \\ z-1 \end{pmatrix} = 2\lambda_1 \begin{pmatrix} x \\ y \\ -z \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$\Rightarrow 2x = 2\lambda_1 x + \lambda_2 \Rightarrow 2x(1-\lambda_1) = \lambda_2 \quad (1)$

$2y = 2\lambda_1 y - 2\lambda_2 \Rightarrow 2y(1-\lambda_1) = -2\lambda_2 \quad (2)$

$2(z-1) = -2\lambda_1 z + \lambda_2 \Rightarrow 2z(1+\lambda_1) = \lambda_2 + 2 \quad (3)$

$(2) + 2 \times (1) \Rightarrow (4x + 2y)(1 - \lambda_1) = 0$

Soit $\lambda_1 = 1$ soit $y = -2x$

$\lambda_1 = 1 \Rightarrow \lambda_2 = 0 \overset{(3)}{\Rightarrow} z = \frac{1}{2}$; $g_2 = 0 \Rightarrow x - 2y = \frac{1}{2}$

$g_1: \; x^2 + y^2 = \frac{1}{4}$

$\Rightarrow \left(\frac{1}{2} + 2y\right)^2 + y^2 = \frac{1}{4} \Rightarrow \frac{1}{4} + 2y + 5y^2 = 1/4$

$\Rightarrow y(2 + 5y) = 0 \qquad y = 0 \Rightarrow x = \frac{1}{2}, \; z = 1/2$ $\qquad \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$

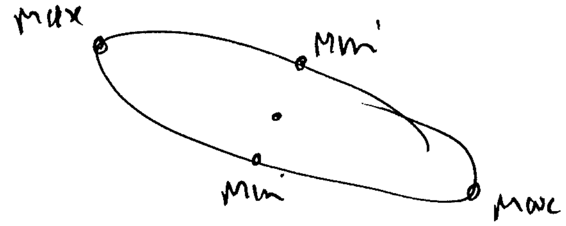
$y = -\frac{2}{5} \Rightarrow x = \frac{1}{2} + \frac{4}{5} = \frac{5+8}{10} = \frac{13}{10}$ $\qquad \begin{pmatrix} 13/10 \\ -2/5 \\ 1/2 \end{pmatrix}$

$y = -2x$: $g_1: x^2 + 4x^2 - z^2 = 0$, $\; g_2: x + 4x + z - 1 = 0$

$\Rightarrow 5x^2 = z^2, \quad 5x + z = 1, \quad x = \pm \frac{z}{\sqrt{5}}$

$x = \frac{z}{\sqrt{5}} \Rightarrow \sqrt{5}z + z = 1 \Rightarrow z = \frac{1}{1+\sqrt{5}} \Rightarrow x = \frac{1}{\sqrt{5}(1+\sqrt{5})}, \; y = -\frac{2}{\sqrt{5}(1+\sqrt{5})}$

$x = -\frac{z}{\sqrt{5}}$ etc



4 points: deux max, deux min.

Il faut calculer f pour en savoir lesquels sont max lesquels sont min

17 (a) Soit $f(x,y) = y^3 + y + e^x - 1$

$$\frac{\partial f}{\partial y} = 3y^2 + 1 \;;\; \frac{\partial f}{\partial y}(0,0) = 1 \neq 0$$

Par le théorème des fonctions implicites $f = 0$ détermine $y = y(x)$ dans un voisinage de $(0,0)$

On dérive $f(x,y) = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y'(x) = 0$

$$\Rightarrow y'(0) = -\frac{\partial f}{\partial x}(0,0) \Big/ \frac{\partial f}{\partial y}(0,0) = -e^0/1 = -1$$

~~On dérive une deuxième fois~~:

Plus specifiquement

$$\frac{\partial}{\partial x} f(x,y) = 3y^2 \, y'(x) + y'(x) + e^x = 0$$

On dérive encore : $6y\, y'(x)^2 + 3y^2 y''(x) + y''(x) + e^x = 0$

En l'origine $y''(0) = -e^0 = -1$

$y'(0) = -1$, $y''(0) = -1$.

$$y(h) = y(0) + h y'(0) + \frac{h^2}{2!} y''(0) + o(h^2)$$

Mais $f = 0 \Rightarrow y(0)^3 + y(0) + e^0 - 1 = 0$

$$\Rightarrow y(0)(y(0)^2 + 1) = 0 \Rightarrow y(0) = 0$$

$$\boxed{y(h) = -h - \frac{h^2}{2!} + o(h^2)}$$

(b) $f(x,y) = x(x+1)^2 - y^2$

$$\frac{\partial f}{\partial x} = (x+1)^2 + 2x(x+1) \;,\; \frac{\partial f}{\partial y} = -2y$$

$y = 0$ et soit $x = -1$, soit $x + 1 + 2x = 0 \Rightarrow x = -1/3$

$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ et $\begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$

$f_{xx} = 2(x+1) + 2(x+1) + 2x$
$= 6x + 4$

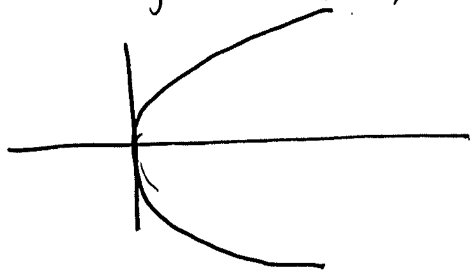
$f_{xy} = 0$, $f_{yy} = -2$

$$H(f) = \begin{pmatrix} 6x+4 & 0 \\ 0 & -2 \end{pmatrix}$$

$H(f)\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$: $\det H > 0 \Rightarrow$ max ou min
$f_{xx} < 0 \Rightarrow$ <u>max</u>

$H(f)\begin{pmatrix} -1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ point de selle.

(b) $y^2 = x(x+1)^2 \Rightarrow y = \pm\sqrt{x}\,(x+1)$



$2yy' = (x+1)^2 + 2x(x+1)$

$= (x+1)(3x+1)$

$x=0$, $y=0$

et $y' = \dfrac{(x+1)(3x+1)}{2y}$

$\to \pm\infty$ lorsque $x \to 0$

(c) On fait $x = -1+\varepsilon$,

$y^2 = (-1+\varepsilon)(\varepsilon^2) < 0$ pour ε assez petit

D'autre part $(-1, 0)$ vérifie à l'équation

d'où c'est un point isolé

(d) Application du théorème de la fonction implicite

(théorie du cours)