Stochastic Differential Games and associated Bellman-Isaacs Equations

Juan Li[1]   Rainer Buckdahn[2]
1) Shandong University, branch of Weihai, P.R. of China
2) Université de Bretagne Occidentale, Brest, France

Roscoff, March 2010

Marie Curie ITN „Controlled Systems"

An Introduction

Deterministic differential games:

Rufus Isaacs (1914 - 1981):
the first to study differential games:
1951 (published: 1956):

a pursuit game, more precisely
"the homocidal chauffeur problem":
two players:
P1: a runner (rather slow, but highly manoeuvrable)
against
P2: a driver of a motor vehicle (much faster, but less manoeuvrable)
Objective of P2: to flatten down the pedestrian,
Objective of P1: to elude the car

for the solution was used:
Level set method, variational calculus.
Rufus Isaacs considered special cases.

Since the pioneering work by Rufus Isaacs: a lot of works on determinist and also stochastic differential games:

Friedman (1971)
Elliot / Kalton (1972)
Fleming
⋮
Fleming / Souganidis (1989)
⋮
Marc Quincampoix (GDR „Games")
Pierre Cardaliaguet / Catherine Rainer

An easy model of a pursuit-evasion game:

Evader (P1): $\dot{x}(t) = u_t,\ t \geq 0,\ u_t \in U,\ x_0 \in R^2$

Pursuer (P2): $\dot{y}(t) = v_t,\ t \geq 0,\ v_t \in V,\ y_0 \in R^2$

+ $\rho \geq 0$;

Objective of P1: $\inf_{t \geq 0} |x(t) - y(t)| > \rho$

Objective of P2: $\inf_{t \geq 0} |x(t) - y(t)| \leq \rho$.

+ If it is not for catching the evador but a competition for money:

Example:

$J(u,v) = |X(T) - y(T)|$ amount P2 has to to pay to P1.

The dynamics can be written as:

$$d\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u_t \\ v_t \end{pmatrix} dt, \ t \geq 0; \ \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{R}^4.$$

More general, for initial data $(t,x) \in [0,T] \times \mathbb{R}^d$:

$$\begin{cases} dX_s^{t,x,u,v} = b(X_s^{t,x,u,v}, u_s, v_s)ds + \sigma(X_s^{t,x,u,v}, u_s, v_s)dW_s, \\ X_t^{t,x,u,v} = x \end{cases}$$

payoff / cost functional:

$$J(t,x;u,v) = E\left[\Phi(X_T^{t,x,u,v}) + \int_t^T f(X_s^{t,x,u,v}, u_s, v_s)ds\right]$$

2-persons zero-sum game:

$P1$: $J(t,x;u,v) \longrightarrow$ max over $u \in \mathcal{U}_{t,T}$

$P2$: $J(t,x;u,v) \longrightarrow$ min over $v \in \mathcal{V}_{t,T}$

Which type of game? Control against control?

In special cases possibles:

+ linear-quadratic SDG:
  existence of a saddle point control $(u^*, v^*) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, characterized by the associated Riccati equation.

+ S. Peng / J.-P. Lepeltier / S. Hamadène (1997 and consecutive works):

If
- $\sigma(x,u,v) = \sigma(x)$, $\exists \sigma(x)^{-1} \in \mathbb{R}^{d\times d}$, $|\sigma(x)^{-1}| \le C$,
- Cost functional given by a BSDE with controls,
- Isaacs condition holds

then S. Peng's Comparison theorem for BSDEs shows: $\exists$ saddle point control (u^*, v^*).

$(u^*, v^*) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ is saddle point iff

$$J(t,x,u,v^*) \le J(t,x,u^*,v^*) \le J(t,x,u^*,v)$$
$$\forall (u,v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$$

Note: If there is a saddle point:

$$\boxed{\begin{aligned} W(t,x) &:= \sup_{u\in\mathcal{U}_{t,T}} \inf_{v\in\mathcal{V}_{t,T}} J(t,x,u,v) \\ &= \\ U(t,x) &= \inf_{v\in\mathcal{V}_{t,T}} \sup_{u\in\mathcal{U}_{t,T}} J(t,x,u,v) \end{aligned}}$$

$\hat{=}$ game has a „value"
(value doesn't depend on who begins the game).

<u>In general</u>: no saddle point, even no value, although Isaacs condition holds.

IS

Example:

A (simplified) pursuit game:

P1 (rabbit): $\dot{X}_1^u(s) = u_s,\ s\in[0,T],\ u_s \in \overline{B_1(0)} \subset \mathbb{R}^2$

P2 (wolf): $\dot{X}_2^v(s) = v_s,\ s\in[0,T],\ v_s \in \overline{B_2(0)} \subset \mathbb{R}^2$

$X_1^u(0) = x^1;\ X_2^v(0) = x^2 \in \mathbb{R}^2;$

rabbit: $\mathcal{U} = L^0_{\mathbb{F}}(0,T;\overline{B_1(0)})$,

wolf: $\mathcal{V} = L^0_{\mathbb{F}}(0,T;\overline{B_2(0)})$.

x^2 x x x^1

„lunch time" T: $\quad x = (x^1, x^2)$:

$$J(0,x;u,v) = E\left[\,|X_1^u(T) - X_2^v(T)|\,\right]$$

rabbit: $J(0,x,u,v) \longrightarrow$ max over $u\in\mathcal{U}$

wolf: $J(0,x,u,v) \longrightarrow$ min over $v\in\mathcal{V}$.

$$W(0,x) := \sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{V}} J(0,x;u,v)$$

$$U(0,x) := \inf_{v\in\mathcal{V}} \sup_{u\in\mathcal{U}} J(0,x;u,v).$$

1) Rabbit begins the match:

Suppose $|x^1 - x^2| \le T$.

rabbit chooses: $u\in\mathcal{U}$;

wolf can then choose: $v_s = \dfrac{x^1 - x^2}{T} + u_s,\ s\in[0,T]$.

$\Rightarrow v\in\mathcal{V}$ and

$J(0,x,u,v) = 0$, i.e., $W(0,x) = 0$.

2) Wolf begins the match:

wolf chooses $v \in \mathcal{V}$,

rabbit then can choose

$$u_s := \begin{cases} 0, \text{ if } E[|x^1 - X_2^v(T)|] \geq \frac{(T\wedge 1)^2}{2}, & a) \\ (T\wedge 1)e, \text{ if } E[|x^1 - X_2^v(T)|] < \frac{(T\wedge 1)^2}{2}, & b) \end{cases}$$

$s \in [0,T]$, for $e \in \mathbb{R}^2$, $|e| = 1$.

{One could take $e = \frac{E[x^1 - X_2^v(T)]}{|E[x^1 - X_2^v(T)]|}$,

but any other direction also works}

Then: $u \in L^0(0,T; \overline{B_1(0)})$

$\subset L^0_{\mathbb{F}}(0,T; \overline{B_1(0)}) = \mathcal{U}$ admissible control

Moreover:

$$J(0,x,u,v) = E[|(X_1^u(T) - X_2^v(T)|]$$

$$= \begin{cases} E[|x^1 - X_2^v(T)|] \geq \frac{(T\wedge 1)^2}{2}, & \text{in case a)} \\ E[|x^1 + (T\wedge 1)T\cdot e - X_2^v(T)|] \\ \quad \geq (T\wedge 1)T|e| - \frac{(T\wedge 1)^2}{2} \geq \frac{(T\wedge 1)^2}{2}. \end{cases}$$

$$\Rightarrow \quad \sup_{u\in\mathcal{U}} J(0,x,u,v) \geq \frac{(T\wedge 1)^2}{2}, \ \forall v \in \mathcal{V}.$$

$$U(0,x) = \inf_{v\in\mathcal{V}} \sup_{u\in\mathcal{U}} J(0,x,u,v) \geq \frac{(T\wedge 1)^2}{2}.$$

$$\Rightarrow \boxed{W(0,x) = 0 < \frac{(T\wedge 1)^2}{2} \leq U(0,x),}$$

$\forall x = (x^1, x^2)$ with $|x^1 - x^2| \leq T$.

Concept used by Fleming / Souganidis (1989) in their pioneering paper on SDG:

strategies – controls.

In our mini-course: generalisation of of their paper

Remark:

P. Cardaliaguet / C. Rainer:

nonanticipating strategy with delay

- Contre- nonanticipating strategy with delay.