An Extension of the Divergence Operator for Gaussian Processes

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Equation

Consider the linear fractional differential equation of the form

$$X_t = \eta + \int_0^t a(s)X_s ds + \int_0^t b(s)X_s dB_s^H, \quad t \in [0, T].$$

Here $\eta \in L^2(\Omega)$, $a, b : [0, T] \to \mathbb{R}$ and $B^H = \{B_t^H : t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$.

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The stochastic integral is an extension of the divergence operator.

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Notation

Let $\mathcal{H}$ and $\mathcal{H}_0$ be two real separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$.

$$W = \{W(h) : h \in \mathcal{H}\}$$

is a Gaussian process on $\mathcal{H}$ such that

$$E(W(h)W(g)) = \langle h, g \rangle_{\mathcal{H}},$$

for $h, g \in \mathcal{H}$.
$\mathcal{F}$ is the σ–field generated by W.

Chaotic representation

Let $F \in L^2(\Omega, \mathcal{F}, P; \mathcal{H}_0)$. Then it has the representation

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \mathcal{H}^{\odot n} \otimes \mathcal{H}_0,$$

where

$$E\Big(\langle h, I_n(f_n) \rangle_{\mathcal{H}_0} (n_{i_1})! H_{n_{i_1}}(W(e_{i_1})) \cdots (n_{i_k})! H_{n_{i_k}}(W(e_{i_k})) \Big)$$

$$= \begin{cases} n! \langle f_n, e_{i_1}^{\otimes n_{i_1}} \otimes \cdots \otimes e_{i_k}^{\otimes n_{i_k}} \otimes h \rangle_{\mathcal{H}^{\otimes n} \otimes \mathcal{H}_0}, & \text{if } \sum_{j=1}^{k} n_{i_j} = n, \\ 0, & \text{otherwise.} \end{cases}$$

Here $h \in \mathcal{H}_0$, $\{e_i : i \in \mathbb{N}\}$ is an OCS of $\mathcal{H}$ and

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}),$$

$x \in \mathbb{R}$ and $n \geq 0$.

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Hypotheses

Throughout we assume that $\mathcal{H}$ is densely and continuously embedded in $\mathcal{H}_0$ and that

$$\mathcal{T} : \mathcal{H} \subset \mathcal{H}_0 \to \mathcal{H}_0$$

is a linear operator (whose domain $\mathcal{D}(\mathcal{T})$ is $\mathcal{H}$) satisfying the following conditions :

(H1) $|\mathcal{T}h|_{\mathcal{H}_0} = |h|_{\mathcal{H}}$, for all $h \in \mathcal{H}$.

(H2) $\mathcal{T}_{\mathcal{H}} := \{h \in \mathcal{H} : \mathcal{T}h \in \mathcal{D}(\mathcal{T}^*)\}$ is a dense subset of $\mathcal{H}$.

(H3) $\mathcal{T}_{\mathcal{H}_0} = \{\mathcal{T}^*\mathcal{T}h : h \in \mathcal{T}_{\mathcal{H}}\}$ is dense in $\mathcal{H}_0$.

$\mathcal{S}_{\mathcal{T}}$ is the family of all the smooth random variables of the form

$$F = f(W(g_1), \dots, W(g_n)),$$

where $\{g_1, \dots, g_n\}$ is in $\mathcal{D}(\mathcal{T}^*\mathcal{T})$, $f \in C_p^\infty(\mathbb{R}^n)$.

Derivative operator

Throughout we assume that $\mathcal{H}$ is densely and continuously embedded in $\mathcal{H}_0$ and that

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where $\{g_1, \ldots, g_n\}$ is in $\mathcal{D}(\mathcal{T}^*\mathcal{T})$, $f \in C_p^\infty(\mathbb{R}^n)$ and

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(g_1), \ldots, W(g_n))g_i.$$

Stochastic integral

Definition

Let $u \in L^2(\Omega, \mathcal{F}, P; \mathcal{H}_0)$. We say that u belongs to Dom δ^* if and only if there exists $\delta(u) \in L^2(\Omega)$ such that

$$
\begin{aligned}
E\langle D_{\mathcal{T}} F, u\rangle_{\mathcal{H}_0} &:= E\langle \mathcal{T}^* \mathcal{T} DF, u\rangle_{\mathcal{H}_0} \\
&= E(F\delta(u)),
\end{aligned}
$$

for every $F \in \mathcal{S}_{\mathcal{T}}$. In this case, the random variable $\delta(u)$ is called the extended divergence of u.

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Remarks.

i) If $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{T} = I_{\mathcal{H}}$, then (1) has the form

$$E\langle DF, u\rangle_{\mathcal{H}} = E(F\delta(u)).$$

Thus, δ is equal to the usual divergence operator.

ii) Let $u \in L^2(\Omega, \mathcal{F}, P; \mathcal{H})$. Then

$$\langle DF, u\rangle_{\mathcal{H}} = \langle \mathcal{T}^* \mathcal{T} DF, u\rangle_{\mathcal{H}_0}.$$

Characterization of δ

Theorem

Assume that (H1)–(H3) hold and that $u \in L^2(\Omega, \mathcal{F}; \mathcal{H}_0)$ has the chaos representation

$$u = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \mathcal{H}^{\odot n} \otimes \mathcal{H}_0.$$

Then $u \in$ Dom δ^* if and only if $\widetilde{f}_n$ (the symmetrization of f_n as an element of $\mathcal{H}_0^{\otimes(n+1)}$) belongs to $\mathcal{H}^{\odot(n+1)}$ for all $n \geq 0$, and

$$\sum_{n=1}^{\infty} n! |\widetilde{f}_{n-1}|^2_{\mathcal{H}^{\otimes n}} < \infty.$$

In this case $\delta(u) = \sum_{n=1}^{\infty} I_n(\widetilde{f}_{n-1})$.

Characterization of δ

Proof : Fix $n \geq 1$. Let $\{n_1, \dots, n_k\}$ be a finite sequence of positive integers such that $n_1 + \cdots + n_k = n$ and $\{g_1, \dots, g_k\} \subset \mathcal{T}_{\mathcal{H}}$ an orthonormal system on $\mathcal{H}$.

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Necessity : We have

$$
\begin{aligned}
& E\left(\langle u, D_T\left(n_1! H_{n_1}(W(g_1)) \dots n_k! H_{n_k}(W(g_k))\right)\rangle_{\mathcal{H}_0}\right) \\
= & \sum_{j=1}^{k} n_j (n-1)! \Big\langle f_{n-1}, (T^* T)^{\otimes(n-1)} (g_1^{\otimes n_1} \\
& \otimes \cdots \otimes g_{j-1}^{\otimes n_{j-1}} \otimes g_j^{\otimes(n_j - 1)} \otimes \cdots \otimes g_{n_k}^{\otimes n_k}) \\
& \otimes T^* T g_j \Big\rangle_{\mathcal{H}_0^{\otimes n}} .
\end{aligned}
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Hence, if $\delta(u)$ has the chaos representation

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\delta(u) = \sum_{n=0}^{\infty} I_n(v_n), \quad v_n \in \mathcal{H}^{\odot n},
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&E\left(\langle u, D_T\left(n_1! H_{n_1}(W(g_1)) \dots n_k! H_{n_k}(W(g_k))\right)\rangle_{\mathcal{H}_0}\right) \\
&= \sum_{j=1}^{k} n_j (n-1)! \langle f_{n-1}, (T^*T)^{\otimes(n-1)} (g_1^{\otimes n_1} \\
&\qquad \otimes \cdots \otimes g_{j-1}^{\otimes n_{j-1}} \otimes g_j^{\otimes(n_j-1)} \otimes \cdots \otimes g_{n_k}^{\otimes n_k}) \\
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\delta(u) = \sum_{n=0}^{\infty} I_n(v_n), \quad v_n \in \mathcal{H}^{\odot n},
$$

then the duality relation (1) and (H3) yield that $v_n = \widetilde{f}_{n-1}$, and therefore $\sum_{n=1}^{\infty} n! |\widetilde{f}_{n-1}|^2_{\mathcal{H}^{\otimes n}} < \infty$.

Characterization of δ

Sufficiency : Let $F = f(W(g_1), \ldots, W(g_k))$ be a random variable in $\mathcal{S}_T$ and $\mathcal{K}$ the linear subspace of $\mathcal{H}$ generated by $\{g_1, \ldots, g_k\}$. Then F has the chaos decompostion given by

$$F = \sum_{n=0}^{\infty} I_n(k_n), \quad k_n \in \mathcal{K}^{\odot n}.$$

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Consequently,

$$\begin{aligned} & E\langle u, D_T F\rangle_{\mathcal{H}_0} \\ = & \sum_{n=0}^{\infty}(n+1)!\langle f_n, (T^* T)^{\otimes(n+1)}(k_{n+1})\rangle_{\mathcal{H}_0^{\otimes(n+1)}} \\ = & \sum_{n=0}^{\infty}(n+1)!\langle \widetilde{f}_n, (T^* T)^{\otimes(n+1)}(k_{n+1})\rangle_{\mathcal{H}_0^{\otimes(n+1)}} \end{aligned}$$

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&= \sum_{n=0}^{\infty}(n+1)!\langle \widetilde{f}_n, k_{n+1}\rangle_{\mathcal{H}^{\otimes(n+1)}}
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That is, the duality relation (1) is satisfied for u and $\delta(u) := \sum_{n=1}^{\infty} I_n(\widetilde{f}_{n-1})$. □

An example

Let $B^H = \{B_t{}^H : t \in [0, \tilde{a}]\}$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1/2)$. From Pipiras and Taqqu, we know that the fBm B^H is a Gaussian process on the Hilbert space

$$\mathcal{H} = \Big\{ f : [0, \tilde{a}] \to \mathbb{R} : \exists \phi_f \in L^2([0, \tilde{a}]) \text{ such that } f(u) = u^{\alpha}(I^{\alpha}_{\tilde{a}-}(s^{-\alpha}\phi_f(s)))(u) \Big\}$$

with

$$\langle f, g \rangle_{\mathcal{H}} = C_H \langle \phi_f, \phi_g \rangle_{L^2([0,\tilde{a}])}.$$

Here $\alpha = \frac{1}{2} - H$ and

$$(I^{\alpha}_{\tilde{a}-} f)(s) = \Gamma(\alpha)^{-1} \int_s^{\tilde{a}} f(u)(u-s)^{\alpha-1} du,$$

for a.a. $s \in [0, \tilde{a}]$.

FBm case

$$\mathcal{H} = \Big\{ f : [0, \tilde{a}] \to \mathbb{R} : \exists \phi_f \in L^2([0, \tilde{a}]) \ \text{such that} \\ f(u) = u^{\alpha}(I^{\alpha}_{\tilde{a}-}(s^{-\alpha}\phi_f(s)))(u) \Big\}$$

Proposition

The space $\mathcal{H}$ is densely and continuously embedded in $L^2([0, \tilde{a}])$.

FBm case

Proposition

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Proof : Let $f \in \mathcal{H}$. Then there exists $\phi_f \in L^2([0, \tilde{a}])$ such that

$$\begin{aligned} &\int_0^{\tilde{a}} (f(u))^2 du \\ &\leq C_\alpha \int_0^{\tilde{a}} \left(\int_u^{\tilde{a}} (r-u)^{\alpha-1} \phi_f(r) dr \right)^2 du \\ &\leq C_{\alpha,\tilde{a}} \int_0^{\tilde{a}} \phi_f(u)^2 du, \end{aligned}$$

which implies that $\mathcal{H}$ is continuously embedded in $L^2([0, \tilde{a}])$.
Finally, from Pipiras and Taqqu, we have that the step functions are included in $\mathcal{H}$. Thus the proof is finished. □

FBm case

Now we introduce the linear operator

$$T : \mathcal{H} \subset \mathcal{H}_0 \to \mathcal{H}_0$$

defined by

$$(Tf)(u) = C_H^{1/2} \phi_f(u), \tag{2}$$

where $f(u) = u^{\alpha}(I^{\alpha}_{\tilde{a}-}(s^{-\alpha}\phi_f(s)))(u)$.

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$$(Tf)(u) = C_H^{1/2} \phi_f(u), \tag{3}$$

where $f(u) = u^\alpha (I^\alpha_{\tilde{a}-}(s^{-\alpha}\phi_f(s)))(u)$. Henceforth, D^α_{0+} is the inverse operator of

$$I^\alpha_{0+}(f)(s) = \Gamma(\alpha)^{-1} \int_0^s f(r)(s-r)^{\alpha-1} dr.$$

FBm case

Proposition

Let $g : [0, \tilde{a}] \to \mathbb{R}$ *be a function such that* $u \mapsto u^{\alpha} g(u)$ *belongs to* $I^{\alpha}_{0+}(L^q([0, \tilde{a}]))$ *for some* $q > \alpha^{-1} \vee H^{-1}$*. Then,* $g \in$ *Dom* T^* *and for* $u \in [0, \tilde{a}]$,

$$(T^* g)(u) = C_H^{1/2} u^{-\alpha} D^{\alpha}_{0+}(s^{\alpha} g(s))(u).$$

FBm case

Let

$$T : \mathcal{H} \subset \mathcal{H}_0 \to \mathcal{H}_0$$

be defined by

$$(Tf)(u) = C_H^{1/2} \phi_f(u), \tag{4}$$

where $f(u) = u^{\alpha}(I^{\alpha}_{\tilde{a}-}(s^{-\alpha}\phi_f(s)))(u)$.

Proposition

The operator T satisfies conditions (H1)–(H3).

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Proof : We define

$$\begin{aligned} \mathcal{H}_* &= \{f \in \mathcal{H} : \exists f^* \in L^\infty([0,\tilde{a}]) \quad \text{such that} \\ &\qquad \phi_f(u) = u^{-\alpha} I^\alpha_{0+}(s^\alpha f^*(s))(u)\}. \end{aligned}$$

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$\mathcal{H}_*$ is a dense set of $\mathcal{H}$ because the family.

$$\begin{aligned} L^2_* &= \{f \in L^2([0, \tilde{a}]) : \ \exists f^* \in L^\infty \text{ such that} \\ &\qquad f(u) = u^{-\alpha} I^\alpha_{0+}(s^\alpha f^*(s))(u)\} \end{aligned}$$

is a dense subset of $L^2([0, \tilde{a}])$.

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Proposition

The operator T satisfies conditions (H1)–(H3).

Proof : Let $g \in L^2([0, \tilde{a}])$ such that for any $f^* \in L^\infty([0, \tilde{a}])$,

$$0 = \int_0^{\tilde{a}} g(u) u^{-\alpha} I_{0+}^{\alpha}(s^\alpha f^*(s))(u) du.$$

Hence,

$$0 = \int_0^{\tilde{a}} I_{\tilde{a}-}^{\alpha}(s^{-\alpha} g(s))(u) u^\alpha f^*(u) du.$$

Consequently, $g = 0$.
Finally, Proposition 6 gives $\mathcal{H}_* \subset \mathcal{T}_{\mathcal{H}}$ and $L^\infty([0, \tilde{a}]) \subset \mathcal{T}_{L^2([0,\tilde{a}])}$. □

FBm

Proposition

Let $H \in (0, 1/4)$. Then B^H belongs to Dom δ^*, but is not in $\mathcal{H}$ w.p.1.

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Let $H \in (0, 1/4)$. Then B^H belongs to Dom δ^*, but is not in $\mathcal{H}$ w.p.1.

Proof : We know

$$B_t^H = I_1(1_{[0,t]}) \in L^2(\Omega; L^2([0, \tilde{a}])).$$

Since $\widetilde{1_{[0,t]}}(\cdot) = \frac{1}{2}(1 \otimes 1)$ (symmetrization as an element of $(L^2([0, \tilde{a}]))^{\otimes 2}$), we get B^H belongs to Dom δ^*.

FBm

Proposition

Let $H \in (0, 1/4)$. Then B^H belongs to Dom δ^*, but is not in $\mathcal{H}$ w.p.1.

Proof :Finally, Cheridito and Nualart have proven that there is a sequence $(t_n)_n$ tending to zero such that

$$t_n^{-2H} \int_0^{\tilde{a}-t_n} (B^H_{s+t_n} - B^H_s)^2 ds \to \tilde{a}, \tag{5}$$

as $n \to \infty$ w.p.1.

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$$t_n^{-2H} \int_0^{\tilde{a}-t_n} (B^H_{s+t_n} - B^H_s)^2 ds \to \tilde{a}, \tag{6}$$

as $n \to \infty$ w.p.1.
On the other hand, if there is $w_0 \in \Omega$ such that $B^H_{\cdot}(\omega_0) \in \mathcal{H}$, then Samko et al. imply

$$\int_0^{\tilde{a}-t} (B^H_{t+s}(\omega_0) - B^H_s(\omega_0))^2 ds = o(t^{2\alpha}). \tag{7}$$

Thus, the fact that $H \in (0, 1/4)$ implies the result.

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Consider the linear fractional differential equation of the form

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s dB_s^H, \quad t \in [0, T].$$

Here $\eta \in L^2(\Omega)$, $a, b : [0, T] \to \mathbb{R}$ and $B^H = \{B_t^H : t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$.

Brownian motion case

Now assume that $H = \frac{1}{2}$ and consider the linear fractional differential equation of the form

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s dW_s, \quad t \in [0, T].$$

1. For $\eta \in \mathbb{R}$, the Itô's formula gives

$$X_t = \eta \exp\left(\int_0^t a(s) ds + \int_0^t b(s) dW_s - \frac{1}{2} \int_0^t b(s) ds \right).$$

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2. Let $(\Omega, \mathcal{F}, P)$ be the canonical Wiener space and $\eta \in L^2(\Omega)$. Then, By Buckdahn, the Girsanov theorem implies

$$X_t = \eta(A_t) \exp\left(\int_0^t (a(s) - \frac{1}{2} b(s)^2) ds + \int_0^t b(s) dW_s \right),$$

where $A_t : \Omega \to \Omega$ is defined by

$$A_t(\omega)_s = \omega_s - \int_0^{t \wedge s} b(u) du.$$

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Now assume that $H = \frac{1}{2}$ and consider the linear fractional differential equation of the form

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s dW_s, \quad t \in [0, T].$$

3. If $X_t = \sum_{n=0}^{\infty} I_n(f_n^t)$, with $f_n \in L^2([0,T]^{n+1})$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} I_n(f_n^t) &= \sum_{n=0}^{\infty} I_n(\eta_n) + \sum_{n=0}^{\infty} I_n\left(\int_0^t a(s) f_n^s ds\right) \\ &\quad + \sum_{n=0}^{\infty} I_{n+1}\left(1_{[0,t]}\widetilde{(\cdot) b(\cdot) f_n^{\cdot}}\right). \end{aligned}$$

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$$\begin{aligned} \sum_{n=0}^{\infty} I_n(f_n^t) &= \sum_{n=0}^{\infty} I_n(\eta_n) + \sum_{n=0}^{\infty} I_n\left(\int_0^t a(s) f_n^s ds\right) \\ &+ \sum_{n=0}^{\infty} I_{n+1}\left(1_{[0,t]}\widetilde{(\cdot)b(\cdot)} f_n^{\cdot}\right). \end{aligned}$$

Remark. Note that, in this case, $1_{[0,t]}b \in L^2([0, T])$ for $b \in L^2([0, T])$.

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consider the linear fractional differential equation of the form

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s dW_s, \quad t \in [0, T].$$

3. If $X_t = \sum_{n=0}^{\infty} I_n(f_n^t)$, with $f_n \in L^2([0, T]^{n+1})$. Then

$$\sum_{n=0}^{\infty} I_n(f_n^t) = \sum_{n=0}^{\infty} I_n(\eta_n) + \sum_{n=0}^{\infty} I_n \left(\int_0^t a(s) f_n^s ds \right) + \sum_{n=0}^{\infty} I_{n+1} \left(1_{[0,t]}\widetilde{(\cdot) b(}\cdot) f_n^{\cdot} \right).$$

Remark. Note that, in this case, $1_{[0,t]} b \in L^2([0, T])$ for $b \in L^2([0, T])$. In the fBm case, we need to show that

$$1_{[0,t]}\widetilde{(\cdot) b(}\cdot) f_n^{\cdot} \in \mathrm{Dom}\delta$$

.

FBm case

Consider the linear fractional differential equation of the form

$$X_t = \eta + \int_0^t a(s)X_s ds + \int_0^t b(s)X_s dB_s^H, \quad t \in [0, T].$$

Let $\eta \in \mathbb{R}$. The Itô's formula gives :

a) For $H > \frac{1}{2}$,

$$X_t = \eta \exp\left(\int_0^t a(s)ds + \int_0^t b(s)dB_s^H - \frac{1}{2}\int_0^t \int_0^t b(s)b(r)|r-s|^{2H-2}dsdr\right).$$

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Let $\eta \in \mathbb{R}$. The Itô's formula gives :

a) For $H > \frac{1}{2}$,

$$\begin{aligned} X_t &= \eta \exp\left(\int_0^t a(s)ds + \int_0^t b(s)dB_s^H \right. \\ &\qquad \left. -\frac{1}{2}\int_0^t \int_0^t b(s)b(r)|r-s|^{2H-2} dsdr\right). \end{aligned}$$

b) For $H < \frac{1}{2}$,

$$X_t = \eta \exp\left(\int_0^t a(s)ds + \int_0^t b(s)dB_s^H - \frac{1}{2}|b1_{[0,t]}|^2_{\mathcal{H}}\right).$$

FBm case

For $H > \frac{1}{2}$, we have that

$$|f_n^t|_{\mathcal{H}^{\otimes n}} \leq c^n |f_n^t|_{L^2([0,T]^n)}.$$

Therefore, $X_t = \sum_{n=0}^{\infty} I_n^{B^H}(f_n^t)$ is solution of

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if $Y_t = \sum_{n=0}^{\infty} I_n^{W}(f_n^t)$ is solution of

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s d W_s, \quad t \in [0, T].$$

Contents

Uniqueness

If $X_t = \sum_{n=0}^{\infty} I_n^{B^H}(f_n^t)$ is solution of

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s dB_s^H, \quad t \in [0, T].$$

Then

$$\begin{aligned} f_n^t(t_1, \ldots, t_n) &= \exp\left(\int_0^t a(s) ds\right) \Bigg[\eta_n(t_1, \ldots, t_n) + \sum_{j=1}^{n} \\ &\quad \times \sum_{\Delta_{j,n}} \frac{(n-j)!}{j! n!} (b 1_{[0,t]})^{\otimes j} (t_{i_1}, \ldots, t_{i_j}) \eta_{n-j}(\hat{t}_{i_1}, \ldots, \hat{t}_{i_j}) \Bigg], \end{aligned}$$

with

$$\Delta_{j,n} = \{\{i_1, \ldots, i_j\} : i_k \neq i_l \text{ if } l \neq k\}.$$

Existence

Assume that

$$f_n^t(t_1,\ldots,t_n) = \exp\left(\int_0^t a(s)ds\right)\Bigg[\eta_n(t_1,\ldots,t_n) + \sum_{j=1}^{n}$$

$$\times \sum_{\Delta_{j,n}} \frac{(n-j)!}{j!n!}(b1_{[0,t]})^{\otimes j}(t_{i_1},\ldots,t_{i_j})\eta_{n-j}(\hat{t}_{i_1},\ldots,\hat{t}_{i_j})\Bigg],$$

satisfies :

1. $f_n \in \mathcal{H}^{\odot n} \otimes L^2([0,T])$.

Existence

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$$f_n^t(t_1,\dots,t_n) = \exp\left(\int_0^t a(s)ds\right)\Bigg[\eta_n(t_1,\dots,t_n)+\sum_{j=1}^n \times \sum_{\Delta_{j,n}} \frac{(n-j)!}{j!n!}(b1_{[0,t]})^{\otimes j}(t_{i_1},\dots,t_{i_j})\eta_{n-j}(\hat{t}_{i_1},\dots,\hat{t}_{i_j})\Bigg],$$

satisfies :

1. $f_n \in \mathcal{H}^{\odot n}\otimes L^2([0,T])$.
2. The process $Y_t=\sum_{n=0}^\infty I_n(f_n^t)$ is in $L^2(\Omega\times[0,T])$. That is

$$\sum_{n=0}^\infty n!\int_0^T ||f_n^t||^2_{\mathcal{H}^{\otimes n}}dt<\infty.$$

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$$f_n^t(t_1,\dots,t_n) = \exp\left(\int_0^t a(s)ds\right)\Bigg[\eta_n(t_1,\dots,t_n)+\sum_{j=1}^{n} \times \sum_{\Delta_{j,n}} \frac{(n-j)!}{j!n!}(b1_{[0,t]})^{\otimes j}(t_{i_1},\dots,t_{i_j})\eta_{n-j}(\hat{t}_{i_1},\dots,\hat{t}_{i_j})\Bigg],$$

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Then Y is a solution of

$$X_t=\eta+\int_0^t a(s)X_s ds+\int_0^t b(s)X_s dB_s^H, \quad t\in[0,T].$$

Main tool

Lemma

Let $f \in \mathcal{H}$ *be such that* $\phi_f \in L^p([0,T])$, *for some* $p \in (2, \frac{1}{1/2-H})$. *Then* $f1_{[0,t]}$ *is also in* $\mathcal{H}$ *and*

$$||\phi_{f1_{[0,t]}}||_{L^{p'}([0,T])} \leq C||\phi_f||_{L^p([0,T])},$$

for $p' \in (2,p)$.

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for $p' \in (2, p)$*.*

Remark. Remember that if $f \in \mathcal{H}$, then

$$f(u) = u^{1/2-H} I_{T-}^{1/2-H}(s^{H-1/2}\phi_f(s))(u), \quad u \in [0,T].$$

Main result

Theorem

Let $a \in L^2([0,T])$, $b \in \mathcal{H}$ *and* $p \in (2, \frac{1}{1/2-H})$ *such that* $\phi_b \in L^p([0,T])$ *and*

$$\sum_{k=0}^{\infty}(k+1)!||\eta_k||^2_{\mathcal{H}^{\otimes k}}(1 + C_H(||\phi_b||^2_{L^2} + \sup_{t\in[0,T]} ||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}})^2)^k < \infty$$

for some $\tilde{p} \in (2,p)$. *Then Conditions 1-3 are satisfied.*

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Remarks.

1. Remember that Conditions 1-3 imply that the equation

$$X_t = \eta + \int_0^t a(s)X_s ds + \int_0^t b(s)X_s dB^H_s, \quad t \in [0,T].$$

has a unique solution.

Main result

Theorem

Let $a \in L^2([0,T])$, $b \in \mathcal{H}$ and $p \in (2, \frac{1}{1/2-H})$ such that $\phi_b \in L^p([0,T])$ and

$$\sum_{k=0}^{\infty}(k+1)!||\eta_k||^2_{\mathcal{H}^{\otimes k}}(1 + C_H(||\phi_b||^2_{L^2} + \sup_{t\in[0,T]} ||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}})^2)^k < \infty$$

for some $\tilde{p} \in (2,p)$. Then Conditions 1-3 are satisfied.

Remarks.

1. Remember that Conditions 1-3 imply that our equation has a unique solution.
2. We have already studied bounds for $||\phi_{b1_{[0,t]}}||_{L^p}$.

Main result : Examples

Theorem

Let $a \in L^2([0,T])$, $b \in \mathcal{H}$ *and* $p \in (2, \frac{1}{1/2-H})$ *such that* $\phi_b \in L^p([0,T])$ *and*

$$\sum_{k=0}^{\infty}(k+1)!||\eta_k||^2_{\mathcal{H}^{\otimes k}}(1+C_H(||\phi_b||^2_{L^2}+\sup_{t\in[0,T]}||\phi_{b1_{[0,t]}}||_{L^{\tilde{p}}})^2)^k<\infty$$

for some $\tilde{p} \in (2,p)$. *Then Conditions 1-3 are satisfied.*

Remarks.

1. η has a finite chaos decomposition
2. $||\eta_n||_{\mathcal{H}^{\otimes n}} \leq \frac{c^n}{n!}$
3. There exists $\varepsilon > 0$ such that

$$\sum_{k=0}^{\infty}(k!)^{1+\varepsilon}||\eta_k||^2_{\mathcal{H}^{\otimes k}} < \infty.$$

Contents

Equation

Now we study the equation

$$X_t = X_0 + \int_0^t \sigma_s X_s dB_s^H + \int_0^t b(s, X_s) ds, \quad t \in [0, T].$$

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Here, $H \in (0, 1)$, $X_0 \in L^p(\Omega)$, for some $p \geq 2$, $\sigma \in \mathbb{L}_W^{1,\infty}$ and $b : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a measurable function

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- $\int_0^T \gamma_s ds \leq M$ and $|b(t, 0, \omega)| \leq M$ for some $t \in [0, T]$.

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Also we assume that $\Omega = C_0([0, T])$ and define

$$(T_t \omega)_s = \omega_s + \int_0^{t \wedge s} K_H(s, r) \sigma_r (T_r \omega) dr, \quad s, t \in [0, T].$$

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Then, by Jien and Ma (2009),

$$X_t = L_t Z_t(A_t, X_0(A_t)), \quad t \in [0, T].$$

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$$Z_t(\omega, x) = x + \int_0^t L_s^{-1} b(s, L_s(T_s\omega)Z_s(\omega, x), T_s\omega)ds$$

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where $A_t T_t = T_t A_t = I$ and

$$E\left(G(A_t)L_t\right) = E(G).$$

Contents

Equation

Consider the semilinear SPDE

$$
\begin{aligned}
du(t,x) &= (Lu(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(x)))ds \\
&\quad +\gamma_t u(t,x)dB_t, \;\; t \in [0,T], \\
u(0,x) &= \Phi(x), \quad x \in \mathbb{R}.
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where

- $L = \frac{1}{2} tr(\sigma\sigma^* \partial_x^2) + b(x)\nabla.$

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- The stochastic integral is the extension of the divergence operator.

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- $L = \frac{1}{2}tr(\sigma\sigma^*\partial_x^2) + b(x)\nabla$.
- $H \in (0, 1/2)$.
- The stochastic integral is the extension of the divergence operator.
- Combining Buckdahn's method and Pardoux and Peng approach, Buckdahn, Jing and León have studied viscosity solutions.

Idea

The fractional backward doubly SDE

$$Y_t = \xi + \int_0^t f(s, Y_s, Z_s)ds - \int_0^t Z_s \downarrow dW_s + \int_0^t \gamma_s Y_s dB_s, \quad t \in [0, T].$$

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Here W is and independent Brownian motion.

Idea

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has the solution

$$(Y_t, Z_t)_{t\in[0,T]} = (\tilde{Y}_t(A_t), \tilde{Z}_t(A_t)L_t)_{t\in[0,T]},$$

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with

$$\tilde{Y}_t = \xi + \int_0^t f(s, \tilde{Y}_s L_s(T_s), \tilde{Z}_s L_s(T_s))L_s^{-1}(T_s)ds - \int_0^t Z_s \downarrow dW_s,$$

for $t \in [0, T]$.