Forward Integral and Fractional Stochastic Differential Equations

Jorge A. León

Departamento de Control Automático
Cinvestav del IPN

Spring School "Stochastic Control in Finance", Roscoff 2010

Jointly with Constantin Tudor

Contents

Contents

Equation

Consider the semilinear fractional differential equation of the form

$$X_t = \eta + \int_0^t b(s, X_s) ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

Here $\eta : \Omega \to \mathbb{R}$, $b : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : \Omega \times [0, T] \to \mathbb{R}$ and $B = \{B_t : t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.

Equation

Consider the semilinear fractional differential equation of the form

$$X_t = \eta + \int_0^t b(s, X_s)ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

Here $\eta : \Omega \to \mathbb{R}$, $b : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : \Omega \times [0, T] \to \mathbb{R}$ and $B = \{B_t : t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$.
The stochastic integral is the Forward integral introduced by Russo and Vallois.

Contents

Fractional Brownian motion

$\{B_t\}_{t\in[0,T]}$ is a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$.

Fractional Brownian motion

$\{B_t\}_{t\in[0,T]}$ is a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$.

$\mathcal{H}$ is the Reproducing Kernel Hilbert Space of the fBm B. That is, $\mathcal{H}$ is the closure of the linear space of step functions defined on $[0, T]$ with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s) = H(2H-1)\int_0^t \int_0^s |r-u|^{2H-2}\, drdu.$$

Fractional Brownian motion

$\{B_t\}_{t\in[0,T]}$ is a fractional Brownian motion of Hurst parameter $H \in (\frac{1}{2}, 1)$.
$\mathcal{H}$ is the closure of the linear space of step functions defined on $[0, T]$ with respect to the scalar product

$$\left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s) = H(2H-1) \int_0^t \int_0^s |r-u|^{2H-2} \, drdu.$$

We consider a subspace of functions included in $\mathcal{H}$ via an isometry. This is the space $|\mathcal{H}|$ of all measurable functions $\varphi : [0, T] \to R$ such that

$$||\varphi||^2_{|\mathcal{H}|} = H(2H-1) \int_0^T \int_0^T |\varphi_r| \, |\varphi_s| \, |r-s|^{2H-2} \, drds < \infty.$$

The space $(|\mathcal{H}|, || \cdot ||_{|\mathcal{H}|})$ is a Banach one and the class of all the step functions defined on $[0, T]$ is dense in it.

Fractional Brownian motion

The space $|\mathcal{H}|$ is the setof all measurable functions $\varphi : [0, T] \to R$ such that

$$||\varphi||^2_{|\mathcal{H}|} = H(2H-1)\int_0^T \int_0^T |\varphi_r|\,|\varphi_s|\,|r-s|^{2H-2}\,drds < \infty$$

and the Banach space $|\mathcal{H}| \otimes |\mathcal{H}|$ is the class of all the measurable functions $\varphi : [0, T]^2 \to R$ such that

$$\begin{aligned}
&||\varphi||^2_{|\mathcal{H}|\otimes|\mathcal{H}|} \\
&= [H(2H-1)]^2 \int_{[0,T]^4} |\varphi_{r,\theta}||\varphi_{u,\eta}||r-u|^{2H-2}|\theta-\eta|^{2H-2} drdud\theta d\eta \\
&\quad < \infty.
\end{aligned}$$

Derivative operator

Let V be a Hilbert space and $\mathcal{S}_V$ the family of V–valued smooth random variables of the form

$$F = \sum_{i=1}^{n} F_i v_i, \quad F_i \in \mathcal{S} \quad \text{and} \quad v_i \in V.$$

Derivative operator

Let V be a Hilbert space and $\mathcal{S}_V$ the family of V–valued smooth random variables of the form

$$F = \sum_{i=1}^{n} F_i v_i, \quad F_i \in \mathcal{S} \quad \text{and} \quad v_i \in V.$$

Set $D^k F = \sum_{i=1}^{n} D^k F_i \otimes v_i$. We define the space $\mathbb{D}^{k,p}(V)$ as the completion of $\mathcal{S}_V$ with respect to the norm

$$||F||_{k,p,V}^{p} = E(||F||_{V}^{p}) + \sum_{i=1}^{k} E(||D^i F||_{\mathcal{H}^{\otimes i} \otimes V}^{p}).$$

Gradient operator

For $p > 1$, $\mathbb{D}^{1,p}(|\mathcal{H}|) \subseteq \mathbb{D}^{1,p}(\mathcal{H})$ is the family of all the elements $u \in |\mathcal{H}|$ a.s. such that $(Du) \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} = E(||u||^p_{|\mathcal{H}|}) + E(||Du||^p_{|\mathcal{H}|\otimes|\mathcal{H}|}) < \infty.$$

Gradient operator

$\mathbb{D}^{1,p}(|\mathcal{H}|) \subseteq \mathbb{D}^{1,p}(\mathcal{H})$ is the family of all the elements $u \in |\mathcal{H}|$ a.s. such that $(Du) \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} = E(||u||^p_{|\mathcal{H}|}) + E(||Du||^p_{|\mathcal{H}|\otimes|\mathcal{H}|}) < \infty.$$

1. $\mathbb{D}^{1,p}(|\mathcal{H}|) \subset Dom\ \delta$.

Gradient operator

$\mathbb{D}^{1,p}(|\mathcal{H}|) \subseteq \mathbb{D}^{1,p}(\mathcal{H})$ is the family of all the elements $u \in |\mathcal{H}|$ a.s. such that $(Du) \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} = E(||u||^p_{|\mathcal{H}|}) + E(||Du||^p_{|\mathcal{H}|\otimes|\mathcal{H}|}) < \infty.$$

1. $\mathbb{D}^{1,p}(|\mathcal{H}|) \subset Dom\ \delta$.
2. $E(|\delta(u)|^2) \leq ||u||^2_{\mathbb{D}^{1,2}(|\mathcal{H}|)}$.

Gradient operator

$\mathbb{D}^{1,p}(|\mathcal{H}|) \subseteq \mathbb{D}^{1,p}(\mathcal{H})$ is the family of all the elements $u \in |\mathcal{H}|$ a.s. such that $(Du) \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} = E(||u||^p_{|\mathcal{H}|}) + E(||Du||^p_{|\mathcal{H}|\otimes|\mathcal{H}|}) < \infty.$$

1. $\mathbb{D}^{1,p}(|\mathcal{H}|) \subset Dom\ \delta$.
2. $E(|\delta(u)|^2) \leq ||u||^2_{\mathbb{D}^{1,2}(|\mathcal{H}|)}$.
3. A process $u \in \mathbb{D}^{1,p}(|\mathcal{H}|)$ belongs to $\mathbb{L}^{1,p}_H$ if

$$||u||^p_{\mathbb{L}^{1,p}_H} = E(||u||^p_{L^{\frac{1}{H}}([0,T])}) + E(||Du||^p_{L^{\frac{1}{H}}([0,T]^2)}) < \infty.$$

Gradient operator

$\mathbb{D}^{1,p}(|\mathcal{H}|) \subseteq \mathbb{D}^{1,p}(\mathcal{H})$ is the family of all the elements $u \in |\mathcal{H}|$ a.s. such that $(Du) \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} = E(||u||^p_{|\mathcal{H}|}) + E(||Du||^p_{|\mathcal{H}|\otimes|\mathcal{H}|}) < \infty.$$

1. $\mathbb{D}^{1,p}(|\mathcal{H}|) \subset Dom\ \delta$.
2. $E(|\delta(u)|^2) \leq ||u||^2_{\mathbb{D}^{1,2}(|\mathcal{H}|)}$.
3. A process $u \in \mathbb{D}^{1,p}(|\mathcal{H}|)$ belongs to $\mathbb{L}^{1,p}_H$ if

$$||u||^p_{\mathbb{L}^{1,p}_H} = E(||u||^p_{L^{\frac{1}{H}}([0,T])}) + E(||Du||^p_{L^{\frac{1}{H}}([0,T]^2)}) < \infty.$$

Then,

$$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} \leq b_H ||u||^p_{\mathbb{L}^{1,p}_H}.$$

Gradient operator

$\mathbb{D}^{1,p}(|\mathcal{H}|) \subseteq \mathbb{D}^{1,p}(\mathcal{H})$ is the family of all the elements $u \in |\mathcal{H}|$ a.s. such that $(Du) \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} = E(||u||^p_{|\mathcal{H}|}) + E(||Du||^p_{|\mathcal{H}|\otimes|\mathcal{H}|}) < \infty.$$

1. $\mathbb{D}^{1,p}(|\mathcal{H}|) \subset Dom\ \delta$.
2. $E(|\delta(u)|^2) \leq ||u||^2_{\mathbb{D}^{1,2}(|\mathcal{H}|)}$.
3. A process $u \in \mathbb{D}^{1,p}(|\mathcal{H}|)$ belongs to $\mathbb{L}^{1,p}_H$ if

   $$||u||^p_{\mathbb{L}^{1,p}_H} = E(||u||^p_{L^{\frac{1}{H}}([0,T])}) + E(||Du||^p_{L^{\frac{1}{H}}([0,T]^2)}) < \infty.$$

   Then,

   $$||u||^p_{\mathbb{D}^{1,p}(|\mathcal{H}|)} \leq b_H ||u||^p_{\mathbb{L}^{1,p}_H}.$$

4. $\mathbb{L}^{1,p}_H \subset Dom\ \delta$.

Gradient operator

Theorem (Alòs and Nualart)

Let $\{u_t\}_{t\in[0,T]}$ *be a process in* $\mathbb{L}^{1,2}_{H-\varepsilon}$ *for some* $0<\varepsilon<H-\frac{1}{2}$. *Then*

$$E\left(\sup_{0\le t\le T}\left|\int_0^t u_s\delta B_s\right|^2\right)$$

$$\le \quad C\left\{\left(\int_0^T |E(u_s)|^{\frac{1}{H-\varepsilon}}\,ds\right)^{2(H-\varepsilon)}\right.$$

$$\left.+E\left(\int_0^T\left(\int_0^T |D_s u_r|^{\frac{1}{H}}\,dr\right)^{\frac{H}{H-\varepsilon}}ds\right)^{2(H-\varepsilon)}\right\},$$

where $C=C(\varepsilon,H,T)$.

Contents

Forward integral

Definition (Russo and Vallois)

Let $\{u_t\}_{t\in[0,T]}$ be a process with integrable paths. We say that u is *forward integrable with respect to* B (or $u \in \mathrm{Dom}\delta^-$) if the stochastic process

$$\left\{\varepsilon^{-1}\int_0^t u_s\left(B_{(s+\varepsilon)\wedge T} - B_s\right) ds\right\}_{t\in[0,T]}$$

converges uniformly on $[0, T]$ in probability as $\varepsilon \to 0$. The limit is denoted by $\int_0^{\cdot} u_s dB_s^-$ and it is called *the forward integral of* u *with respect to* B.

Forward integral

Proposition

Assume that $u \in \mathbb{L}^{1,2}_{H-\rho}$*, for some* $0 < \rho < H - \frac{1}{2}$*, and that the trace condition*

$$\int_0^T \int_0^T |D_s u_t| \, |t-s|^{2H-2} \, dsdt < \infty \qquad a.s.$$

holds. Then $u \in Dom\delta^-$ *and for every* $t \in [0, T]$*,*

$$\int_0^t u_s dB_s^- = \int_0^t u_s \delta B_s + H(2H-1) \int_0^t \int_0^T D_s u_r \, |r-s|^{2H-2} \, dsdr.$$

Relation between forward integral and divergence operator

Proposition

Assume that $u \in \mathbb{L}^{1,2}_{H-\rho}$, *for some* $0 < \rho < H - \frac{1}{2}$, *and that the trace condition*

$$\int_0^T \int_0^T |D_s u_t| \, |t-s|^{2H-2} \, dsdt < \infty \qquad a.s.$$

holds. Then $u \in Dom\delta^-$ *and for every* $t \in [0, T]$,

$$\int_0^t u_s dB_s^- = \int_0^t u_s \delta B_s + H(2H-1) \int_0^t \int_0^T D_s u_r \, |r-s|^{2H-2} \, dsdr.$$

Remark This relation was obtained by Alòs and Nualart when in last Definition we only have convergence in probability.

Proof

Proposition

Assume that $u \in \mathbb{L}^{1,2}_{H-\rho}$*, holds. Then*

$$\int_0^t u_s dB_s^- = \int_0^t u_s \delta B_s + H(2H-1)\int_0^t \int_0^T D_s u_r \, |r-s|^{2H-2} \, dsdr.$$

Lemma

Let $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$*. Then for every* $\varepsilon > 0$ *and* $t \in [0,T]$*, we have*

$$\begin{aligned}
&\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s\right) ds \\
&= \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^r u_s ds\right] \delta B_r + \int_{(t-\varepsilon)\vee 0}^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_t\right) ds \\
&\quad - \int_{(t-\varepsilon)\vee 0}^t \left\langle Du_s, 1_{[t,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds.
\end{aligned}$$

Proof

We have

$$\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s \right) ds$$
$$= \int_0^t u_s \int_s^{(s+\varepsilon)\wedge T} \delta B_r ds$$

Proof

We have

$$\begin{aligned}
&\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s \right) ds \\
&= \int_0^t u_s \int_s^{(s+\varepsilon)\wedge T} \delta B_r ds \\
&= \int_0^t \left[\int_s^{(s+\varepsilon)\wedge T} u_s \delta B_r \right] ds + \int_0^t \left\langle D u_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds
\end{aligned}$$

Proof

We have

$$\begin{aligned}
&\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s\right) ds \\
&= \int_0^t u_s \int_s^{(s+\varepsilon)\wedge T} \delta B_r ds \\
&= \int_0^t \left[\int_s^{(s+\varepsilon)\wedge T} u_s \delta B_r\right] ds + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds \\
&= \int_0^{(t+\varepsilon)\wedge T} \left[\int_{(r-\varepsilon)\vee 0}^{r\wedge t} u_s ds\right] \delta B_r + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds
\end{aligned}$$

Proof

We have

$$
\begin{aligned}
&\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s \right) ds \\
&= \int_0^t u_s \int_s^{(s+\varepsilon)\wedge T} \delta B_r ds \\
&= \int_0^t \left[\int_s^{(s+\varepsilon)\wedge T} u_s \delta B_r \right] ds + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \\
&= \int_0^{(t+\varepsilon)\wedge T} \left[\int_{(r-\varepsilon)\vee 0}^{r\wedge t} u_s ds \right] \delta B_r + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \\
&= \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^{r} u_s ds \right] \delta B_r + \int_t^{(t+\varepsilon)\wedge T} \left[\int_{(r-\varepsilon)\vee 0}^{t} u_s ds \right] \delta B_r \\
&\quad + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds
\end{aligned}
$$

Proof

$$
\begin{aligned}
&\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s \right) ds \\
&= \int_0^t \left[\int_s^{(s+\varepsilon)\wedge T} u_s \delta B_r \right] ds + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \\
&= \int_0^{(t+\varepsilon)\wedge T} \left[\int_{(r-\varepsilon)\vee 0}^{r\wedge t} u_s ds \right] \delta B_r + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \\
&= \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^{r} u_s ds \right] \delta B_r + \int_t^{(t+\varepsilon)\wedge T} \left[\int_{(r-\varepsilon)\vee 0}^{t} u_s ds \right] \delta B_r \\
&\quad + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \\
&= \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^{r} u_s ds \right] \delta B_r + \int_{(t-\varepsilon)\vee 0}^{t} \left[\int_t^{(s+\varepsilon)\wedge T} u_s \delta B_r \right] ds \\
&\quad + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds.
\end{aligned}
$$

Proof

$$
\begin{aligned}
\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s \right) ds \\
= & \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^r u_s ds \right] \delta B_r + \int_t^{(t+\varepsilon)\wedge T} \left[\int_{(r-\varepsilon)\vee 0}^t u_s ds \right] \delta B_r \\
& + \int_0^t \left\langle D u_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \\
= & \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^r u_s ds \right] \delta B_r + \int_{(t-\varepsilon)\vee 0}^t \left[\int_t^{(s+\varepsilon)\wedge T} u_s \delta B_r \right] ds \\
& + \int_0^t \left\langle D u_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \\
= & \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^r u_s ds \right] \delta B_r + \int_{(t-\varepsilon)\vee 0}^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_t \right) ds \\
& - \int_{(t-\varepsilon)\vee 0}^t \left\langle D u_s, 1_{[t,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds + \int_0^t \left\langle D u_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds.
\end{aligned}
$$

Proof

Lemma

Let $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ *satisfy the trace condition*

$$\int_0^T \int_0^T |D_s u_t| \, |t-s|^{2H-2} \, dsdt < \infty. \qquad a.s.$$

Then

$$\sup_{0 \le t \le T} \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^{t} \left\langle Du_s, 1_{[t,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds \longrightarrow 0 \qquad a.s. \ as \quad \varepsilon \to 0.$$

Proof

Lemma

Let $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$ *satisfy the trace condition*

$$\int_0^T \int_0^T |D_s u_t|\,|t-s|^{2H-2}\,dsdt < \infty. \qquad a.s.$$

Then $\sup_{0\le t\le T} \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^{t} \left\langle Du_s, 1_{[t,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds \to 0$ *as* $\varepsilon \to 0$.

Proof. We have

$$\begin{aligned}
&\left|\varepsilon^{-1}\int_{(t-\varepsilon)\vee 0}^{t} \left\langle Du_s, 1_{[t,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds\right| \\
&\le \int_{(t-\varepsilon)\vee 0}^{t}\int_0^T |D_r u_s| \left[\int_{-\varepsilon}^{\varepsilon} \varepsilon^{-1}|s-r+u|^{2H-2}du\right] drds \\
&\le C_H \int_{(t-\varepsilon)\vee 0}^{t}\int_0^T |D_r u_s|\,|r-s|^{2H-2}\,drds.
\end{aligned}$$

Proof

Lemma

If $u \in \mathbb{L}^{1,2}_{H-\rho}$ *for some* $0 < \rho < H - \frac{1}{2}$, *then*

$$\sup_{0\le t\le T}\left|\varepsilon^{-1}\int_{(t-\varepsilon)\vee 0}^{t} u_s\left(B_{(s+\varepsilon)\wedge T} - B_t\right)ds\right| \longrightarrow 0 \quad a.s.\ as\ \varepsilon \to 0.$$

Proof

Lemma

If $u \in \mathbb{L}^{1,2}_{H-\rho}$ *for some* $0 < \rho < H - \frac{1}{2}$, *then*

$$\sup_{0\le t\le T} \left| \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^{t} u_s \left(B_{(s+\varepsilon)\wedge T} - B_t \right) ds \right| \longrightarrow 0 \quad a.s. \text{ as } \varepsilon \to 0.$$

Proof.

$$\begin{aligned} &\left| \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^{t} u_s \left(B_{(s+\varepsilon)\wedge T} - B_t \right) ds \right| \\ &\le \left(\sup_{|r-s|\le\varepsilon} |B_r - B_s| \right) \int_{(t-\varepsilon)\vee 0}^{t} \frac{|u_s|}{\varepsilon} ds \\ &\le \left(\sup_{|r-s|\le\varepsilon} |B_r - B_s| \right) \left[\int_{(t-\varepsilon)\vee 0}^{t} |u_s|^{\frac{1}{H-\rho}} \frac{ds}{\varepsilon} \right]^{H-\rho} \end{aligned}$$

Proof

Lemma

If $u \in \mathbb{L}^{1,2}_{H-\rho}$ *for some* $0 < \rho < H - \frac{1}{2}$, *then*

$$\sup_{0 \le t \le T} \left| \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^{t} u_s \left(B_{(s+\varepsilon)\wedge T} - B_t \right) ds \right| \longrightarrow 0 \quad a.s. \text{ as } \varepsilon \to 0.$$

Proof. Using that B has Hölder continuous paths,

$$\begin{aligned} & \left| \varepsilon^{-1} \int_{(t-\varepsilon)\vee 0}^{t} u_s \left(B_{(s+\varepsilon)\wedge T} - B_t \right) ds \right| \\ & \le \left(\sup_{|r-s| \le \varepsilon} |B_r - B_s| \right) \left[\int_{(t-\varepsilon)\vee 0}^{t} |u_s|^{\frac{1}{H-\rho}} \frac{ds}{\varepsilon} \right]^{H-\rho} \\ & \le C(\omega) \varepsilon^{\rho-\rho'} \left[\int_0^T |u_s|^{\frac{1}{H-\rho}} ds \right]^{H-\rho}. \end{aligned}$$

Proof

Proposition

Assume that $u \in \mathbb{L}^{1,2}_{H-\rho}$, *holds. Then*

$$\int_0^t u_s dB_s^- = \int_0^t u_s \delta B_s + H(2H-1) \int_0^t \int_0^T D_s u_r \left| r - s \right|^{2H-2} dsdr.$$

Lemma

Let $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$. *Then for every* $\varepsilon > 0$ *and* $t \in [0, T]$, *we have*

$$\begin{aligned} &\int_0^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_s \right) ds \\ &= \int_0^t \left[\int_{(r-\varepsilon)\vee 0}^r u_s ds \right] \delta B_r + \int_{(t-\varepsilon)\vee 0}^t u_s \left(B_{(s+\varepsilon)\wedge T} - B_t \right) ds \\ &\quad - \int_{(t-\varepsilon)\vee 0}^t \left\langle Du_s, 1_{[t,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds + \int_0^t \left\langle Du_s, 1_{[s,(s+\varepsilon)\wedge T]} \right\rangle_{\mathcal{H}} ds. \end{aligned}$$

Proof

$$E\left[\sup_{0\le t\le T}\left|\int_0^t\left(u_s-\varepsilon^{-1}\int_{(s-\varepsilon)\vee 0}^s u_r dr\right)\delta B_s\right|^2\right]$$

$$\le \quad C\left\{\left[\int_0^T\left|E\left(u_s-\varepsilon^{-1}\int_{(s-\varepsilon)\vee 0}^s u_r dr\right)\right|^{\frac{1}{H-\rho}} ds\right]^{2(H-\rho)}\right.$$

$$\left.+E\left[\int_0^T\left(\int_0^T\left|D_s u_r-\varepsilon^{-1}\int_{(r-\varepsilon)\vee 0}^r D_s u_\theta d\theta\right|^{\frac{1}{H}} dr\right)^{\frac{H}{H-\rho}} ds\right]^{2(H-\rho)}\right\}$$

which goes to zero since

$$\int_0^T [E(|u_s|)]^{\frac{1}{H-\rho}} ds \le \left\{E\left[\int_0^T |u_s|^{\frac{1}{H-\rho}} ds\right]^{2(H-\rho)}\right\}^{\frac{1}{2(H-\rho)}} < \infty.$$

Relation between the Stratonovich and forward integrals

Proposition

Assume that $u \in \mathbb{L}^{1,2}_{H-\rho}$, *for some* $\rho \in (0, H - \frac{1}{2})$, *and the trace condition*

$$\int_0^T \int_0^T D_s u_r \left| r - s \right|^{2H-2} dsdr < \infty.$$

holds. Then

$$\begin{aligned} \int_0^t u_s \circ dB_s &= \int_0^t u_s dB_s^- \\ &= \int_0^t u_s \delta B_s + H(2H-1) \int_0^t \int_0^T D_s u_r \left| r - s \right|^{2H-2} dsdr. \end{aligned}$$

Contents

Semilinear fractional equations

We consider the semilinear stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

Semilinear fractional equations

We consider the semilinear stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

The coefficients $b : \Omega \times [0, T] \times R \to R$ and $\sigma : \Omega \times [0, T] \to R$ are measurable, and X_0 is a random variable.

Semilinear fractional equations

We consider the semilinear stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

(H1) For all $\omega \in \Omega$, $t \in [0, T]$ and $x, y \in R$,

$$\begin{aligned} |b(\omega, t, x) - b(\omega, t, y)| &\leq K(\omega)\,|x - y|, \\ |b(\omega, t, 0)| &\leq K(\omega), \end{aligned}$$

for some random variable K.

Semilinear fractional equations

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

(H2) σ is forward integrable and there is $\varepsilon_1 > 0$ such that the family of random variables

$$\eta_\varepsilon = \int_0^T \left| \int_0^r \sigma_s \varepsilon^{-1}(B_{(s+\varepsilon)\wedge T} - B_s)ds - \int_0^r \sigma_s dB_s^- \right| \times \left| \sigma_r \varepsilon^{-1}(B_{(r+\varepsilon)\wedge T} - B_r) \right| dr, \quad 0 < \varepsilon < \varepsilon_1,$$

is bounded in probability ($\lim_{C\to\infty} \sup_{0<\varepsilon<\varepsilon_1} P(\eta_\varepsilon > C) = 0$).

(H3) σ is forward integrable and for all $\theta > 0$,

$$\lim_{\varepsilon\to 0} P\left(\sup_{0\le t\le T} \left| \int_0^t \left[\int_0^r \sigma_s \varepsilon^{-1}(B_{(s+\varepsilon)\wedge T} - B_s)ds - \int_0^r \sigma_s dB_s^- \right] \times \sigma_r \varepsilon^{-1}(B_{(r+\varepsilon)\wedge T} - B_r)dr \right| > \theta \right) = 0.$$

Semilinear fractional equations

We consider the semilinear stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

$\mathcal{A}$ will be the class of all the processes X such that $(\sigma X) \in \mathrm{Dom}\delta^-$ and for any $\theta > 0$ and $t \in [0, T]$,

$$\lim_{\varepsilon \to 0} \lim_{\eta \to 0} P\left(\left| \int_0^t \sigma_s X_s \exp\left\{ -\int_0^s \sigma_r \varepsilon^{-1}(B_{(r+\varepsilon)\wedge T} - B_r)dr \right\} \right.\right.$$
$$\left.\left. \times \left[\eta^{-1}(B_{(s+\eta)\wedge T} - B_s) - \varepsilon^{-1}(B_{(s+\varepsilon)\wedge T} - B_s) \right] ds \right| > \theta \right) = 0.$$

Semilinear fractional equations

We consider the semilinear stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma_s X_s dB_s^-, \quad t \in [0, T].$$

Theorem

Above equation has a unique solution in $\mathcal{A}$ that is given by the unique solution of the equation

$$\begin{aligned} X_t &= \exp\{\int_0^t \sigma_s dB_s^-\} X_0 \\ &\quad + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\} b(u, X_u)du, \quad t \in [0, T]. \end{aligned}$$

Semilinear fractional equations

Lemma

Suppose that Hypotheses (H2) and (H3) hold. Then

$$\lim_{\varepsilon\to 0} P\left(\sup_{0\le t\le T}\left|\int_0^t \Phi_s\sigma_s\left[\exp\left\{\int_0^s \sigma_r\varepsilon^{-1}(\Delta_{r,\varepsilon}B)dr\right\}\right.\right.\right.$$
$$\left.\left.\left. - \quad \exp\left\{\int_0^s \sigma_r dB_r^-\right\}\right]\varepsilon^{-1}(\Delta_{s,\varepsilon}B)ds\right| > \theta\right) = 0$$

for any $\theta > 0$ and any continuous process Φ.

Semilinear fractional equations

Lemma

Suppose that Hypotheses (H2) and (H3) hold. Then

$$\lim_{\varepsilon\to 0} P\left(\sup_{0\le t\le T}\left|\int_0^t \Phi_s\sigma_s\left[\exp\left\{\int_0^s \sigma_r\varepsilon^{-1}(\Delta_{r,\varepsilon}B)dr\right\}\right.\right.\right.$$
$$\left.\left.\left. - \quad \exp\left\{\int_0^s \sigma_r dB_r^-\right\}\right]\varepsilon^{-1}(\Delta_{s,\varepsilon}B)ds\right| > \theta\right) = 0$$

for any $\theta > 0$ *and any continuous process* Φ.

Let $\Phi_t = X_0 + \int_0^t \exp\left(-\int_0^u \sigma_r dB_r^-\right) b(u, X_u)du$ and

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \;\; t \in [0, T].$$

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \ \ t \in [0, T].$$

We have

$$\sup_{0\le t\le T} \left|\varepsilon^{-1} \int_0^t \sigma_s X_s (B_{(s+\varepsilon)\wedge T} - B_s)ds - Y_t^\varepsilon\right| \to 0$$

in probability.

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \ \ t \in [0, T].$$

We have

$$\sup_{0\le t\le T}\left|\varepsilon^{-1}\int_0^t \sigma_s X_s(B_{(s+\varepsilon)\wedge T} - B_s)ds - Y_t^\varepsilon\right| \to 0$$

in probability. Here,

$$Y_t^\varepsilon = \varepsilon^{-1}\int_0^t \sigma_s \exp\left(\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) \times \left(X_0 + \int_0^s \exp\{-\int_0^u \sigma_r dB_r^-\}b(u, X_u)du\right)(B_{(s+\varepsilon)\wedge T} - B_s)ds.$$

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \ \ t \in [0, T].$$

We have

$$\sup_{0\le t\le T}\left|\varepsilon^{-1}\int_0^t \sigma_s X_s(B_{(s+\varepsilon)\wedge T} - B_s)ds - Y_t^\varepsilon\right| \to 0$$

in probability. Here,

$$\varepsilon^{-1}\int_0^t \sigma_r X_r(B_{(r+\varepsilon)\wedge T} - B_r)dr = \varepsilon^{-1}\int_0^t \sigma_s \exp\left(\int_0^s \sigma_r dB_r^-\right)$$
$$\times\left(X_0 + \int_0^s \exp\{-\int_0^u \sigma_r dB_r^-\}b(u, X_u)du\right)(B_{(s+\varepsilon)\wedge T} - B_s)ds.$$

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \;\; t \in [0, T].$$

We have

$$\sup_{0\le t\le T} \left| \varepsilon^{-1} \int_0^t \sigma_s X_s (B_{(s+\varepsilon)\wedge T} - B_s)ds - Y_t^\varepsilon \right| \to 0$$

in probability. Here,

$$Y_t^\varepsilon = \varepsilon^{-1} \int_0^t \sigma_s \exp\left(\varepsilon^{-1} \int_0^s \sigma_r (B_{(r+\varepsilon)\wedge T} - B_r)dr\right) \times \left(X_0 + \int_0^s \exp\{-\int_0^u \sigma_r dB_r^-\}b(u, X_u)du\right)(B_{(s+\varepsilon)\wedge T} - B_s)ds.$$

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \ \ t \in [0, T].$$

Using integration by parts,

$$\int_0^t \sigma_s X_s dB_s^- = X_t - X_0 - \int_0^t b(u, X_u)du.$$

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\} b(u, X_u)du, \;\; t \in [0, T].$$

Using integration by parts,

$$\int_0^t \sigma_s X_s dB_s^- = X_t - X_0 - \int_0^t b(u, X_u)du.$$

So

$$\int_0^t \sigma_s X_s \exp\left(-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) dB_s^-$$
$$= \int_0^t \exp\left(-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right)(dX_s - b(s, X_s)ds)$$

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \ \ t \in [0, T].$$

Using integration by parts,

$$\int_0^t \sigma_s X_s dB_s^- = X_t - X_0 - \int_0^t b(u, X_u)du.$$

$$\begin{aligned}
&\int_0^t \sigma_s X_s \exp\left(-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) dB_s^- \\
&= \ \varepsilon^{-1}\int_0^t \sigma_s X_s e^{-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr}(B_{(s+\varepsilon)\wedge T} - B_s)ds \\
&\quad -X_0 + X_t \exp\left(-\varepsilon^{-1}\int_0^t \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) \\
&\quad - \int_0^t \exp\left(-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) b(s, X_s)ds.
\end{aligned}$$

Semilinear fractional equations

$$X_t = \exp\{\int_0^t \sigma_s dB_s^-\}X_0 + \int_0^t \exp\{\int_u^t \sigma_s dB_s^-\}b(u, X_u)du, \;\; t \in [0, T].$$

$$\int_0^t \sigma_s X_s dB_s^- = X_t - X_0 - \int_0^t b(u, X_u)du.$$

$$\begin{aligned}
&\int_0^t \sigma_s X_s \exp\left(-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) dB_s^- \\
&= \;\; \varepsilon^{-1}\int_0^t \sigma_s X_s e^{-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T}-B_r)dr}(B_{(s+\varepsilon)\wedge T} - B_s)ds \\
&\quad -X_0 + X_t \exp\left(-\varepsilon^{-1}\int_0^t \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) \\
&\quad -\int_0^t \exp\left(-\varepsilon^{-1}\int_0^s \sigma_r(B_{(r+\varepsilon)\wedge T} - B_r)dr\right) b(s, X_s)ds.
\end{aligned}$$

So $X \in \mathcal{A}$.

Examples

Proposition (Hölder continuous case)

Assume that the stochastic process $\{\sigma_t\}_{t\in[0,T]}$ *satisfies :*

(a) $\sigma \in \mathbb{L}^{1,2}_{H-\rho}$, *for some* $0 < \rho < H - \frac{1}{2}$, *and for some* $r_0 \in [0, T]$,

$$E\left[\int_0^T |D_s \sigma_{r_0}|^{\frac{1}{H-\rho}} ds\right]^{2(H-\rho)} < \infty.$$

(b) There exists $0 < \beta \leq 1$ *such that for all* $r, s \in [0, T]$,

$$E\left[|\sigma_r - \sigma_s|\right] \leq C\,|r-s|^{\frac{\beta}{2}}\,, \qquad (1)$$

and

$$E\left[\int_0^T |D_u(\sigma_r - \sigma_s)|^{\frac{1}{H-\rho}} du\right]^{2(H-\rho)} \leq C\,|r-s|^{\beta}\,.$$

Examples

Proposition (Hölder continuous case)

Assume that the stochastic process $\{\sigma_t\}_{t\in[0,T]}$ *satisfies :*

(c) *The stochastic processes* $\{\sigma_t\}_{t\in[0,T]}$ *and* $\{\int_0^T |D_r\sigma_t||t-r|^{2H-2}dr\}_{t\in[0,T]}$ *have square integrable paths.*

(d) *There are* $\alpha, a \in (0, H)$ *such that :*

- (d_1) *The family* $\{\theta_\varepsilon\}_{0<\varepsilon<\varepsilon_1}$ *is bounded in probability, where* $\theta_\varepsilon = \varepsilon^{-1+H-a}\left\{\int_0^T \left[\int_0^r \int_{(s-\varepsilon^\alpha)\vee 0}^{(s+\varepsilon^\alpha)\wedge T} |D_u\sigma_s|\,|s-u|^{2H-2}\,duds\right]^2 dr\right\}^{\frac{1}{2}}$.
- (d_2) *The set* $\{\theta_\varepsilon\}_{0<\varepsilon<\varepsilon_1}$ *converges in probability to* 0 *as* $\varepsilon\to 0$.
- (d_3) $\beta > 2(1-H+a)$ *and* $H - a > \max(\frac{1}{2}, \alpha)$.

Then σ *satisfies Assumptions (H2) and (H3).*

Examples : Proof

(H2) σ is forward integrable and there is $\varepsilon_1 > 0$ such that the family of random variables

$$\eta_\varepsilon = \int_0^T \left| \int_0^r \sigma_s \varepsilon^{-1} (B_{(s+\varepsilon)\wedge T} - B_s) ds - \int_0^r \sigma_s dB_s^- \right| \times \left| \sigma_r \varepsilon^{-1} (B_{(r+\varepsilon)\wedge T} - B_r) \right| dr, \quad 0 < \varepsilon < \varepsilon_1,$$

is bounded in probability ($\lim_{C\to\infty} \sup_{0<\varepsilon<\varepsilon_1} P(\eta_\varepsilon > C) = 0$).

Examples : Proof

We have

$$\varepsilon^{-1}\int_0^t \sigma_s(B_{(s+\varepsilon)\wedge T}-B_s)ds=\sum_{i=1}^4 J_i^\varepsilon(t),$$

with

$$\begin{aligned}
J_1^\varepsilon(t) &= \varepsilon^{-1}\int_0^t\left[\int_{(r-\varepsilon)\vee 0}^r \sigma_s ds\right]\delta B_r,\\
J_2^\varepsilon(t) &= \varepsilon^{-1}\int_{(t-\varepsilon)\vee 0}^t \sigma_s(B_{(s+\varepsilon)\wedge T}-B_t)ds,\\
J_3^\varepsilon(t) &= -\varepsilon^{-1}\int_{(t-\varepsilon)\vee 0}^t \left\langle D\sigma_s, 1_{[t,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds,\\
J_4^\varepsilon(t) &= \varepsilon^{-1}\int_0^t \left\langle D\sigma_s, 1_{[s,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds.
\end{aligned}$$

Examples : Proof

$$\varepsilon^{-1}\int_0^t \sigma_s(B_{(s+\varepsilon)\wedge T}-B_s)ds=\sum_{i=1}^4 J_i^\varepsilon(t),$$

with

$$\begin{aligned}
J_1^\varepsilon(t) &= \varepsilon^{-1}\int_0^t\left[\int_{(r-\varepsilon)\vee 0}^r \sigma_s ds\right]\delta B_r,\\
J_2^\varepsilon(t) &= \varepsilon^{-1}\int_{(t-\varepsilon)\vee 0}^t \sigma_s(B_{(s+\varepsilon)\wedge T}-B_t)ds,\\
J_3^\varepsilon(t) &= -\varepsilon^{-1}\int_{(t-\varepsilon)\vee 0}^t \left\langle D\sigma_s, 1_{[t,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds,\\
J_4^\varepsilon(t) &= \varepsilon^{-1}\int_0^t \left\langle D\sigma_s, 1_{[s,(s+\varepsilon)\wedge T]}\right\rangle_{\mathcal{H}} ds.
\end{aligned}$$

and

$$\int_0^t \sigma_s dB_s^- = \int_0^t \sigma_s\delta B_s + H(2H-1)\int_0^T\int_0^t D_s\sigma_r\,|r-s|^{2H-2}\,drds.$$

Examples : Proof

$$
\begin{aligned}
\eta_\varepsilon \leq & \int_0^T \left| J_1^\varepsilon(r) - \int_0^r \sigma_s \delta B_s \right| \left| \varepsilon^{-1} \sigma_r (\Delta B_{r,\varepsilon}) \right| dr \\
& + \int_0^T \left| J_2^\varepsilon(r) \right| \left| \varepsilon^{-1} \sigma_r (\Delta_{r,\varepsilon} B) \right| dr + \int_0^T \left| J_3^\varepsilon(r) \right| \left| \varepsilon^{-1} \sigma_r (\Delta_{r,\varepsilon} B) \right| dr \\
& + \int_0^T \left| J_4^\varepsilon(r) - H(2H-1) \int_0^r \int_0^T D_s \sigma_u \left| u - s \right|^{2H-2} ds du \right| \\
& \quad \times \left| \varepsilon^{-1} \sigma_r (\Delta_{r,\varepsilon} B) \right| dr = \sum_{j=1}^4 A_j^\varepsilon .
\end{aligned}
$$

Examples : Proof

$$A_3^{\varepsilon} \leq C\varepsilon^{-1} \int_0^T \int_{(r-\varepsilon)\vee 0}^{r} \int_0^T |D_u\sigma_s| \, |u-s|^{2H-2} \, duds \, |\sigma_r| \, |\Delta_{r,\varepsilon} B| \, dr$$

Examples : Proof

$$
\begin{aligned}
A_3^{\varepsilon} &\leq C\varepsilon^{-1}\int_0^T\int_{(r-\varepsilon)\vee 0}^{r}\int_0^T |D_u\sigma_s|\,|u-s|^{2H-2}\,duds\,|\sigma_r|\,|\Delta_{r,\varepsilon}B|\,dr\\
&\leq CU_a\varepsilon^{H-a}\int_0^T\left[\int_{(r-\varepsilon)\vee 0}^{r}\int_0^T \varepsilon^{-1}\,|D_u\sigma_s|\,|u-s|^{2H-2}\,duds\right]|\sigma_r|\,dr,
\end{aligned}
$$

Examples : Proof

$$
\begin{aligned}
A_3^{\varepsilon} &\leq C\varepsilon^{-1}\int_0^T \int_{(r-\varepsilon)\vee 0}^{r} \int_0^T |D_u\sigma_s|\,|u-s|^{2H-2}\,duds\,|\sigma_r|\,|\Delta_{r,\varepsilon}B|\,dr \\
&\leq CU_a\varepsilon^{H-a}\int_0^T \left[\int_{(r-\varepsilon)\vee 0}^{r}\int_0^T \varepsilon^{-1}\,|D_u\sigma_s|\,|u-s|^{2H-2}\,duds\right]|\sigma_r|\,dr,
\end{aligned}
$$

which implies

$$
A_3^{\varepsilon} \leq C_H U_a \varepsilon^{H-a}\left[\int_0^T |\sigma_r|^2\,dr\right]^{\frac{1}{2}}\left[\int_0^T \left|\int_0^T |D_u\sigma_s|\,|u-s|^{2H-2}\,du\right|^2 ds\right]^{\frac{1}{2}}.
$$

Examples

Proposition (The absolutely continuous case)

Let $\{\sigma_t\}_{t\in[0,T]}$ *be an absolutely continuous process of the form*

$$\sigma_t = \sigma_0 + \int_0^t \dot{\sigma}_s ds, \qquad t \in [0,T],$$

with $\sigma_0, \dot{\sigma} \in \mathbb{L}^{1,2}_{H-\rho}$ *for some* $0 < \rho < H - \frac{1}{2}$*. Then Hypotheses (H2) and (H3) are satisfied for the process* $\{\sigma_t\}_{t\in[0,T]}$*.*

Contents

Forward and Young Integrals

Proposition (Russo and Vallois)

Let Y and X be two processes with paths in $C^{\alpha}([0,T])$ and $C^{\beta}([0,T])$, respectively, where $\alpha+\beta>1$. Then

$$\int_0^T Y_s dX_s^- = \int_0^T Y_s \circ dX_s = \int_0^T Y_s dX_s^{(y)}.$$

Forward and Young Integrals

Proposition (Russo and Vallois)

Let Y and X be two processes with paths in $C^{\alpha}([0,T])$ and $C^{\beta}([0,T])$, respectively, where $\alpha+\beta>1$. Then

$$\int_0^T Y_s dX_s^- = \int_0^T Y_s \circ dX_s = \int_0^T Y_s dX_s^{(y)}.$$

Proof Let

$$X_{\varepsilon}(t) = \frac{1}{\varepsilon}\int_0^t (X(u+\varepsilon)-X(u))\,du, \quad t\in[0,T].$$

Forward and Young Integrals

Proposition (Russo and Vallois)

Let Y and X be two processes with paths in $C^\alpha([0,T])$ and $C^\beta([0,T])$, respectively, where $\alpha+\beta>1$. Then

$$\int_0^T Y_s dX_s^- = \int_0^T Y_s \circ dX_s = \int_0^T Y_s dX_s^{(y)}.$$

Proof Let

$$X_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t (X(u+\varepsilon) - X(u))\, du, \quad t\in[0,T],$$

which has paths in $C^1([0,T])$. Then,

$$\int_0^T Y_s dX_\varepsilon(s) = \int_0^T Y_s dX_\varepsilon^{(y)}(s).$$

Forward and Young Integrals

Proposition (Russo and Vallois)

Let Y and X be two processes with paths in $C^{\alpha}([0,T])$ and $C^{\beta}([0,T])$, respectively, where $\alpha+\beta>1$. Then

$$\int_0^T Y_s dX_s^- = \int_0^T Y_s \circ dX_s = \int_0^T Y_s dX_s^{(y)}.$$

Proof Let

$$X_{\varepsilon}(t) = \frac{1}{\varepsilon}\int_0^t (X(u+\varepsilon)-X(u))\,du, \quad t\in[0,T],$$

So,

$$\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_0^T Y_s dX_s^{(y)} - \int_0^T Y_s dX_{\varepsilon}(s)\right| \\
&= \sup_{t\in[0,T]}\left|\int_0^T Y_s dX_s^{(y)} - \int_0^T Y_s dX_{\varepsilon}^{(y)}(s)\right|
\end{aligned}$$

Forward and Young Integrals

Proposition (Russo and Vallois)

Let Y and X be two processes with paths in $C^\alpha([0,T])$ and $C^\beta([0,T])$, respectively, where $\alpha+\beta>1$. Then

$$\int_0^T Y_s dX_s^- = \int_0^T Y_s \circ dX_s = \int_0^T Y_s dX_s^{(y)}.$$

Proof

$$\begin{aligned}
&\sup_{t\in[0,T]}\left|\int_0^T Y_s dX_s^{(y)} - \int_0^T Y_s dX_\varepsilon(s)\right| \\
&= \sup_{t\in[0,T]}\left|\int_0^T Y_s dX_s^{(y)} - \int_0^T Y_s dX_\varepsilon^{(y)}(s)\right| \\
&\leq C||Y||_\alpha ||X-X_\varepsilon||_\beta .
\end{aligned}$$

Forward and Young Integrals

Step 1 Case $0 \leq s < s + \varepsilon < t$.
Set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

Forward and Young Integrals

Step 1 Case $0 \leq s < s + \varepsilon < t$.
Set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

Hence

$$\begin{aligned} & |\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \\ \leq \ & \frac{1}{\varepsilon}\int_t^{t+\varepsilon} |X_u - X_t| du + \frac{1}{\varepsilon}\int_s^{s+\varepsilon} |X_u - X_t| du. \end{aligned}$$

Forward and Young Integrals

Step 1 Case $0 \le s < s + \varepsilon < t$.
Set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

Hence

$$\begin{aligned} & |\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \\ \le \quad & \frac{1}{\varepsilon}\int_t^{t+\varepsilon} |X_u - X_t| du + \frac{1}{\varepsilon}\int_s^{s+\varepsilon} |X_u - X_t| du \\ \le \quad & ||X||_\beta \frac{1}{\varepsilon}\left(\int_t^{t+\varepsilon} (u-t)^\beta du - \int_s^{s+\varepsilon} (u-t)^\beta du\right) \end{aligned}$$

Forward and Young Integrals

Step 1 Case $0 \leq s < s + \varepsilon < t$.
Set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

Hence

$$\begin{aligned}
&|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \\
&\leq \frac{1}{\varepsilon}\int_t^{t+\varepsilon} |X_u - X_t| du + \frac{1}{\varepsilon}\int_s^{s+\varepsilon} |X_u - X_t| du \\
&\leq ||X||_\beta \frac{1}{\varepsilon}\left(\int_t^{t+\varepsilon} (u-t)^\beta du - \int_s^{s+\varepsilon} (u-t)^\beta du\right) \\
&\leq C\varepsilon^\beta
\end{aligned}$$

Forward and Young Integrals

Step 1 Case $0 \leq s < s+\varepsilon < t$.
Set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

Hence

$$\begin{aligned}
&|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \\
&\leq \frac{1}{\varepsilon}\int_t^{t+\varepsilon} |X_u - X_t| du + \frac{1}{\varepsilon}\int_s^{s+\varepsilon} |X_u - X_t| du \\
&\leq ||X||_\beta \frac{1}{\varepsilon}\left(\int_t^{t+\varepsilon} (u-t)^\beta du - \int_s^{s+\varepsilon} (u-t)^\beta du\right) \\
&\leq C\varepsilon^\beta \leq C\varepsilon^{\beta-\beta'}|t-s|^{\beta'}.
\end{aligned}$$

Forward and Young Integrals

Step 2 Case $0 \leq s < t < s + \varepsilon$.
Set

$$\Delta_{\varepsilon}(t) = X_{\varepsilon}(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

Forward and Young Integrals

Step 2 Case $0 \leq s < t < s+\varepsilon$.
Set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

In this case

$$\begin{aligned} &\Delta_\varepsilon(t) - \Delta_\varepsilon(s) \\ &= \frac{1}{\varepsilon}\int_{s+\varepsilon}^{t+\varepsilon}(X_u - X_{s+\varepsilon})du - \frac{1}{\varepsilon}\int_s^t (X_u - X_s)du \\ &\quad + \frac{t-s}{\varepsilon}(X_{s+\varepsilon} - X_s) + X_s - X_t. \end{aligned}$$

Forward and Young Integrals

Step 2 Case $0 \leq s < t < s+\varepsilon$.
Set

$$\Delta_\varepsilon(t) = X_\varepsilon(t) - X_t = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} X_u du - \frac{1}{\varepsilon}\int_0^{\varepsilon} X_u du.$$

In this case

$$\begin{aligned}
&\Delta_\varepsilon(t) - \Delta_\varepsilon(s) \\
&= \frac{1}{\varepsilon}\int_{s+\varepsilon}^{t+\varepsilon}(X_u - X_{s+\varepsilon})du - \frac{1}{\varepsilon}\int_s^t (X_u - X_s)du \\
&\quad + \frac{t-s}{\varepsilon}(X_{s+\varepsilon} - X_s) + X_s - X_t.
\end{aligned}$$

Hence, using $0 \leq t-s < \varepsilon$,

$$|\Delta_\varepsilon(t) - \Delta_\varepsilon(s)| \leq C\varepsilon^{\beta-\beta'}|t-s|^{\beta'}.$$