Fractional Delay Equations in the Young Sense

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Contents

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Equation

We consider the equation

$$\begin{aligned} y_t &= \xi_0 + \int_0^t f(\mathcal{Z}_s^y)dx_s, \quad t \in [0, T], \\ \mathcal{Z}_0 &= \xi. \end{aligned}$$

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$$\nu > 1/2 \quad \text{and} \quad \mathcal{Z}_s^y(\theta) = y(s + \theta), \; \theta \in [-h, 0].$$

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The integral is a Young one

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Increments

We consider

$$C_k(\mathbb{R}) = \left\{ g : [0, T]^k \to \mathbb{R} : \begin{array}{l} g_{t_1,\ldots,t_k} = 0 \text{ if } t_i = t_{i+1} \\ \text{for some } \; i \in \{1, \ldots, k-1\} \end{array} \right\}$$

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$$(\delta g)_{t_1,\dots,t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{t_i,\dots,\hat{t}_i,\dots,t_{k+1}}.$$

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$$\mathcal{Z}C_k(\mathbb{R}) = \mathcal{B}C_k(\mathbb{R}).$$
- Let $k \geq 1$ and $h \in \mathcal{Z}C_{k+1}(\mathbb{R})$. Then there exists a (nonunique) $f \in C_k(\mathbb{R})$ such that
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- Let $k \geq 1$ and $h \in \mathcal{Z}C_{k+1}(\mathbb{R})$. Then there exists a (nonunique) $f \in C_k(\mathbb{R})$ such that
$$h = \delta f.$$
- For $g \in C_1(\mathbb{R})$ and $h \in C_2(\mathbb{R})$,
$$(\delta g)_{st} = g_t - g_s \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}.$$

Notation

For $v \in \mathbb{R}$,

$$C^{\mu}_{v,a_1,a_2}(\mathbb{R}) = \{g : [a_1, a_2] \to \mathbb{R} : g_{a_1} = v,\ ||g||_{\mu,[a_1,a_2]} < \infty\}.$$

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and, for $\rho \in C^{\mu}_1([a_1 - h, a_1])$,

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These are complete metric spaces with respec to

$$d_{\mu}(f, g) = ||f - g||_{\mu}.$$

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For $f \in C_2([a_1, a_2]; \mathbb{R})$, we define

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Similarly, for $h \in C_3([a_1, a_2])$, we define

$$||h||_{\nu,\rho,[a_1,a_2]} = \sup_{s,u,t\in[a_1,a_2]} \frac{|h_{sut}|}{|u-s|^{\nu}|t-u|^{\rho}}.$$

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the norm

$$||h||_{\mu,[a_1,a_2]} = \inf\{\sum_i ||h_i||_{\rho_i,\mu-\rho_1}; h = \sum_i h_i, \quad 0 < \rho_i < \mu\}.$$

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We use the notation

$$C_k^{1+} = \bigcup_{\mu>1} C_k^{\mu}([a_1, a_2]), \quad k = 2, 3.$$

Inverse of δ

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Proposition (Gubinelli)

Let $0 \leq a_1 < a_2 \leq T$. Then, there exists a unique linear map $\Lambda : \mathcal{Z} C_3^{1+}([a_1, a_2]) \to C_2^{1+}([a_1, a_2])$ such that

$$\delta \Lambda = Id_{\mathcal{Z} C_3^{1+}([a_1, a_2])}$$

and

$$||\Lambda h||_{\mu, [a_1, a_2]} \leq \frac{||h||_{\mu, [a_1, a_2]}}{2^{\mu} - 2}.$$

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Proposition (Gubinelli)

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and

$$||\Lambda h||_{\mu,[a_1,a_2]} \leq \frac{||h||_{\mu;[a_1,a_2]}}{2^\mu - 2}.$$

Remark For any $h \in C_3^{1+}([a_1,a_2])$ such that $\delta h = 0$, there exists a unique $g = \Lambda(h) \in C_2^{1+}$ such that $\delta g = h$.

Inverse of δ

Corollary

For $g \in C_2(\mathbb{R})$ *such that* $\delta g \in C_3^{1+}$, *we have*

$$[(I_d - \Lambda\delta)g]_{st} = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n-1} g_{t_i t_{i+1}},$$

where

$$\Pi_{st} = \{t_0 = s < t_1 < \ldots < t_n = t\}.$$

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Proof : Note that

$$\delta(I_d - \Lambda\delta)g = \delta g - \delta g = 0.$$

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Proof : Note that

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Then, there exists $f \in C_1(\mathbb{R})$ such that

$$\delta f = (I_d - \Lambda\delta)g.$$

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Proof : There exists $f \in C_1(\mathbb{R})$ such that

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$$\begin{aligned}
[(I_d - \Lambda\delta)g]_{st} &= \sum_{i=0}^{n-1} (f_{t_{i+1}} - f_{t_i}) = \sum_{i=0}^{n-1} (\delta f)_{t_i t_{i+1}} \\
&= \sum_{i=0}^{n-1} g_{t_i t_{i+1}} - \sum_{i=0}^{n-1} (\Lambda\delta g)_{t_i, t_{i+1}}.
\end{aligned}$$

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Proof :

$$[(I_d - \Lambda\delta)g]_{st} = \sum_{i=0}^{n-1} g_{t_i t_{i+1}} - \sum_{i=0}^{n-1} (\Lambda\delta g)_{t_i, t_{i+1}}.$$

Finally, there is $\mu > 1$ such that

$$\sum_{i=0}^{n-1} |(\Lambda\delta g)_{t_i, t_{i+1}}| \leq \sum_{i=0}^{n-1} ||\Lambda\delta g||_\mu (t_{i+1} - t_i)^\mu \to 0.$$

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Young integral

We want to define

$$\int_0^T f_r dg_r,$$

with $f \in C^\nu(\mathbb{R})$ and $g \in C^\mu(\mathbb{R})$, where

$$\nu + \mu > 1.$$

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To do so, we first assume that f y g are two smooth functions.

Young integral

Assume that f y g are two smooth functions. Then

$$\begin{aligned} J_{st}(fdg) &:= \int_s^t f r d g r = f_s(\delta g)_{st} + \int_s^t (\delta f)_{su} dg_u \\ &= f_s(\delta g)_{st} + J_{s,t}((\delta f)_{s\cdot} dg). \end{aligned}$$

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On the other hand,

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On the other hand,

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Therefore $h \in C_3^{1+}([0,t];\mathbb{R})$ and $\delta h = 0$ because $\delta \circ \delta = 0$.

Young integral

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Young integral

Definition

Let $f \in C_1^{\nu}([0, T])$ and $g \in C_1^{\mu}([0, T])$ be such that $\mu + \nu > 1$. Then, we define the Young integral of f with respec to g as

$$J_{st}(fdg) := \int_s^t f r d g r = f_s(\delta g)_{st} + \Lambda_{st}(\delta f \delta g).$$

Properties of the Young Integral

Theorem

Let $f \in C^{\nu}$ y $f \in C^{\mu}$, $\nu + \mu > 1$. Then,

- $|J_{st}(fdg)| \leq ||f||_{\infty}||g||_{\mu}(t-s)^{\mu} + C||f||_{\nu}||g||_{\mu}(t-s)^{\mu+\nu}$.

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- *Let* $\Pi_{s,t} = \{s = t_0 < t_1 < \ldots < t_n = t\}$,

$$J_{st} = \lim_{|\Pi_{s,t}|\to 0} \sum_{i=0}^{n-1} f_{t_i}(\delta g)_{t_i,t_{i+1}}.$$

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- *For* $\Pi_{s,t} = \{s = t_0 < t_1 < \ldots < t_n = t\}$,

$$J_{st} = \lim_{|\Pi_{s,t}| \to 0} \sum_{i=0}^{n-1} f_{t_i}(\delta g)_{t_i, t_{i+1}}.$$

Proof : The first statement is a consequence of the properties of Λ. The second one follows from

$$J(fdg) = [Id - \Lambda d](f\delta g).$$

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where $x \in C^\nu$ and $y \in C^\lambda$, with $1/2 < \lambda < \nu$.

Hypothesis

(H1) There exists $\lambda \in (1/2, \nu)$ such that

$$|f(z)| \leq M \quad \text{y} \quad |f(z_1) - f(z_2)| \leq M \sup_{\theta \in [-h,0]} |(z_1 - z_2)(\theta)|,$$

where $z, z_1, z_2 \in C_1^{\lambda}([-h, 0])$.

Auxiliary result

(H1) There exists $\lambda \in (1/2, \nu)$ such that

$$|f(z)| \leq M \quad \text{y} \quad |f(z_1) - f(z_2)| \leq M \sup_{\theta\in[-h,0]} |(z_1 - z_2)(\theta)|,$$

where $z, z_1, z_2 \in C_1^\lambda([-h, 0])$.

Lemma

Let $a = (a_1, a_2)$ *and*

$$[\mathcal{U}^{(a)} z]_s = f(\mathcal{Z}_s^z), \quad s \in [a_1, a_2].$$

Then

$$||\mathcal{U}^{(a)} z||_{\lambda,[a_1,a_2]} \leq M||z||_{\lambda,[a_1-h,a_2]}.$$

Hypotheses

(H1) There exists $\lambda \in (1/2, \nu)$ such that

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where $z, z_1, z_2 \in C_1^\lambda([-h, 0])$.

(H2) For any positive integer N,

$$\begin{aligned} & ||U^{(a)}(z_1) - U^{(a)}(z_2)||_{\lambda,[a_1,a_2]} \\ \leq \quad & C_N ||z_1 - z_2||_{\lambda,[a_1-h,a_2]} \\ & ||z_1||_{\lambda,[a_1-h,a_2]}, ||z_2||_{\lambda,[a_1-h,a_2]} \leq N. \end{aligned}$$

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Let m be a finite measure on $[-h, 0]$ and $\sigma : \mathbb{R} \to \mathbb{R}$ a function with two derivatives.

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$$f(z) = \sigma\left(\int_{-h}^{0} z(\theta) m(d\theta)\right)$$

satisfies **(H1)** and **(H2)**.

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Then

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Proof : By the mean valued theorem,

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq M \int_{-h}^{0} |z_1(\theta) - z_2(\theta)| m(d\theta) \\ &\leq Mm([-h, 0]) \sup_{\theta \in [-h,0]} |z_1(\theta) - z_2(\theta)|. \end{aligned}$$

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$$\begin{aligned} y_t &= \xi_0 + \int_0^t f(\mathcal{Z}_u^y) d x_u, \quad t \in [0, T], \\ \mathcal{Z}_0^y &= \xi, \quad \text{on} \quad [-h, 0], \end{aligned}$$

where $x \in C^{\nu}$.

Theorem

*Under Hypotheses **(H1)** and **(H2)**, above equation has a unique solution in $C^{\lambda}_{\xi,0,T}$.*

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Proof (an idea) : We first consider the interval $[0,\eta]$, where η is as follows :

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*Under Hypotheses **(H1)** and **(H2)**, our equation has a unique solution in* $C^{\lambda}_{\xi,0,T}$.

Proof (an idea) : We first consider the interval $[0,\eta]$, where η is as follows :
Set

$$\begin{aligned}\Gamma : C^{\lambda}_{\xi,0,\eta} &\to C^{\lambda}_{\xi,0,\eta}\\ z &\mapsto \xi_0 + \int_0^t f(\mathcal{Z}^z_u)dx_u.\end{aligned}$$

Proof

Set

$$\begin{aligned} \Gamma : C^{\lambda}_{\xi,0,\eta} &\rightarrow C^{\lambda}_{\xi,0,\eta} \\ z &\mapsto \xi_0 + \int_0^t f(\mathcal{Z}^z_u) dx_u. \end{aligned}$$

Then

$$\begin{aligned} &||\Gamma(z_1) - \Gamma(z_2)||_{\lambda,[-h,0]} \\ &\leq ||f(\mathcal{Z}^{z_1}) - f(\mathcal{Z}^{z_2})||_{\infty,[0,\eta]} ||x||_{\nu,[0,\eta]} \eta^{\nu-\lambda} \\ &\quad + C||f(\mathcal{Z}^{z_1}) - f(\mathcal{Z}^{z_2})||_{\lambda,[0,\eta]} ||x||_{\nu} \eta^{\nu} \\ &\leq (1 + C)||x||_{\nu,[0,\eta]} ||f(\mathcal{Z}^{z_1}) - f(\mathcal{Z}^{z_2})||_{\lambda,[0,\eta]} \eta^{\nu} \\ &\leq (1 + C)||x||_{\nu_1[0,\eta]} \eta^{\nu} C_1 ||z_1 - z_2||_{\lambda,[-h,\eta]}. \end{aligned}$$

Fractional delay equations

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Proposition

Let $f(z) = \sigma\left(\int_{-h}^0 z(\theta) m(d\theta)\right)$, *with* $\sigma \in C_b^\infty(\mathbb{R})$ *and* $\sigma(\eta_1)\sigma(\eta_2) > \varepsilon$ *for all* $\eta_1, \eta_2 \in \mathbb{R}$. *Then* y_t *has a* C^∞*-density for any* $t \in (0, T]$.

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Using this approach, Nualart and Rascanu, Nualart and Saussereau, and Nualart and Hu have seen that the equation

$$X_t = a + \int_0^t f(X_s) dB_s$$

has a unique solution with a smooth density under non-degeneracy conditions.