Stochastic Integration with Respect to FBM

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Contents

Contents

Introduction

In this section we introduce the framework that we use in this course.

Contents

Divergence operator

In this section we introduce two of the main tools of the Malliavin calculus. Namely, the divergence and derivative operators.

Divergence operator

In this section we introduce two of the main tools of the Malliavin calculus. Namely, the divergence and derivative operators.

The divergence operator δ is a generalization of the Itô integral to anticipating integrands. Even in the general case, several authors have obtained some properties of δ similars to those of the Itô integral.

Equation

In this section we introduce two of the main tools of the Malliavin calculus. Namely, the divergence and derivative operators.
Also we consider

$$X_t = \eta + \int_0^t a(s) X_s ds + \int_0^t b(s) X_s dB_s^H, \quad t \in [0, T].$$

Here $\eta \in L^2(\Omega)$, $a, b : [0, T] \to \mathbb{R}$ and $B^H = \{B_t^H : t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

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The stochastic integral is an extension of the divergence operator.

Contents

Young integral

In this part we first introduce the Young integral for Hölder continuos functions using the framework established by Gubinelli.
M. Gubinelli, *Controlling rough path*. J. Funct. Anal. **216**, 86-140, 2004.

Young delay equations

In this part we first introduce the Young integral for Hölder continuos functions using the framework established by Gubinelli.
Also we consider

$$dy_t = b(\mathcal{Z}_t^y)dt + f(\mathcal{Z}_t^y)dB_t^H, \quad t \in [0, T],$$

where $b, f : \mathcal{C}^\nu([-h,0];\mathbb{R}) \to \mathbb{R}$, $\mathcal{Z}_t^y : [-h,0] \to \mathbb{R}$ is given by $\mathcal{Z}_t^y(s) = y_{t+s}$, $B^H = \{B_t^H : t \in [0, T]\}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and $\nu > 1/2$.

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Finally, we introduce the Young integral via the fractional calculus, which was given by Zähle ("Integration with respect to fractal functions and stochastic calculus". PTRF **111**, 1998), and use it to study fractional stochastic differential equations. This approach is based on a priori estimate by Nualart and Rășcanu.

Contents

Stratonovich integral

We first introduce a Stratonovich type stochastic integral with respect to a fBm with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ via the Malliavin Calculus.

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We use the Itô formula to study

$$X_t = x + \int_0^t a(X_s) \circ dB^H{}_s + \int_0^t b(X_s) ds, \quad t \in [0, T].$$

Here $x \in \mathbb{R}$, $a, b : \mathbb{R} \to \mathbb{R}$.

Forward integral

We first introduce a Stratonovich type stochastic integral with respect to a fBm with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ via the Malliavin Calculus.

In the second part of this talk we introduce the forward integral and compare it with the Stratonovich integral.

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We first introduce a Stratonovich type stochastic integral with respect to a fBm with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ via the Malliavin Calculus.

In the second part of this talk we introduce the forward integral and compare it with the Stratonovich integral. We also consider

$$X_t = x + \int_0^t a(X_s) d B_s^{H-} + \int_0^t b(X_s) ds, \quad t \in [0, T].$$

and

$$Y_t = X_0 + \int_0^t c(s, Y_s) ds + \int_0^t \sigma_s Y_s d B_s^{H-}, \quad t \in [0, T].$$

Here $x \in \mathbb{R}$, $a, b : \mathbb{R} \to \mathbb{R}$, $c : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : \Omega \times [0, T] \to \mathbb{R}$ and $H \in (1/2, 1)$.

Contents

Transport processes

We introduce a sequence of processes which converges strongly to FBM uniformly on bounded intervals.

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We introduce a sequence of processes which converges strongly to FBM uniformly on bounded intervals.
This processes allow us to obtain a method for simulating the paths of a stochastic differential equation

$$X_t = x + \int_0^t a(X_s) \circ dB^H{}_s + \int_0^t b(X_s)ds, \quad t \in [0, T].$$

Here $x \in \mathbb{R}$, $a, b : \mathbb{R} \to \mathbb{R}$ and $H \in (1/4, 1)$.

Contents

Semimartingale method

Here we define the stochastic integral with respect to FBM as the limit of stochastic integrals with respect to a semimartingale that converges to FBM.

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Here we define the stochastic integral with respect to FBM as the limit of stochastic integrals with respect to a semimartingale that converges to FBM. Hence we can approximate the solution of

$$X_t = x + \int_0^t a(s) X_s dB_s^H + \int_0^t b(s)(X_s) ds, \quad t \in [0, T].$$

by solutions of SDE driven by semimartingales.