Semimartingale Approach for Stochastic Integration

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Equation

In this part, we define

$$\int_0^T \cdot dB_s$$

as the limit of intregrals with respect to semimartingales.

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Fractional integrals and derivatives

The right-sided fractional Riemann-Liouville integral of f of order α on $[0, T]$ is defined as

$$I_{T-}^{\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \int_s^T (r-s)^{\alpha-1} f(r) dr, \quad \text{for a.a. } s \in [0, T].$$

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Fractional diferentiation is introduced as an inverse operation : We will denote by $I_{T-}^{\alpha}(L^p([0, T]))$ the class of functions $f \in L^p([0, T])$ such that

$$f = I_{T-}^{\alpha} \varphi \quad \text{for some } \varphi \in L^p([0, T]).$$

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$$f = I^{\alpha}_{T-}\varphi \quad \text{for some } \varphi \in L^p([0,T]).$$

In this case

$$\varphi = D^{\alpha}_{T-}f(s) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(s)}{(T-s)^{\alpha}} - \alpha \int_s^T \frac{f(r)-f(s)}{(r-s)^{\alpha+1}} dr \right).$$

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Proposition (Samko et al., 1993)

Let $f \in L^p([0,T])$. Then $f \in I^{\alpha}_{T-}(L^p([0,T]))$ if and only if

$$\int_0^T \frac{|f(s)|^p}{(T-s)^{p\alpha}}ds < \infty.$$

and the integral

$$\int_{s+\varepsilon}^T \frac{f(r)-f(s)}{(r-s)^{\alpha+1}}dr$$

converges in $L^p([0,T])$ as $\varepsilon \to 0$.

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Case $H < \frac{1}{2}$

For $H < \frac{1}{2}$,

$$B_t^H = \frac{1}{\Gamma(1-\alpha)} \left(Z_t + \int_0^t (t-s)^{-\alpha} dW_s \right),$$

where $\{W_t : t \in \mathbb{R}\}$ is a Bm, $\alpha = \frac{1}{2} - H \in (0, \frac{1}{2})$ and

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has absolutely continuous paths. Hence we only need to consider the term

$$B_t = \int_0^t (t-s)^{-\alpha} dW_s.$$

Case $H < \frac{1}{2}$

$$B_t = \int_0^t (t-s)^{-\alpha} dW_s, t \in [0, T],$$

and for $\varepsilon > 0$,

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Lemma (Alòs, Mazet and Nualart)

Let $\phi \in \mathbb{L}^{1,2}$. *Then*

$$\begin{aligned} \int_0^T \phi_t dB_t^{\varepsilon} &= \int_0^T \Big(\phi_s (T-s+\varepsilon)^{-\alpha} \\ &\qquad -\alpha \int_s^T (\phi_r - \phi_s)(r-s+\varepsilon)^{-\alpha-1} dr\Big) dW_s \\ &\quad -\alpha \int_0^T \int_0^r D_s\phi_r (r-s+\varepsilon)^{-\alpha-1} ds dr. \end{aligned}$$

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Proof

$$\begin{aligned}\int_0^T \phi_t dB_t^\varepsilon &= \varepsilon^{-\alpha} \int_0^T \phi_t dW_t \\ &\quad -\alpha \int_0^T \left(\int_0^t (t-s+\varepsilon)^{-\alpha-1} dW_s\right) \phi_t dt\end{aligned}$$

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&\quad -\alpha \int_0^T \left(\int_0^t (t-s+\varepsilon)^{-\alpha-1} dW_s \right) \phi_t dt \\
&= \varepsilon^{-\alpha} \int_0^T \phi_t dW_t - \alpha \int_0^T \left(\int_0^t \phi_t (t-s+\varepsilon)^{-\alpha-1} dW_s \right) dt \\
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\begin{aligned}
\int_0^T \phi_t dB_t^{\varepsilon} &= \varepsilon^{-\alpha}\int_0^T \phi_t dW_t - \alpha \int_0^T \left(\int_0^t \phi_t (t-s+\varepsilon)^{-\alpha-1} dW_s\right) dt \\
&\quad -\alpha \int_0^T \int_0^t D_s\phi_t (t-s+\varepsilon)^{-\alpha-1} ds dt \\
&= \varepsilon^{-\alpha}\int_0^T \phi_t dW_t \\
&\quad -\alpha \int_0^T \left(\int_0^t (\phi_t-\phi_s)(t-s+\varepsilon)^{-\alpha-1} dW_s\right) dt \\
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&\quad -\alpha \int_0^T \int_0^t D_s\phi_t (t-s+\varepsilon)^{-\alpha-1} ds dt
\end{aligned}
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Proof Finally, Fubini theorem gives

$$\alpha \int_0^T \left(\int_0^t \phi_s (t - s + \varepsilon)^{-\alpha - 1} dW_s \right) dt$$
$$= \varepsilon^{-\alpha} \int_0^T \phi_s dW_s - \int_0^T \phi_s (T - s + \varepsilon)^{-\alpha} dW_s.$$

Case $H < \frac{1}{2}$

Definition

Let $\phi \in$Dom δ. We say that ϕ is integrable with respect to B if $\int_0^T \phi_t dB_t^\varepsilon$ converges in probability as $\varepsilon \to 0$.

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Theorem

Let $\phi \in I_{T-}^\alpha(\mathbb{L}^{1,2})$ be such that

$$\int_0^T \int_0^r |D_s\phi_r|(r-s)^{-\alpha-1} dsdr < \infty.$$

Then, ϕ is integrable with respect to B and

$$\int_0^T \phi_t dB_t = \Gamma(1-\alpha)\int_0^T D_{T-}^\alpha \phi_t dW_t - \alpha \int_0^T \int_0^r D_s\phi_r (r-s)^{-\alpha-1} dsdr.$$

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Let $\phi \in I^{\alpha}_{T-}(\mathbb{L}^{1,2})$ be such that $\int_0^T \int_0^r |D_s\phi_r|(r-s)^{-\alpha-1}dsdr < \infty$. Then, ϕ is integrable with respect to B and

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Proof

$$\begin{aligned}\int_0^T \phi_t dB_t^{\varepsilon} &= \int_0^T \Big(\phi_s(T-s+\varepsilon)^{-\alpha} \\ &\qquad -\alpha\int_s^T(\phi_r-\phi_s)(r-s+\varepsilon)^{-\alpha-1}dr\Big)\,dW_s \\ &\quad -\alpha\int_0^T\int_0^r D_s\phi_r(r-s+\varepsilon)^{-\alpha-1}dsdr.\end{aligned}$$

General case

In the remaining, we present the ideas of Carmona, Coutin and Monseny (2003).

General case

Fix a time interval $[0, T]$ and consider the fBm $B = \{B_t; t \in [0, T]\}$. Then there exists a Bm $\{W_t; t \in [0, T]\}$ such that

$$B_t = \int_0^t K_H(t, s) dW_s,$$

where

- For $H > 1/2$,

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad s < t.$$

- For $H < 1/2$,

$$K_H(t, s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t - s)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u - s)^{H-\frac{1}{2}} du \right], \quad s < t.$$

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Then,

$$B_t^\varepsilon = \int_0^t K_H(s + \varepsilon, s) dW_s + \int_0^t \int_0^u \partial_1 K(u + \varepsilon, s) dW_s du, \quad t \in [0, T].$$

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- $\alpha + H > 1/2$ and $p > 1/H$.

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- $\sup_{s<u} \frac{E[(a_u - a_s)^2 + \int_0^T (D_r a_u - D_r a_s)^2 dr]}{|u-s|^{2\alpha}} < \infty$.

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Theorem (Carmona, Coutin and Monseny)

Let a be an adapted process satisfying above hypotheses. Then,

$$\begin{aligned}\int_0^T a_t dB_t &= \int_0^T a_s K(T,s) dW_s \\ &+ \int_0^T \int_s^T [a_u - a_s] \partial_1 K(u,s) du \delta W_s \\ &+ \int_0^T \int_0^u D_s a_u \partial_1 K(u,s) ds du.\end{aligned}$$

General case

Proposition (Carmona, Coutin and Monseny)

Let $H > 1/4$ *and* $f \in C^5(\mathbb{R})$. *Then,*

$$\begin{aligned} f(B_t) &= f(0) + \int_0^t f'(B_s)k(t,s)dW_s \\ &\quad + \int_0^t \int_s^t \left[f'(B_u) - f'(B_s)\right] \partial_1 K(u,s)du\delta W_s \\ &\quad + H\int_0^t f''(B_s)s^{2H-1}ds. \end{aligned}$$

Contents

Stochastic differential equations

Let $\alpha = H - \frac{1}{2}$ be with $H > 1/2$ and $B_t = \int_0^t (t-s)^\alpha dW_s$.

Proposition (Thao)

Let $S_0 \in L^2(\Omega)$. Then,

$$dS_t = S_t(\mu dt + \nu dB_t), \quad 0 < t \leq T,$$

has a unique solution given by

$$S_t = S_0 \exp(\mu t + \nu B_t).$$