Finance, Insurance, and Stochastic Control (IV)

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Outline

A Different Look at the Reinsurance Problem

Recall the general form of a reserve equation with reinsurance:

$$dX_t = b(X_t, \alpha_t, \pi_t)dt + \sigma(\pi_t)dB_t - \int_{\mathbb{R}_+} [\alpha f](t,x)\tilde{N}(dxdt),$$
$$X_0 = x.$$

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$$X_0 = x.$$

Suppose that the random field α is such that there exists some predictable pair (β, u) so that the martingale

$$M_t^u \triangleq \int_0^t \beta_s dB_s + \int_0^t \int_{\mathbb{R}_+} [\alpha f](s,x)\tilde{N}(dxds)$$

satisfies the following property:

$$d[M^u]_t = dt + u_t dM_t^u, \qquad t \geq 0. \tag{1}$$

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Note: Since $\Delta[M^u]_t = (\Delta M_t^u)^2 = u_t \Delta M_t^u$, u exactly controls the jumps of the reserve, that is, the claim size!

A Different Look at the Reinsurance Problem

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- One can show (and will do) that for any bounded, predictable process u there are always such α and β, at least when the probability space is "nice".
- Then, noting that the Brownian motion B itself satisfies (1) with $u \equiv 0$, one can rewrite (1) as

$$X_t = x + \int_0^t b(X_s, u_s, \pi_s)ds + \int_0^t \tilde{\sigma}(\pi_s)dM_s^u, \quad t \geq 0, \quad (2)$$

where M^u is a (possibly multi-dimensional) martingale satisfying the Structure Equation.

A Different Look at the Reinsurance Problem

Note

- The process u "controls" exactly the jump sizes of M^u (whence that of X)
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- The system (2) provides a new model for stochastic control problems in which the control of the jump size is essential.

Some references:

- Ma-Protter-San Martin (1998) — Anticipating calculus and an Ocone-Haussmann-Clark type formula for normal martingales
- Dritschel-Protter (1999) — complete market with discontinuous security prices.
- Buckdahn-Ma-Rainer (2008) — Stochastic Control for Systems driven by normal mg.

Normal Martingales and Structure Equation

- $(\Omega, \mathscr{F}, P; \mathbf{F})$— a filtered probability space, and $\mathbf{F} \stackrel{\triangle}{=} \{\mathscr{F}_t\}_{t\geq 0}$ satisfies the "*usual hypotheses*".
- $\mathscr{M}_0^2(\mathbf{F}, P)$ — the space of all L^2 P-martingales s.t. $X_0 = 0$.
- $X \in \mathscr{M}_0^2(\mathbf{F}, P)$ is "normal" if $\langle X \rangle_t = t$, (i.e., $[X]_t = t + \text{mg}$.)
- If a normal martingale also has the "Representation Property", then there exists an $\mathbf{F}$-predictable process u such that

$$[X]_t = t + \int_0^t u_s dX_s, \qquad \forall t \geq 0. \tag{3}$$

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Warning

A solution to the structure equation must be "normal" but the converse it not necessarily true!

Some examples

Example

- $u \equiv 0$ — Brownian motion
- $u \equiv \alpha \in \mathbb{R}^*(\stackrel{\triangle}{=} \mathbb{R} \setminus \{0\})$ — compensated Poisson process
- $u_t = -X_t$ — Azéma's martingale
- $u_t = -2X_t$ — "*Parabolic martingale*" (Protter-Sharpe (1979)).

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Characteristics of the solutions to structure equations

Let $X \in \mathscr{M}_0^2(\mathbf{F}, P)$ be a solution to (3), and denote $\mathscr{D}_X(\omega) \stackrel{\triangle}{=} \{t > 0; \Delta X_t(\omega) \neq 0\}$, $\omega \in \Omega$. Then,

- $\Delta X_t = u_t$, for all $t \in \mathscr{D}_X$, P-a.s.
- Decomposing $X = X^c + X^d$, one has

$$dX_t^c = \mathbf{1}_{\{u_t=0\}} dX_t, \text{ and } dX_t^d = \mathbf{1}_{\{u_t \neq 0\}} dX_t, \ t \geq 0.$$

Note that in general the well-posedness of higher dimensional structure equation is not trivial. References include Meyer (1989), Kurtz-Protter (1991), and Phan (2001), ...

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"Wiener-Poisson" space

Assume that on $(\Omega, \mathscr{F}, P)$ there exist

- B — a d-dimensional standard Brownian motion
- μ— a Poisson random measure, such that $B \perp\!\!\!\perp \mu$, and with the compensator $\widehat{\mu}(dtdx) \stackrel{\triangle}{=} \nu(dx)dt$, where ν is the Lévy measure of μ.

Normal martingales in a Wiener-Poisson space

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Denote

- $\mathbf{F}^{B,\mu} = \{\mathscr{F}_t^{B,\mu}\}_{t\geq 0}$ to be the natural filtration generated by B and μ, and let
- $\mathbf{F} \stackrel{\triangle}{=} \overline{\mathbf{F}^{B,\mu}}^P$ (augmentation) satisfies the usual hypothses.

Martingale Representation Theorem (Jacod-Shiryaev (1987/2003))

For any $X \in \mathscr{M}_0^2(\mathbf{F}, P; \mathbb{R}^d)$, There exists a unique pair $(\alpha, \beta) \in L^2_{\mathbf{F}}([0, T]; \mathbb{R}^{d\times d}) \times L^2_{\mathbf{F}}([0, T] \times \mathbb{R}^*; dt \times d\nu; \mathbb{R}^d)$, such that

$$X_t = \int_0^t \alpha_s dB_s + \int_{[0,t]\times\mathbb{R}^*} \beta_s(x)\tilde{\mu}(dsdx), \qquad t \geq 0. \qquad (4)$$

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Question:

If $X \in \mathscr{M}_0^2(\mathbf{F}, P; \mathbb{R}^d)$ is a normal martingale driven by $u = \{u_t\}_{t\geq 0}$ on a Wiener-Poisson space $(\Omega, \mathscr{F}, P)$, what would be the relations between u and (α, β)?

Theorem

X is a solution to the structure equation driven by u:

$$\begin{cases} d[X^i]_t = dt + u^i_t dX^i_t, & 1 \le i \le d, \\ d[X^i, X^k]_t = 0, & 1 \le i < k \le d, \quad t \ge 0, \end{cases} \tag{5}$$

$\iff \quad \exists\ A^i_s = \{(x,\omega) : \beta^i_s(x,\omega) \neq 0\} \in \mathscr{B}(\mathbb{R}^*) \otimes \mathscr{F}_s$, s.t.

$$\begin{cases} \sum_{j=1}^d \alpha^{i,j}_s \alpha^{k,j}_s = \delta_{i,k} \mathbf{1}_{\{u^i_s = 0\}}, \quad \alpha^{i,j}_s \mathbf{1}_{\{u^i_s \neq 0\}} = 0, ds \times dP \text{ a.e.;} \\ \beta^i_s(x) = u^i_s \mathbf{1}_{A^i_s}(x), ds \times d\nu \times dP \text{ a.e.;} \\ \nu(A^i_s \cap A^k_s)\mathbf{1}_{\{u^i_s \neq 0, u^k_s \neq 0\}} = \delta_{i,k} \frac{1}{(u^i_s)^2} \mathbf{1}_{\{u^i_s \neq 0\}}, \quad ds \times dP \text{ a.e.} \end{cases} \tag{6}$$

Here $\{\delta_{ik}\}$ is the Kronecker's delta.

Existence of Solution to Structure Equation

- Denote $\Gamma = \{\text{all atoms of } \nu\}$ and define

$$\nu^{cont}(A) = \nu(A) - \sum_{x\in\Gamma\cap A} \nu(\{x\}), \qquad A \in \mathscr{B}(\mathbb{R}\setminus\{0\}),\ 0 \notin \overline{A},$$

where $\overline{A}$ is the closure of A in $\mathbb{R}$.

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  where $\overline{A}$ is the closure of A in $\mathbb{R}$.
- Assume $\nu^{cont}([-1,1]) = +\infty$ (e.g., $\nu(dx) = C|x|^{-(1+\alpha)}dx!$), and let $u = (u^1, \ldots, u^d)$ be any bdd, **F**-predictable process.

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- Then $\forall t \geq 0$, $\exists\ 1 = \tau_t^0 > \tau_t^1 > \tau_t^2 > \cdots > \tau_t^d > 0$, all $\mathscr{F}_t$-measurable, s.t.,

$$A_t^i \stackrel{\triangle}{=} \{(-\tau_t^{i-1}, -\tau_t^i] \cup [\tau_t^i, \tau_t^{i-1})\} \cap \Gamma^c, \quad i = 1, \cdots, d$$

and $\alpha_t^{i,j} = \delta_{ij}\mathbf{1}_{\{u_t^i=0\}}$, $\beta_t^i(x) = u_t^i \mathbf{1}_{A_t^i}(x)$, $i,j = 1, \cdots, d$, $x \in \mathbb{R}$, satisfy (6). Hence $dX = \alpha dB + \int \beta d\tilde{\mu}$ solves the struction equation on $(\Omega, \mathscr{F}, P, \mathbf{F}, B, \mu)$!

Uniqueness

It turns out that the solution to (5) is not unique, not even in law!

Example

Assume $d = 1$, $\nu(dx) = x^{-2}\mathbf{1}_{\{x>0\}}$, and $B \perp\!\!\!\perp \mu$. Set $\tau = \inf\{t \geq 0, B_t = 1\}$. Define $u_t = \mathbf{1}_{(\tau,+\infty)}(t)$, $\alpha_t = -\alpha'_t = \mathbf{1}_{[0,\tau]}(t)$, and $\beta_t(x) = \beta'_t(x) = \mathbf{1}_{A_t}(x)$, where

$$A_t(\omega) = \begin{cases} \emptyset & \text{on } 0 \leq t \leq \tau(\omega); \\ [1,\infty) & \text{on } t > \tau(\omega). \end{cases}$$

Then $N_t \stackrel{\triangle}{=} \mu([0,t] \times [1,\infty))$ is a standard Poisson, $\perp\!\!\!\perp B$, and denoting $\tilde{N}_t = N_t - t$, by the characterization theorem,

$$X_t = B_{t\wedge\tau} + \tilde{N}_t - \tilde{N}_{t\wedge\tau} \quad \text{and} \quad X'_t = -B_{\tau\wedge t} + \tilde{N}_t - \tilde{N}_{t\wedge\tau},$$

both solves the structure equation driven by u, but X and X' are not identical in law! (Indeed write $\tau = \inf\{t; X_t = 1\}$ and define $\tau' = \inf\{t; X'_t = 1\}$. Then look at $X^\tau_t = X_{\tau\wedge t}$ and $X'^{\tau'}_t = X'_{\tau'\wedge t}$!)

Itô's formula

Because of the special structure of a normal martingale, the Itô's formula takes a unusual form. We first define the following *Differential-Difference operators*:

$$\mathscr{A}_u^i[\varphi](s,x) \triangleq \mathbf{1}_{\{u^i=0\}}\nabla_{x_i}\varphi(s,x) + \mathbf{1}_{\{u^i\neq 0\}}\frac{\varphi(s,x+u^ie_i)-\varphi(s,x)}{u^i},$$

$$\mathscr{L}_u[\varphi](s,x) \triangleq \sum_{i=1}^d \Big\{\mathbf{1}_{\{u^i=0\}}\frac{1}{2}D^2_{x_ix_i}\varphi(s,x) + \mathbf{1}_{\{u^i\neq 0\}}\frac{\varphi(s,x+u^ie_i)-\varphi(s,x)-u^i\nabla_{x^i}\varphi(s,x)}{(u^i)^2}\Big\},$$

where $\{e_1,\dots,e_d\}$ is the canonical orthonormal basis in $\mathbb{R}^d$.

Itô's formula

Theorem

Let $u = \{u_t; t \geq 0\}$ be a bounded **F**-predictable process with values in $\mathbb{R}^d$ and $X \in \mathscr{M}_0^2(\mathbf{F}, P; \mathbb{R}^d)$ a solution to the structure equation (5) driven by u.
Then, for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$, the following formula holds :

$$\begin{aligned}\varphi(t, X_t) - \varphi(0, X_0) &= \sum_{i=1}^d \int_0^t \mathscr{A}^i_{u_s}[\varphi](s, X_{s-}) dX^i_s \\ &+ \int_0^t \Big(\partial_s \varphi(s, X_s) + \mathscr{L}_{u_s}[\varphi](s, X_s) \Big) ds,\end{aligned}$$

Proof. Apply the general Itô formula, and note that $[X^i, X^k] = 0$, for $i \neq k$; $\Delta X^i_t = u^i_t$ on $u^i_t \neq 0$, and the structure equation ... ∎

The Stochastic Control Problem

The "non-uniqueness" of the solution to the structure equation indicates that a "weak" form of stochastic control is necessary.

Definition ("weak controls")

Let $U \in \mathbb{R}$, $U_1 \in \mathbb{R}^k$ be compact. A "*weak control* at time $t \in [0, T]$" is a 7-tuple $(\Omega, \mathscr{F}, P, \mathbf{F}^t, \pi, u, X^u)$ such that

- $(\Omega, \mathscr{F}, P; \mathbf{F}^t = \{\mathscr{F}_s\}_{s \geq t})$ satisfies the usual hypotheses;
- (π, u) is $\mathbf{F}^t$-predictable, with values in $\overline{U} \triangleq U_1 \times U^d$;
- $X = X^u \in \mathscr{M}_0^2(\mathbf{F}^t, P; \mathbb{R}^d)$ satisfies the structure equation

$$\begin{cases} d[X^i]_s = ds + u_s^i dX_s^i, & 1 \leq i \leq d, \quad s \in [t, T] \\ d[X^i, X^k]_s = 0, & 1 \leq i < k \leq d, \quad s \in [t, T], \\ X_s = 0, \quad s \in [0, t]. \end{cases} \tag{7}$$

We denote the set of all weak controls at t by $\mathscr{U}(t)$

The Stochastic Control Problem

Note

If $0 \leq t \leq t' \leq T$, then $\mathscr{U}(t) \subseteq \mathscr{U}(t')$ in the following sense:
$(\Omega, \mathscr{F}, P, \mathbf{F}^t, \pi, u, X) \in \mathscr{U}(t) \Longrightarrow$
$(\Omega, \mathscr{F}, P, \mathbf{F}^{t'}, (\pi_s, u_s)_{s \geq t'}, (X_s - X_{t'})_{s \geq t'}) \in \mathscr{U}(t')$.

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Assume that $b = b(y, \pi, u)$ and $\sigma = \sigma(y, \pi, u)$ are

- uniformly continuous in (y, π, u);
- Lipschitz in y, uniformly in (u, π).

For $(t, y) \in [0, T] \times \mathbb{R}^m$ and $\mu = (\Omega, \mathscr{F}, P, \mathbf{F}, \pi, u, X^u) \in \mathscr{U}(t)$, consider the controlled dynamics

$$Y_s = y + \int_t^s b(Y_r, \pi_r, u_r)dr + \int_t^s \sigma(Y_{r-}, \pi_r, u_r)dX_r^u, s \ge t. \quad (8)$$

Denote the (unique) solution of (8) by $Y^{t,y}(\mu) = \{Y_s^{t,y}(\mu)\}_{s \in [t,T]}$.

The Dynamic Programming (Bellman) Principle

Define

- The cost functional:

$$J(t,y;\mu) \stackrel{\triangle}{=} E\{g(Y_T^{t,y}(\mu))\}, \qquad (t,y) \in [0,T] \times \mathbb{R}^*,$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is, say, bounded and continuous.

- The value function:

$$V(t,y) = \inf_{\mu \in \mathscr{U}(t)} E\{g(Y_T^{t,y}(\mu))\}, \qquad (t,x) \in [0,T] \times \mathbb{R}^m.$$

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Theorem

For any $(t, y) \in [0, T] \times \mathbb{R}^m$ and $0 < h \leq T - t$, it holds that

$$V(t, y) = \inf_{\mu \in \mathscr{U}(t)} E[V(t + h, Y_{t+h}^{t,y}(\mu))]. \tag{9}$$

Sketch of the Proof.

(I) Show that $V(t,y) \geq \inf_{\mu\in\mathscr{U}(t)} E[V(t+h, Y^{t,y}_{t+h}(\mu))]$.

- Since $\mu \in \mathscr{U}(t) \subset \mathscr{U}(t+h) \Longrightarrow Y^{t,y}_T(\mu) = Y_T^{t+h, Y^{t,y}_{t+h}}(\mu)$,

$$
\begin{aligned}
&\Longrightarrow E\left\{g\left(Y_T^{t+h,Y_{t+h}}(\mu)\right)\right\} \\
&= \int_{\mathbb{R}^m} E\left\{g\left(Y_T^{t+h,Y_{t+h}}\right)\Big| Y_{t+h}(\mu)=z\right\} P\circ[Y_{t+h}(\mu)]^{-1}(dz) \\
&\geq \int_{\mathbb{R}^m} E\left\{V(t+h, Y_{t+h})| Y_{t+h}(\mu)=z\right\} P\circ[Y_{t+h}](\mu)]^{-1}(dz) \\
&= E\left\{V\left(t+h, Y_{t+h}(\mu)\right)\right\}.
\end{aligned}
$$

Warning

The real argument involves the decomposition of the Wiener-Poisson canonical space on $[t,T]$ into $[t,t+h]]$ part and $[t+h,T]$ part, following the idea of Fleming-Souganidis (1989).

The Dynamic Programming (Bellman) Principle

(II) Show $V(t,y) \le \inf_{\mu \in \mathscr{U}(t)} E[V(t+h, Y^{t,y}_{t+h}(\mu))]$.

- Fix $0 \le t \le t+h \le T$, $y \in \mathbb{R}^m$
- For $n = 1, 2, \cdots$, define $\Gamma^n = \{k2^{-n}; k \in \mathbf{Z}^m\}$, and

$$I(z) \stackrel{\triangle}{=} \Pi_{i=1}^m [(k_i - 1)2^{-n}, k_i 2^{-n}], \qquad z = k2^{-n} \in \Gamma^n.$$

- Define $Y^{(n)}_{t+h}(\mu) = \sum_{z \in \Gamma^n} z \mathbf{1}_{I(z)}(Y^{t,y}_{t+h}(\mu))$ ($\Longrightarrow$ $Y^{(n)}_{t+h}(\mu) - Y^{t,y}_{t+h}(\mu) \in [0, 2^{-n}]^m$) and

$$P^z(\cdot) \stackrel{\triangle}{=} P\{\cdot \,|\, Y^{(n)}_{t+h}(\mu) = z\}, \text{whenever} \quad P\{Y^{(n)}_{t+h}(\mu) = z\} > 0.$$

- $\forall \varepsilon > 0$, $n \ge 1$, and $z \in \Gamma^n$, let $\mu^z = (\Omega^z, \mathscr{F}^z, P^z, \mathbf{F}^z, u^z, X^z) \in \mathscr{U}(t+h)$ be such that $E^z[g(Y^{t+h,z}_T)(\mu^z)] \le V(t+h, z) + \varepsilon$.

The Dynamic Programming (Bellman) Principle

- Define $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{P}) = (\Omega, \mathscr{F}, P) \otimes \left(\otimes_{z\in\Gamma^n}(\Omega^z, \mathscr{F}^z, P^z)\right)$,
- $\widehat{\mathscr{F}}_s = \begin{cases} \mathscr{F}_s, & s \in (t, t+h), \\ \mathscr{F}_s \otimes \left(\otimes_{z\in\Gamma^n}\mathscr{F}_s^z\right), & s \in [t+h, T], \end{cases}$ with usual augmentation.
- $(\widehat{\pi}_s, \widehat{u}_s) = \begin{cases} (\pi_s, u_s), & \text{if} \quad s \in (t, t+h), \\ (\pi_s^z, u_s^z), & \text{if} \quad \left\{Y_{t+h}^{t,y} \in I(z)\right\} \\ & \text{and } s \in [t+h, T], \end{cases}$
- $\widehat{X}_s = \begin{cases} X_s, & \text{if} \quad s \in (t, t+h), \\ X_{t+h} + (X_s^z - z), & \text{if} \quad \left\{Y_{t+h}^{t,y} \in I(z)\right\} \\ & \text{and} \quad s \in [t+h, T], \end{cases}$

The Dynamic Programming (Bellman) Principle

Then, show that

- $\widehat{X}$ is a solution of structure equation (5) driven by $\widehat{u}$;
- $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{P}, \widehat{\mathbf{F}}, \pi, u, X)$, $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{P}, \widehat{\mathbf{F}}, \widehat{\pi}, \widehat{u}, \widehat{X}) \in \mathscr{U}(t)$.
- $(\widehat{\Omega}, \widehat{\mathscr{F}}, \widehat{P}, \widehat{\mathbf{F}}^{t+h}, \pi^z, u^z, X^z) \in \mathscr{U}(t+h)$.
- for large n, the solution $\widehat{Y}_s = Y_s^{t,y}$ satisfies

$$\begin{aligned}
\widehat{E}\{g(\widehat{Y}_T)\} &\le \sum_{z\in\Gamma^n} E\Big\{g(Y_T^z)\Big\}P\Big\{Y_{t+h}^{t,y} \in I(z)\Big\} + \varepsilon \\
&\le \sum_{z\in\Gamma^n} V(t+h,z)P\Big\{Y_{t+h}^{t,y} \in I(z)\Big\} + 2\varepsilon \\
&= E\Big\{\sum_{z\in\Gamma^n} V(t+h,z)\mathbf{1}_{I(z)}\Big(Y_{t+h}^{t,y}\Big)\Big\} + 2\varepsilon.
\end{aligned}$$

- Letting $\varepsilon \searrow 0$ and $n \to \infty \implies$ Done! ∎

Controlled Differential-Difference Operators

For each $(\pi, u) \in U_1 \times U^d$ we define

$$\begin{aligned}
\mathscr{A}^i_{\pi,u}[\varphi](s,y) &= \mathbf{1}_{\{u^i=0\}} \nabla_y \varphi(s,y)\sigma^i(y,\pi,u) \\
&\quad + \mathbf{1}_{\{u^i\neq 0\}} \frac{\varphi(s, y+u^i\sigma^i(y,\pi,u)) - \varphi(s,y)}{u^i}, \\
\mathscr{L}_{\pi,u}[\varphi](s,y) &= \nabla_y \varphi(s,y) b(y,\pi,u) \\
&\quad + \sum_{i=1}^d \Big\{ \mathbf{1}_{\{u^i=0\}} \frac{1}{2} [D^2_{yy}\varphi(s,y)\sigma^i(y,\pi,u), \sigma^i(y,\pi,u)] \\
&\quad + \mathbf{1}_{\{u^i\neq 0\}} \frac{\Delta^i[\varphi](s,y,u^i) - u^i \nabla_y \varphi(s,y)\sigma^i(y,\pi,u)}{(u^i)^2} \Big\},
\end{aligned}$$

where $\Delta^i[\varphi](s,y,u^i) \stackrel{\triangle}{=} \varphi(s, y+u^i\sigma^i(y,\pi,u))) - \varphi(s,y)$.

The HJB Equations

If $(\Omega, \mathscr{F}, P, \mathbf{F}^t, \pi, u, X) \in \mathscr{U}(t)$ and $Y = Y^{t,y}$ is the corresponding system dynamics, then by Itô's formula, for any $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^m)$ it holds that

$$\begin{aligned} & \varphi(s, Y_s) - \varphi(t, y) \\ = \; & \sum_{i=1}^{d} \int_0^t \mathscr{A}^i_{\pi_s, u_s}[\varphi](s, Y_{s-}) dX^i_s \\ & + \int_0^t \Big(\partial_s \varphi(s, Y_s) + \mathscr{L}_{\pi_s, u_s}[\varphi](s, Y_s) \Big) ds. \end{aligned}$$

The HJB Equations

Consider the following fully nonlinear partial differential-difference equation (PDDE):

$$\begin{cases} -\partial_t V(t,y) - \inf_{(\pi,u)\in\overline{U}} \mathscr{L}_{\pi,u}[V](t,y) = 0, \\ \qquad\qquad\qquad\qquad (t,y) \in [0,T) \times \mathbb{R}^m, \\ V(T,y) = g(y), \qquad\qquad y \in \mathbb{R}^m, \end{cases} \tag{10}$$

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Purpose:

Show that the value function $V(\cdot,\cdot)$ is the unique viscosity solution to the HJB equation (10).

The HJB Equations

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Purpose:

Show that the value function $V(\cdot,\cdot)$ is the unique viscosity solution to the HJB equation (10).

Note: The PDDE (10) has not been studied in the literature, therefore a thorough investigation is needed, starting from the definition of "viscosity solution"(!)

The HJB Equations

To overcome the difficulty caused by the "small jumps", we adopt the idea of Barles-Buckdahn-Pardoux (1997) for the IPDEs.

Definition

For each $\delta > 0$, define the following operator:

$$\begin{aligned}\mathscr{L}^{\delta}_{\pi,u}[V,\varphi](t,y) &\stackrel{\triangle}{=} \nabla_y\varphi(t,y)b(y,\pi,u)\\ &+\sum_{i=1}^{d}\Big\{\mathbf{1}_{\{u^i=0\}}\frac{1}{2}(D^2_{yy}\varphi(t,y)\sigma^i(y,\pi,u),\sigma^i(y,\pi,u))\\ &+\mathbf{1}_{\{0<|u^i|\le\delta\}}\frac{\Delta^i[\varphi](t,y,u^i)-u^i\nabla_y\varphi(t,y)\sigma^i(y,\pi,u)}{(u^i)^2}\\ &+\mathbf{1}_{\{|u^i|>\delta\}}\frac{\Delta^i[V](t,y,u^i)-u^i\nabla_y\varphi(t,y)\sigma^i(y,\pi,u)}{(u^i)^2}\Big\},\end{aligned}$$

where $\Delta^i[\varphi](t,y,u^i) \stackrel{\triangle}{=} \varphi(t,y+u^i\sigma^i(y,\pi,u)))-\varphi(t,y)$.

Definition

A continuous function $V : [0, T] \times \mathbb{R}^m \to \mathbb{R}$ is called a viscosity "subsolution" (resp. "supersolution") of the PDDE (10) if

(i) $V(T, y) \leq$ (resp. $\geq$) $g(y)$, $y \in \mathbb{R}^m$; and

(ii) for any $(t, y) \in [0, T) \times \mathbb{R}^m$ and $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^m)$ such that $V - \varphi$ attains a local maximum (resp. minimum) at (t, y), it holds that

$$-\frac{\partial}{\partial t}\varphi(t, y) - \inf_{(\pi,u)\in\overline{U}} \mathscr{L}^{\delta}_{\pi,u}[V, \varphi](t, y) \leq \text{(resp. } \geq\text{) } 0,$$

for all sufficiently small $\delta > 0$.

A function V is called a viscosity solution of (10) if it is both a viscosity subsolution and a supersolution of (10).

Remark

One can show that

- the definition is equivalent to one in which the "*local* maximum (minimum)" is replaced by "*global* maximum (minimum)" or even '*strict global* maximum (minimum)";
- the operator $\mathscr{L}_{\pi,u}[V,\varphi](t,y)$ is replaced by $\mathscr{L}_{\pi,u}[\varphi](t,y)$.

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(The idea is similar to that of Barles-Buckdahn-Pardoux (1997), but a little more complicated because of the "difference" operators!)

Theorem

The value function $V(t,y)$ is a viscosity solution of (10).

Sketch of the proof.

(I) "*Subsolution*":

- Fix (t, y). Let $\mu \stackrel{\triangle}{=} (\Omega, \mathscr{F}, P, \mathbf{F}, \pi, u, X) \in \mathscr{U}(t)$ with deterministic $(\pi, u) \in \overline{U}$. Let $\varphi \in C^{1,2}$ be such that $V - \varphi$ achieves a global maximum at (t, y).
- Applying Itô's formula and the Bellman principle:

$$\begin{aligned}
0 &\le E\left\{V(t+h, Y_{t+h}^{t,y}) - V(t,y)\right\} \\
&\le \left\{\varphi(t+h, Y_{t+h}^{t,y}) - \varphi(t,y)\right\} \\
&= \int_t^{t+h} E\left\{\frac{\partial}{\partial s}\varphi(s, Y_s^{t,y}) + \mathscr{L}_{\pi,u}[\varphi](s, Y_s^{t,y})\right\} ds.
\end{aligned}$$

- Dividing both sides by h and letting $h \longrightarrow 0$ we obtain

$$-\partial_t \varphi(t,y) - \mathscr{L}_{\pi,u}[\varphi](t,y) \le 0, \quad \forall (\pi, u) \in \overline{U}.$$

Namely V is a viscosity subsolution.

Sketch of the proof.

(II) "*Supersolution*":

- Fix (t, y). Let $\varphi \in C^{1,2}$ be such that $V - \varphi$ attains a global minimum at (t, y).
- Fix an arbitrary $h > 0$. For any $\varepsilon > 0$, applying the Bellman Principle to find $\mu^{\varepsilon,h} = (\Omega, \mathscr{F}, \mathbf{F}^t, P, u, X)^{\varepsilon,h} \in \mathscr{U}(t)$ s.t.

$$V(t, y) + \varepsilon h \geq E^{\varepsilon,h}\{V(t + h, Y^{t,y}_{t+h}(\mu^{\varepsilon,h}))\}.$$

- following the similar argument as before we have

$$E^{\varepsilon,h}\left\{\int_t^{t+h}\{\partial_s\varphi + \mathscr{L}_{\pi_s,u_s}[\varphi]\}(s, Y^{t,y}_s(\mu^{\varepsilon,h}))ds\right\} \leq \varepsilon h.$$

- Find $C > 0$ and $\delta > 0$ such that

$$|\{\partial_s\varphi(s, z) + \mathscr{L}_{\pi,u}[\varphi](s, z)| \leq C, \qquad \forall(s, z)$$

and for all $|(s, z) - (t, y)| \leq 2\delta$ and $(\pi, u) \in \overline{U}$,

$$|\{\partial_s\varphi + \mathscr{L}_{\pi,u}[\varphi]\}(s, z) - \{\partial_s\varphi + \mathscr{L}_{\pi,u}[\varphi]\}(t, y)| \leq \varepsilon.$$

Sketch of the proof.

- Consequently,

$$
\begin{aligned}
\varepsilon h &\geq h\{\partial_t\varphi + \mathscr{L}_{\pi_s,u_s}[\varphi]\}(t,y) - h\varepsilon \\
&\quad -ChP^{\varepsilon,h}\left\{\sup_{s\in[t,t+h]} |Y_s^{t,y} - y| \geq \delta\right\} \\
&\geq h\Big(\partial_t\varphi(t,y) + \inf_{(\pi,u)\in\overline{U}} \mathscr{L}_{\pi,u}\varphi(t,y)\Big) - h\varepsilon \\
&\quad -ChP^{\varepsilon,h}\left\{\sup_{s\in[t,t+h]} |Y_s^{t,y} - y| \geq \delta\right\}
\end{aligned}
$$

Sketch of the proof.

- Since

$$
\begin{aligned}
& P^{\varepsilon,h}\{\sup_{s\in[t,t+h]} |Y_s^{t,y} - y| \geq \delta\} \\
\leq \quad & \frac{4}{\delta^2} E^{\varepsilon,h}\left[\sum_{i=1}^{d} \int_t^{t+h} |\sigma^{\cdot,i}(Y_{s\wedge\tau-}^{t,y}, \pi_s, u_s)|^2 ds\right] \\
\leq \quad & C(1+|y|^2)\frac{1}{\delta^2}h.
\end{aligned}
$$

$$
\Longrightarrow \frac{\partial}{\partial t}\varphi(t,y) + \inf_{(\pi,u)\in\overline{U}} \mathscr{L}_{\pi,u}\varphi(t,y) \leq 2\varepsilon + C(1+|y|^2)\frac{1}{\delta^2}h.
$$

$$
\Longrightarrow \frac{\partial}{\partial t}\varphi(t,y) + \inf_{(\pi,u)\in\overline{U}} \mathscr{L}_{\pi,u}\varphi(t,y) \leq 0. \qquad \blacksquare
$$

Theorem

The value function $V(\cdot,\cdot)$ is the unique viscosity solution of (10) among all bounded, continuous functions.

Sketch of the proof. (Assume $d = 1$, no π, $\sigma = 1$, $b = 0$)

- Change $V(t,x) \mapsto e^{\gamma t} V(T-t,x)$, for $\gamma > 0$, then need only consider the equation

$$\partial_t V(t,x) + \gamma V(t,x) - \inf_{u \in U} \mathscr{L}_u[V](t,x) = 0,$$
$$V(0,x) = g(x).$$

- Let V be a sub- and W a super-solution of (11) (want: $V \le W$). Suppose that

$$\theta \stackrel{\triangle}{=} \sup_{(t,x)\in[0,T]\times\mathbb{R}} (V(t,x) - W(t,x)) > 0.$$

Uniqueness

- For $\varepsilon, \alpha > 0$, set

$$\Psi_{\varepsilon,\alpha} \triangleq V(t,x) - W(s,y) - \frac{\alpha}{2}\left(\frac{1}{T-t} + \frac{1}{T-s}\right) - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{1}{2\varepsilon}|x-y|^2 - \frac{1}{2\varepsilon}|s-t|^2,$$

and $(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \in \operatorname{argmax}\Psi_{\varepsilon,\alpha}$. (Note: $(\hat{t}, \hat{x}, \hat{s}, \hat{y})$ depend on ε, α, of course!)

- $\forall \eta > 0$, $\exists (t_\eta, x_\eta) \in [0, T] \times \mathbb{R}$, $\alpha_\eta > 0$ such that $\forall \alpha \in (0, \alpha_\eta)$,

$$V(t_\eta, x_\eta) - W(t_\eta, x_\eta) \geq \theta - \eta/2; \quad \Psi_{\varepsilon,\alpha}(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \theta - \eta > 0.$$

$\Longrightarrow \forall \alpha \in (0, \alpha_\eta)$, $\exists (t_\alpha, x_\alpha, t_\alpha, x_a)$ such that (possibly along a subsequence),

$$V(\hat{t}, \hat{x}) - W(\hat{s}, \hat{y}) \to V(t_\alpha, x_\alpha) - W(t_\alpha, x_\alpha), \qquad \text{as} \quad \varepsilon \to 0.$$

Uniqueness

- Applying Ishii's lemma to get: $\exists\ (\mathscr{X}, \mathscr{Y}) \in \mathbb{R}^{2m}$ such that

$$\begin{cases} (\dfrac{\hat{t}-\hat{s}}{\varepsilon} + \dfrac{\alpha}{2}\dfrac{1}{(T-\hat{t})^2}, \dfrac{\hat{x}-\hat{y}}{\varepsilon} + \alpha\hat{x}, \mathscr{X}) \in \overline{\mathscr{P}}^{1,2,+}V(\hat{t},\hat{x}), \\ (\dfrac{\hat{t}-\hat{s}}{\varepsilon} - \dfrac{\alpha}{2}\dfrac{1}{(T-\hat{s})^2}, \dfrac{\hat{x}-\hat{y}}{\varepsilon} - \alpha\hat{y}, \mathscr{Y}) \in \overline{\mathscr{P}}^{1,2,-}W(\hat{s},\hat{y}), \end{cases}$$

and

$$\begin{pmatrix} \mathscr{X} & 0 \\ 0 & -\mathscr{Y} \end{pmatrix} \le A+\rho A^2, \text{ with } A = \frac{1}{\varepsilon}\begin{pmatrix} I_m & -I_m \\ -I_m & I_m \end{pmatrix} + \alpha I_{2m};$$

where $\overline{\mathscr{P}}^{1,2,+}V(\hat{t},\hat{x})$ (resp. $\overline{\mathscr{P}}^{1,2,-}W(\hat{t},\hat{x})$) denotes the "parabolic superjet" (resp. "subjets").

Uniqueness

- By definition of viscosity solution (via "jets") we then have

$$
\begin{aligned}
0 &\geq \frac{\alpha}{2}\Big(\frac{1}{(T-\hat{t})^2}+\frac{1}{(T-\hat{s})^2}\Big)+\gamma\Big(V(\hat{t},\hat{x})-W(\hat{s},\hat{y})\Big) \\
&+\inf_{u\in U}\Big\{\mathbf{1}_{\{u\neq 0\}}\frac{W(\hat{s},\hat{y}+u)-W(\hat{s},\hat{y})}{u^2} \\
&\quad -\mathbf{1}_{\{u\neq 0\}}\frac{V(\hat{t},\hat{x}+u)+V(\hat{t},\hat{x})+\alpha u(\hat{x}+\hat{y})}{u^2} \\
&\quad +\mathbf{1}_{\{u=0\}}(\mathscr{Y}-\mathscr{X})\Big\}
\end{aligned}
$$

Uniqueness

- By definition of viscosity solution (via "jets") we then have

$$0 \geq \frac{\alpha}{2}\Big(\frac{1}{(T-\hat{t})^2} + \frac{1}{(T-\hat{s})^2}\Big) + \gamma\Big(V(\hat{t},\hat{x}) - W(\hat{s},\hat{y})\Big)$$
$$+ \inf_{u\in U}\Big\{\mathbf{1}_{\{u\neq 0\}}\frac{W(\hat{s},\hat{y}+u) - W(\hat{s},\hat{y})}{u^2}$$
$$-\mathbf{1}_{\{u\neq 0\}}\frac{V(\hat{t},\hat{x}+u) + V(\hat{t},\hat{x}) + \alpha u(\hat{x}+\hat{y})}{u^2}$$
$$+\mathbf{1}_{\{u=0\}}(\mathscr{Y} - \mathscr{X})\Big\}$$

Warning:

Since $\mathscr{H}(t,y,v,p,S) \stackrel{\triangle}{=} \inf_{(\pi,u)} \mathscr{L}_{\pi,u}[\varphi]$ is discontinuous on (p,S), this conclusion could be wrong, unless U takes some special form! We need to assume that $U = \{0\} \cup U_1$ where U_1 is compact. (Consider, e.g., the insurance model where there is "deductible".)

Uniqueness

- Now recall the definition of $(\hat{t}, \hat{x}, \hat{s}, \hat{y})$ we have $\Psi_{\varepsilon,\alpha}(\hat{t}, \hat{x}, \hat{s}, \hat{y}) \geq \Psi_{\varepsilon,\alpha}(\hat{t}, \hat{x} + u, \hat{s}, \hat{y} + u)$
- We obtain that

$$\begin{aligned} 0 &\geq \gamma(\theta - \eta) + \inf_{u \in U} \Big\{ \mathbf{1}_{\{u=0\}}(\mathscr{Y} - \mathscr{X}) \\ &\quad + \mathbf{1}_{\{u \neq 0\}} \frac{-\alpha(\hat{x} + \hat{y})u - \alpha u^2 + \alpha(\hat{x} + \hat{y})u}{u^2} \Big\} \\ &= \inf_{u \in U} \{ -\alpha \mathbf{1}_{\{u \neq 0\}} + \mathbf{1}_{\{u=0\}}(\mathscr{Y} - \mathscr{X}) \}. \end{aligned}$$

- Thus

$$\begin{aligned} 0 &\geq \gamma(\theta - \eta) + \inf_{u \in U} (-\alpha \mathbf{1}_{\{u \neq 0\}} - 4\alpha \mathbf{1}_{\{u=0\}}) \\ &= \gamma(\theta - \eta) - 4\alpha. \end{aligned}$$

- Choose $0 < \eta < \theta$ and $\gamma > \frac{4\alpha}{\theta - \eta}$, we have $0 \geq \gamma(\theta - \eta) - 4\alpha > 0$, a contradiction. ■

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