FILIPPOV TYPE ESTIMATES FOR FULLY NONLINEAR DIFFERENTIAL INCLUSIONS

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PRELIMINARIES

Let X be a Banach space.
Let $A : D(A) \subseteq X \rightsquigarrow X$ be an m-dissipative operator, $\xi \in \overline{D(A)}$, $f \in L^1(0, T; X)$ and let us consider the differential inclusion

$$u'(t) \in Au(t) + f(t). \tag{1}$$

PRELIMINARIES

Definition

A C^0 – solution to the problem (1) is a function u in $C([0,T];X)$ satisfying: for each $0 < c < T$ and $\varepsilon > 0$ there exist

(i) $0 = t_0 < t_1 < \cdots < c \leq t_n < T,\ t_k - t_{k-1} \leq \varepsilon$ for $k = 1, 2, ..., n$;

$$(ii)\quad f_1, ..., f_n \in X \text{ with } \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|f(t) - f_k\| \leq \varepsilon;$$

$$(iii)\quad v_0, ..., v_n \in X \text{ satisfying :}$$

$$\frac{v_k - v_{k-1}}{t_k - t_{k-1}} \in Av_k + f_k \text{ for } k = 1, 2, ..., n \text{ and such that}$$

$$\|u(t) - v_k\| \leq \varepsilon \text{ for } t \in [t_{k-1}, t_k),\ \ k = 1, 2, ..., n.$$

PRELIMINARIES

Theorem

[Lakshmikantham, Leela, p.116] *Let X be a Banach space and let $A : D(A) \subseteq X \rightsquigarrow X$ be m-dissipative. Then, for each $\xi \in \overline{D(A)}$ and $f \in L^1(0, T; X)$, there exists a unique C^0-solution $u : [0, T] \to \overline{D(A)}$, to (1), which satisfies $u(0) = \xi$.*

We denote by $u(\cdot, 0, \xi, f) : [0, T] \to \overline{D(A)}$ the unique C^0-solution to (1) satisfying $u(0, 0, \xi, f) = \xi$.

PRELIMINARIES

Theorem

[Carja, Necula, Vrabie, p.18] *If* $A : D(A) \subseteq X \rightsquigarrow X$ *is m-dissipative,* $\xi, \eta \in \overline{D(A)}$ *and* $f, g \in L^1(0, T; X)$, *then* $\tilde{u} = u(\cdot, 0, \xi, f)$ *and* $\tilde{v} = u(\cdot, 0, \eta, g)$ *satisfy*

$$\|\tilde{u}(t) - \tilde{v}(t)\| \leq \|\xi - \eta\| + \int_0^t \|f(s) - g(s)\| ds, \tag{2}$$

for each $t \in [0, T]$.

PRELIMINARIES

Let $A : D(A) \subseteq X \mapsto X$ be the generator of a nonlinear semigroup of contractions, $\{S(t) : \overline{D(A)} \mapsto \overline{D(A)} |\ t \geq 0\}$, K a nonempty subset in $\overline{D(A)}$ and $F : K \rightsquigarrow X$ a nonempty, closed, bounded and convex valued multi-function. We consider the Cauchy problem for the nonlinear perturbed differential inclusion

$$\begin{cases} u'(t) \in Au(t) + F(u(t)) \\ u(0) = \xi \end{cases}$$

PRELIMINARIES

Definition

The function $u : [0, T] \mapsto K$ is a C^0 – solution to the above problem if $u(0) = \xi$ and there exists $f \in L^1(0, T; X)$ with $f(t) \in F(u(t))$ *a.e.* $t \in [0, T]$, and such that u is a C^0-solution on $[0, T]$ to the equation (1) in the sense of Definition 1.

Definition

The subset $K \subset D$ is C^0 – viable with respect to $A + F$ if for each $\xi \in K$ there exist $T^0 > 0$ and a C^0-solution $u : [0, T^0] \mapsto K$ to the above problem.

PRELIMINARIES

Let $E \subseteq X$ be nonempty and bounded. We denote by

$$\mathcal{E} = \{f \in L^1_{loc}(\mathbb{R}_+; X)|\ f(s) \in E \ a.e. \ s \in \mathbb{R}_+\}.$$

Definition

Let K a subset in X and $\xi \in K$. The set $E \subseteq X$ is $A-$ quasi-tangent to the set K at the point $\xi \in K$ if we have

$$\liminf_{h \downarrow 0} \frac{1}{h} \, dist(S_{\mathcal{E}}(h)\xi; K) = 0,$$

where

$$S_{\mathcal{E}}(h)\xi = \{u(h, 0, \xi, f)|\ f \in \mathcal{E}\}.$$

PRELIMINARIES

We denote by $\mathcal{QTS}_K^A(\xi)$ the class of all A-quasi-tangent sets to K at $\xi \in K$.

Remark. Let $K \subseteq X$, $\xi \in K$ and $E \subseteq X$. Then $E \in \mathcal{QTS}_K^A(\xi)$ if and only if for each $\varepsilon > 0$ there exist $h \in (0, \varepsilon]$, $p \in X$ with $\|p\| \leq \varepsilon$ and $f \in \mathcal{E}$ such that

$$u(h, 0, \xi, f) + hp \in K.$$

Equivalently, $E \in \mathcal{QTS}_K^A(\xi)$ if and only if there exist three sequences, $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$, $(p_n)_n$ in X with $\lim_n p_n = 0$ and $(f_n)_n$ in $\mathcal{E}$, such that

$$u(h_n, 0, \xi, f_n) + h_n p_n \in K,$$

for $n = 1, 2, \ldots$.

PRELIMINARIES

Definition

An m-dissipative operator $A : D(A) \subseteq X \rightsquigarrow X$ is called of $\overline{D(A)}$ complete continuous type if for each fixed $(\tau, \xi) \in \mathbb{R} \times \overline{D(A)}$ the graph of the C^0-solution operator, $f \mapsto u(\cdot, \tau, \xi, f)$, is weakly$\times$strongly sequentially closed in $L^1(\tau, T; X) \times C([\tau, T]; X)$.

Theorem

[Carja, Necula, Vrabie, p.226] *Let X be a Banach space, $A : D(A) \subseteq X \rightsquigarrow X$ an m-dissipative operator of complete continuous type which generates a compact semigroup of contractions, let K be a nonempty, locally closed subset in $\overline{D(A)}$ and $F : K \rightsquigarrow X$ a nonempty, weakly compact and convex valued strongly-weakly u.s.c. multi-function. Then a sufficient condition in order that K be C^0-viable with respect to $A + F$ is that, for each $\xi \in K$, $F(\xi) \in \mathcal{QTS}_K^A(\xi)$.*

PRELIMINARIES

Let X be a separable Banach space.

Theorem

[Frankowska, p.105]. *Let* $U : [0, T] \rightsquigarrow X$ *be a measurable multi-function with closed nonempty values and* $g : [0, T] \mapsto X$*,* $k : [0, T] \mapsto \mathbb{R}_+$ *be measurable single-valued maps. Assume that*

$$W(t) := U(t) \cap (g(t) + k(t)B(0, 1)) \neq \emptyset \quad a.e. \ in \ [0, T].$$

Then there exists a mesurable function $u : [0, T] \mapsto X$ *such that* $u(t) \in W(t)$ *almost everywhere.*

Definition

The multi-function $F : X \rightsquigarrow X$ is $L-$ lipschitzian, if there exists $L > 0$, such that

$$F(x) \subset F(y) + L\|x - y\|B(0, 1), \quad \forall x, y \in X.$$

MAIN RESULT

Theorem

Let X be a separable Banach space, $A : D(A) \subseteq X \rightsquigarrow X$ an m-dissipative operator of complete continuous type which generates a compact semigroup of contractions, $F : \overline{D(A)} \rightsquigarrow X$ a nonempty, weakly compact and convex valued L-lipschitzian strongly-weakly u.s.c. multi-function. Let $x_0, \overline{x}_0 \in \overline{D(A)}$, $T > 0$, $y_0 : [0, T] \longrightarrow \overline{D(A)}$ a fixed C^0-solution to the problem

$$\begin{cases} y_0'(t) \in Ay_0(t) + F(y_0(t)) \\ y_0(0) = \overline{x}_0. \end{cases}$$

MAIN RESULT

Theorem

Then, there exists a C^0-solution $y : [0, T] \to \overline{D(A)}$ *to the problem*

$$\begin{cases} y'(t) \in Ay(t) + F(y(t)) \\ y(0) = x_0, \end{cases}$$

such that

$$\|y(t) - y_0(t)\| \leq \|x_0 - \overline{x}_0\| e^{Lt}, \quad \forall t \in [0, T].$$

PROOF

Let us consider the problem

$$\begin{cases} t'(s) = 1 \\ y'(s) \in Ay(s) + F(y(s)) \\ z'(s) = L\|y - y_0(t)\| \\ t(0) = t_0 \\ y(0) = y_0 \\ z(0) = z_0. \end{cases}$$

PROOF

We define the following set in the space $\mathbb{R} \times X \times \mathbb{R}$

$$\mathcal{A} = \{(t, y, z) \in [0, T) \times \overline{D(A)} \times \mathbb{R}_+ |\ \|y - y_0(t)\| \leq z\},$$

the operator $\mathbb{A} = (0, A, 0)$ and the multi-function
$\mathcal{F} : \mathcal{A} \rightsquigarrow \mathbb{R} \times X \times \mathbb{R}$,

$$\mathcal{F}(t, y, z) = (1, F(y), L\|y - y_0(t)\|).$$

1. We will prove that the set $\mathcal{A}$ is C^0-viable with respect to $\mathbb{A} + \mathcal{F}$. For this purpose, we take the point $(t_0, y_0, z_0) \in \mathcal{A}$, i. e., $\|y_0 - y_0(t_0)\| \leq z_0$.

PROOF

2. Let us prove that the tangency condition is satisfied, i.e., $(1, F(y_0), L\|y_0 - y_0(t_0)\|) \in \mathcal{QTS}_{\mathcal{A}}^{\mathbb{A}}(t_0, y_0, z_0)$. We should prove that there exist the sequences $(h_n)_n, (p_n)_n, (p_n')_n, (p_n'')_n, (f_n)_n$, such that $h_n \downarrow 0$, $p_n, p_n'' \to 0$ in $\mathbb{R}$, $p_n' \to 0$ in X, $(f_n)_n \subset F(y_0)$ and

$$(t_0 + h_n(1 + p_n), y(h_n(1 + p_n), 0, y_0, f_n) + h_n p_n',$$

$$z_0 + h_n L\|y_0 - y_0(t_0)\| + h_n p_n'') \in \mathcal{A}.$$

We can choose $(p_n)_n = (0)_n$.

PROOF

We will prove that there exist the sequences $(h_n)_n, (p_n)_n, (p'_n)_n, (p''_n)_n, (f_n)_n$, like above, such that

$$\|y(h_n, 0, y_0, f_n) + h_n p'_n - y_0(t_0 + h_n, 0, \overline{x}_0, g)\| \leq z_0 + h_n L \|y_0 - y_0(t_0)\| + h_n p''_n,$$

which is equivalent with

$$\|y(h_n, 0, y_0, f_n) + h_n p'_n - y_0(h_n, 0, y_0(t_0), \tilde{g})\| \leq z_0 + h_n L \|y_0 - y_0(t_0)\| + h_n p''_n,$$

where $\tilde{g}(s) = g(s + t_0) \in F(y_0(s + t_0))$ a.e. $s \in [0, T - t_0]$.

PROOF

3. In accordance with definition of a C^0-solution, $\exists g \in L^1(0, T; X)$, with $g(t) \in F(y_0(t))$ a.e. $t \in [0, T]$, such that y_0 is a C^0-solution to the problem

$$\begin{cases} y_0'(t) \in Ay_0(t) + g(t) \\ y_0(0) = \overline{x}_0. \end{cases}$$

Taking into account that the multi-function F is L-lipschitzian, we have

$$F(y_0(t+t_0)) \subset F(y_0) + L\|y_0(t+t_0) - y_0\| B(0,1), \quad \forall t \geq 0.$$

PROOF

Then, applying Theorem [Frankowska] for g, we obtain that there exist two measurable functions f, $w \in L^1(0, T; X)$, such that $f(t) \in F(y_0)$, $w(t) \in L\|y_0(t + t_0) - y_0\| B(0, 1)$ and $\tilde{g}(t) = f(t) + w(t)$ a.e. $t \in [0, T - t_0]$. We put $(f_n)_n = (f)_n$.

PROOF

4. Next, we use the inequality (2) to obtain

$$\|y_0 - y_0(t_0)\| + \int_0^{h_n} \|f(s) - \tilde{g}(s)\| ds + h_n \|p_n'\| \leq \|y_0 - y_0(t_0)\|$$

$$+L \int_0^{h_n} \|y_0(s+t_0) - y_0\| ds + h_n \|p_n'\| = \|y_0 - y_0(t_0)\| + L h_n \|y_0(t_0) - y_0\|$$

$$+h_n \left(L \left[\frac{1}{h_n} \int_0^{h_n} \|y_0(s+t_0) - y_0\| ds - \|y_0(t_0) - y_0\| \right] \right) + h_n \|p_n'\|$$

$$\leq h_n \left(L \left[\frac{1}{h_n} \int_0^{h_n} \|y_0(s+t_0) - y_0\| ds - \|y_0(t_0) - y_0\| \right] \right)$$

$$z_0 + L h_n \|y_0(t_0) - y_0\| + h_n \|p_n'\|.$$

PROOF

We put $(p'_n)_n = (0)_n$,
$(p''_n)_n = \left(\frac{1}{h_n} \int_0^{h_n} \|y_0(s+t_0) - y_0\| ds - \|y_0(t_0) - y_0\| \right)_n$.
Taking into account that the solution $y_0(t)$, defined on $[0, T]$, is from the space $C([0, T], X)$, choosing $(h_n)_n$ to be sufficiently small, we have

$$\lim_{h_n \downarrow 0} \left(\frac{1}{h_n} \int_0^{h_n} \|y_0(s+t_0) - y_0\| ds \right) = \|y_0(t_0) - y_0\|.$$

Hence, $p''_n \to 0$ in $\mathbb{R}$.

PROOF

5. We mention that under the hypotheses on F and A, the multi-function $\mathcal{F}$ is a nonempty, weakly compact and convex valued strongly-weakly u.s.c. multi-function and the operator $\mathbb{A}$ is a m-dissipative operator of complete continuous type which generates a compact semigroup of contractions.
It is obvious that the subset $\mathcal{A}$ of $\mathbb{R} \times \overline{D(A)} \times \mathbb{R}$ is nonempty $((0, x_0, \|x_0 - \overline{x}_0\|) \in \mathcal{A})$ and locally closed.

PROOF

Therefore, taking into account that the tangency condition is satisfied, we can apply Theorem 3 (about the sufficient conditions for C^0-viability of the set $\mathcal{A}$ with respect to $\mathbb{A} + \mathcal{F}$) and conclude that there exist $0 < \tilde{T} < T$ and a C^0-solution $(\hat{t}, \hat{y}, \hat{z})$ to the above problem, such that

$$(\hat{t}, \hat{y}(\hat{t}), \hat{z}(\hat{t})) \in \mathcal{A}, \quad \forall \hat{t} \in [0, \tilde{T}].$$

Equivalently,

$$\|\hat{y}(t) - y_0(t)\| \leq \hat{z}(t), \quad \forall t \in [0, \tilde{T}].$$

PROOF

That is, taking into account the form of the solution $\hat{z}(t)$,

$$\|\hat{y}(t) - y_0(t)\| \leq \|x_0 - \overline{x}_0\| + L \int_0^t \|\hat{y}(s) - y_0(s)\| ds, \quad \forall t \in [0, \tilde{T}].$$

Which implies, if we use Gronwall's Lemma,

$$\|\hat{y}(t) - y_0(t)\| \leq \|x_0 - \overline{x}_0\| e^{Lt}, \quad \forall t \in [0, \tilde{T}].$$

PROOF

The fact that the set $\mathcal{A}$ is C^0-viable with respect to $\mathbb{A}+\mathcal{F}$ assures the existence of a noncontinuable C^0-solution $(t, y, z) : [0, a) \to \mathcal{A}$, where $0 < a \leq T$ (see Theorem 11.7.1 [Carja, Necula, Vrabie]) and

$$\|y(t) - y_0(t)\| \leq \|x_0 - \overline{x}_0\| e^{Lt}, \quad \forall t \in [0, a). \tag{3}$$

In the following we prove that $a = T$.
Let us assume, by contradiction, that $a < T$.

PROOF

We define the function $y_1 : [0, T-a] \to \overline{D(A)}$

$$y_1(t) \equiv y_0(t+a), \quad \forall t \in [0, T-a],$$

which, obviously, is a C^0-solution to the problem

$$\begin{cases} y_1'(t) \in Ay_1(t) + F(y_1(t)) \\ y_1(0) = y_0(a). \end{cases}$$

Let us prove that $\exists \lim_{t\uparrow a} y(t) \in \overline{D(A)}$.

PROOF

From definition of a C^0-solution, $\exists \varphi \in L^1([0,a), X)$, with $\varphi(t) \in F(y(t))$ a.e. $t \in [0,a)$, such that $y : [0,a) \to \overline{D(A)}$ is a C^0-solution to the problem

$$\begin{cases} y'(t) \in Ay(t) + \varphi(t) \\ y(0) = x_0. \end{cases}$$

The estimation (3) implies that y is bounded on $[0,a)$.

PROOF

F, being L-lipschitzian, maps bounded subsets from $\overline{D(A)}$ to bounded subsets from X. Therefore, φ is bounded on $[0, a)$, which implies the existence of $\lim_{t \uparrow a} y(t) \in \overline{D(A)}$. Let us put

$$y(a) \equiv \lim_{t \uparrow a} y(t).$$

Using the fact that $y \in C([0, a), \overline{D(A)})$ and $y_0 \in C([0, T], \overline{D(A)})$, from (3) we have

$$\|y(a) - y_0(a)\| \leq \|x_0 - \overline{x}_0\| e^{La}.$$

PROOF

Applying the similar procedure like above for y_1, we state that there exist $0 < b < T - a$ and a C^0-solution $y_2 : [0, b] \to \overline{D(A)}$ to the problem

$$\begin{cases} y_2'(t) \in Ay_2(t) + F(y_2(t)) \\ y_2(0) = y(a), \end{cases}$$

such that

$$\|y_2(t) - y_1(t)\| \le \|y_0(a) - y(a)\| e^{Lt}, \quad \forall t \in [0, b].$$

PROOF

Let us define the function $y_3 : [0, a+b] \to \overline{D(A)}$

$$y_3(t) \equiv \begin{cases} y(t), & \forall t \in [0, a], \\ y_2(t-a), & \forall t \in [a, a+b], \end{cases}$$

which, obviously, is a C^0-solution to the problem

$$\begin{cases} y_3'(t) \in Ay_3(t) + F(y_3(t)) \\ y_3(0) = x_0. \end{cases}$$

PROOF

Let us mention that

$$\|y_3(t) - y_0(t)\| = \|y(t) - y_0(t)\| \le \|x_0 - \overline{x}_0\|e^{Lt}, \quad \forall t \in [0, a].$$

Moreover, for all $t \in [a, a+b]$ we have

$$\|y_3(t) - y_0(t)\| = \|y_2(t-a) - y_1(t-a)\| \le \|y_0(a) - y(a)\|e^{L(t-a)}$$

$$\le \|x_0 - \overline{x}_0\|e^{La}e^{L(t-a)} = \|x_0 - \overline{x}_0\|e^{Lt}.$$

PROOF

Hence,

$$\|y_3(t) - y_0(t)\| \leq \|x_0 - \overline{x}_0\| e^{Lt}, \quad \forall t \in [0, a+b],$$

which contradicts the statement that $(t, y, z) : [0, a) \to \mathcal{A}$ is a noncontinuable C^0-solution. The contradiction is eliminated, if $a = T$.
Similarly, we prove that $\exists \lim_{t \uparrow T} y(t) \in \overline{D(A)}$ and, if we put $y(T) \equiv \lim_{t \uparrow T} y(t)$, (3) implies

$$\|y(T) - y_0(T)\| \leq \|x_0 - \overline{x}_0\| e^{LT}.$$

BIBLIOGRAPHY

1. Carja, O., Necula, M., Vrabie, I. I., Viability, invariance and applications, North-Holland Mathematics Studies, 207, *Elsevier Science B.V., Amsterdam*, 2007, xii+344 pp.

2. Frankowska, H., A priori estimates for operational differential inclusions, *J. Differential Equations* 84 (1990), no. 1, 100–128.

3. Lakshmikantham, V., Leela, S., Nonlinear differential equations in abstract spaces, International Series in Nonlinear Mathematics: Theory, Methods and Applications, 2, *Pergamon Press, Oxford-New York*, 1981, x+258 pp.