A Canonical Setting and Stochastic Exponentials for Continuous Local Martingales

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- Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probabiltiy space equipped with a filtration $\mathbb{F}$ satisfying the usual conditions.
- Consider a continuous local martingale (in short, CLM) $(X, \mathbb{F})$ such that $X_0 = 0$ and its associated increasing process $\langle X \rangle$.
- The *stochastic exponential* or *Doléans exponential* of X is defined by

$$\mathcal{E}(X) := \exp\left(X - \frac{1}{2}\langle X \rangle\right) . \tag{1}$$

- Applying Itô's formula it can easily be seen that $(\mathcal{E}(X), \mathbb{F})$ is again a CLM, with $\mathcal{E}(X)_0 = 1$.

- Using Fatou's lemma, we see that the nonnegative local martingale $(\mathcal{E}(X), \mathbb{F})$ is a supermartingale. It follows that:

$$(2) \quad (\mathcal{E}(X), \mathbb{F}) \text{ is a martingale } \iff \mathbf{E}[\mathcal{E}(X)_t] = 1,\ t \geq 0\,.$$

- A.A. Novikov (1972) proved that the condition

$$(3) \qquad \mathbf{E}\left[\exp\left(\frac{1}{2}\langle X\rangle_t\right)\right] < +\infty, \qquad t \geq 0\,,$$

is sufficient for $(\mathcal{E}(X), \mathbb{F})$ to be a martingale.

- N. Kazamaki (1977) showed that the condition

$$(4) \qquad \mathbf{E}\left[\exp\left(\frac{1}{2}X_t\right)\right] < +\infty, \qquad t \geq 0\,,$$

also implies that $(\mathcal{E}(X), \mathbb{F})$ is a martingale if $(X, \mathbb{F})$ is a martingale, otherwise it should be required that $(\exp(\frac{1}{2}X), \mathbb{F})$ is a submartingale.

Introduction

- The KAZAMAKI condition (4) follows from the NOVIKOV condition (3) by the Schwarz inequality, but the converse does not hold.
- N. KAZAMAKI and T. SEKIGUCHI (1979) gave another sufficient condition for $(\mathcal{E}(X), \mathbb{F})$ to be a martingale: $(X, \mathbb{F})$ belongs locally to BMO, i.e., if

$$(5) \qquad \mathbf{E}(\langle X\rangle_t - \langle X\rangle_s \mid \mathcal{F}_s) \leq c(t), \qquad s \leq t, \ \mathbf{P}\text{-f.s.}$$

  for all $t \geq 0$, where $c(t)$ is a constant.
- Note that, in general, the KAZAMAKI condition (4) and the BMO-condition (5) are not comparable each other.
- In the case that $\langle X\rangle$ is bounded it is clear that all conditions (3), (4) and (5) are satisfied.
- For most applications, however, $\langle X\rangle$ is not bounded and in concrete cases it is more difficult and, as a rule, hardly possible to verify that one of the sufficient conditions (3), (4) or (5) is fulfilled.

Introduction

The main goal of this talk is to derive conditions on a CLM $(X, \mathbb{F})$ such that:

- $(\mathcal{E}(X), \mathbb{F})$ is a martingale
- The conditions are necessary and sufficient for this property to be true.
- The conditions can effectively be verified in concrete situations.

The Basic Observation

The following useful result is taken from:

Engelbert, H.-J.; Senf, T.:
On Functionals of a Wiener Process with Drift and Exponential Local Martingales.
Proceedings of the 8th Winter School on Stochastic Processes and Optimal Control, Georgenthal, January 22–26 (1990), pp. 45–58, Akademie-Verlag, Berlin 1991

- Let $(X, \mathbb{F})$ to be a CLM and $A = \langle X \rangle$ the associated increasing process.
- It is well-known that there exists a Brownian motion $(W, \mathbb{G})$ (on a, possibly, enlarged probability space) such that $A = (A_t)_{t \geq 0}$ is a $\mathbb{G}$-time change and

$$X_t = W_{A_t}, \quad t \geq 0 .$$

- Of course, $(\mathcal{E}(W), \mathbb{G})$ is a nonnegative martingale with expectation $\mathbf{E}[\mathcal{E}(W)_t] = 1, \; t \geq 0$.

The Basic Observation

- Hence we can define probability measures $\mathbf{Q}_t$ on $\mathcal{G}_t$ by

$$d\mathbf{Q}_t = \mathcal{E}(W)_t \, d\mathbf{P}, \quad t \geq 0 .$$

- The consistent family $(\mathbf{Q}_t)_{t\geq 0}$ can be extended to an additive set function $\mathbf{Q}$ on the algebra $\bigcup_{t\geq 0} \mathcal{G}_t$.

Theorem

The process $(\mathcal{E}(X), \mathbb{F})$ is a martingale if and only if

$$\lim_{n\to+\infty} \mathbf{Q}(\{A_t < n\}) = 1 \qquad \text{for all } t \geq 0 . \tag{6}$$

Proof.

$$\begin{aligned}
\lim_{n\to+\infty} \mathbf{Q}(\{A_t < n\}) &= \sum_{k=0}^{\infty} \mathbf{Q}(\{k \le A_t < k+1\}) \\
&= \sum_{k=0}^{\infty} \int_{\{k\le A_t<k+1\}} \mathcal{E}(W)_{k+1}\, d\mathbf{P} \\
&= \sum_{k=0}^{\infty} \int_{\{k\le A_t<k+1\}} \mathcal{E}(W)_{A_t}\, d\mathbf{P} \\
&= \int_{\{A_t<\infty\}} \mathcal{E}(W)_{A_t}\, d\mathbf{P} \\
&= \int_{\Omega} \mathcal{E}(W)_{A_t}\, d\mathbf{P} \\
&= \mathbf{E}\mathcal{E}(W)_{A_t} = \mathbf{E}\mathcal{E}(X)_t
\end{aligned}$$

which proves the equivalence. □

The Basic Observation: An Equivalent Formulation

Now we consider the right inverse $T = (T_t)_{t \geq 0}$ of A, i.e.,

$$T_t := \inf\{s \geq 0 : A_s > t\}, \qquad t \geq 0\,,$$

We also put $T_\infty := \sup_{t \geq 0} T_t$.

Theorem

Suppose that $\mathbf{Q}$ *is* σ*-additive on the algebra* $\bigcup_{t \geq 0} \mathcal{G}_t$ *and, hence, can be extended to a probability measure on* $\mathcal{G} = \sigma(\bigcup_{t \geq 0} \mathcal{G}_t)$. *Then the following statements are equivalent:*

(i) $(\mathcal{E}(X), \mathbb{F})$ *is a martingale.*

(ii) $\mathbf{Q}(\{A_t < \infty\}) = 1, \forall t \geq 0\,.$

(iii) $\mathbf{Q}(\{T_\infty = +\infty\}) = 1\,.$

Generalization:

The theorem remains true if $(X, \mathbb{F})$ is only a CLM up to T_∞. This means that, possibly, $T_\infty < \infty$ with strictly positive probability and $(X, \mathbb{F})$ is exploding at explosion time T_∞.

The stochastic exponential $(\mathcal{E}(X), \mathbb{F})$ of $(X, \mathbb{F})$ is then defined as

$$\mathcal{E}(X)_t = \begin{cases} \exp(X_t - \frac{1}{2}\langle X\rangle_t)\,, & \text{if} \quad t < T_\infty\,, \\ \lim_{t\uparrow T_\infty} \exp(X_t - \frac{1}{2}\langle X\rangle_t) = 0\,, & \text{if} \quad T_\infty \leq t\,. \end{cases}$$

Then $(\mathcal{E}(X), \mathbb{F})$ is again a proper CLM (without explosion).

The Basic Observation: The Crucial Problem

Problem

Can we always extend $\mathbf{Q}$ *to a probability measure on* $\mathcal{G}$*?*

- This mainly depends on the choice of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the Brownian $(W, \mathbb{G})$ on it. But the martingale property of $(\mathcal{E}(X), \mathbb{F})$ is a distributional property and does not depend on this choice.
- Before solving the problem in a general way, we will discuss several examples.

The First Example: Solutions of an SDE

This example was given in:

Engelbert, H.-J.; Senf, T.:
On Functionals of a Wiener Process with Drift and Exponential Local Martingales.
Proceedings of the 8th Winter School on Stochastic Processes and Optimal Control, Georgenthal, January 22–26 (1990), pp. 45–58, Akademie-Verlag, Berlin 1991

- We consider the one-dimensional SDE

$$(7) \qquad dX_t = b(X_t)dB_t \,, \quad t \geq 0 \,, \quad X_0 = 0 \,,$$

where $(B, \mathbb{F})$ is a Brownian motion and b some real Borel function.

- Note that every solution $(X, \mathbb{F})$ is a CLM.
- For the sake of simplicity, let us assume that $b^{-2} := \frac{1}{b^2}$ is locally integrable.

The First Example: Solutions of an SDE

- Then there exists a solution $(X, \mathbb{F})$ which has no sojourn time in the set $\{b = 0\}$:

$$\int_0^\infty \mathbf{1}_{\{b=0\}}(X_u)\, du = 0 \quad \mathbf{P}\text{-a.s.}$$

- Such a solution is called fundamental solution, and the fundamental solution is unique in law.

Construction of the Solution: Let $(C, \mathcal{C})$ be the space of continuous real functions on $[0, \infty)$ equipped with the Wiener measure **P**. Let $W = (W_t)_{t \geq 0}$ be the coordinate mapping on C. Then $(W, \mathbb{G})$ is a Brownian motion where $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the smallest right-continuous filtration (not completed!) with respect to which W is adapted. We define

$$T_t = \int_0^{t+} b^{-2}(W_u)\,du, \quad t \in [0, \infty],$$

and let $A = (A_t)_{t \geq 0}$ be the right inverse of T:

$$A_t = \inf\{s \geq 0 : T_s > t\}, \quad t \geq 0.$$

Because T is strictly increasing and $T_\infty = \infty$ **P**-a.s., A is a continuous and finite $\mathbb{G}$-time change. It can be shown that the process $(X, \mathbb{F})$ defined by $X_t = W_{A_t}$, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0} := (\mathcal{G}_{A_t})_{t \geq 0}$ is a fundamental solution of Eq. (7).

The First Example: The Result

Now we can apply our second theorem from above:

- Defining $\mathbf{Q}_t = \mathcal{E}(W)_t \, d\mathbf{P}$ on $\mathcal{G}_t$, we observe that $(\mathbf{Q}_t)_{t\geq 0}$ can be extended to a probability measure $\mathbf{Q}$ on $\mathcal{C} = \sigma(\bigcup_{t\geq 0} \mathcal{G}_t)$.
- The probability measure $\mathbf{Q}$ on $(C, \mathcal{C})$ is just the $\mathbf{P}$-distribution of $(W_t + t)_{t\geq 0}$.
- This yields

$$T_\infty = \int_0^\infty b^{-2}(W_u)\, du = \int_0^\infty b^{-2}(\widetilde{W}_u + u)\, du \quad \mathbf{Q}\text{-a.s.}$$

  where $\widetilde{W} = (\widetilde{W}_t)_{t\geq 0})$ is a $\mathbf{Q}$-Brownian motion. Hence W is a $\mathbf{Q}$-Brownian motion with drift.
- Integral functionals of a Brownian motion with drift have been studied in the paper cited above. The result is

$$\mathbf{Q}(\{T_\infty = \infty\}) = 1 \iff \int_{-\varepsilon}^\infty b^{-2}(x)\, dx = \infty \quad \forall \varepsilon > 0 .$$

The First Example: The Result

Summarizing we obtain the following purely analytical criterion.

Theorem

The stochastic exponential $(\mathcal{E}(X), \mathbb{F})$ *associated with a fundamental solution of SDE (7) is a martingale if and only if*

$$\int_{-\varepsilon}^{\infty} b^{-2}(x)\, dx = \infty \quad \forall \varepsilon > 0\,.$$

The Second Example: Strong Markov CLM

The following result is taken from:

Blei, S.; Engelbert, H.-J.:
On Exponential Local Martingales Associated with Strong Markov Continuous Local Martingales.
Stoch. Proc. Appl. **119** (2009), 2859–2880

- A strong Markov CLM $(X, \mathbb{F})$ is in law uniquely determined by its speed measure m.
- m can be an arbitrary measure on the real line which assigns strictly positive measure to every non-empty open set.

The Second Example: The Construction

The **construction** of a strong Markov CLM $(X, \mathbb{F})$ with given speed measure m is similar as above: Let $(C, \mathcal{C}, \mathbf{P})$ be the Wiener space, $W = (W_t)_{t \geq 0}$ the coordinate mappings and $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ the smallest right-continuous filtration (not completed!) with respect to which W is adapted. We define

$$T_t = \int_{\mathbb{R}} L^W(t, a)\, m(da), \quad t \in [0, \infty],$$

where $L^W(t, a)$ is the local time of W, and let $A = (A_t)_{t \geq 0}$ be the right inverse of T:

$$A_t = \inf\{s \geq 0 : T_s > t\}, \quad t \geq 0.$$

The Second Example: Strong Markov CLM

Because T is strictly increasing and $T_\infty = \infty$ **P**-a.s., A is a continuous and finite $\mathbb{G}$-time change. It can be shown that the process $(X, \mathbb{F})$ defined by

$$X_t = W_{A_t}, \quad \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} := (\mathcal{G}_{A_t})_{t \geq 0}$$

is a strong Markov CLM with speed measure m. As above, we define **Q** on $(C, \mathcal{C})$ as the **P**-distribution of $(W_t + t)_{t \geq 0}$.

Integral functionals of type T_∞ with

$$T_t = \int_{\mathbb{R}} L^W(t, a)\, m(da)\,,$$

where W is a **Q**-Brownian motion with drift have been studied in the paper cited above. The result is

$$\mathbf{Q}(\{T_\infty = \infty\}) = 1 \iff m((-\varepsilon, \infty)) = \infty \quad \forall \varepsilon > 0\,.$$

Summarizing we obtain the following purely analytical criterion.

Theorem

The stochastic exponential $(\mathcal{E}(X), \mathbb{F})$ *associated with a strong Markov CLM* $(X, \mathbb{F})$, $X_0 = 0$, *having speed measure* m *is a martingale if and only if*

$$m((-\varepsilon, \infty)) = \infty \quad \forall \varepsilon > 0 .$$

Problem

What happens for general CLMs $(X, \mathbb{F})$?

A Canonical Setting: Definitions

We now prepare the general case and introduce a canonical representation for CLMs.

- As before, by $(C, \mathcal{C})$ we denote the space of continuous functions on $[0, \infty)$ and by $W = (W_t)_{t \geq 0}$ the coordinate mappings. By $\mathbb{C} = (\mathcal{C}_t)_{t \geq 0}$ we denote the filtration generated by W. Let μ be the Wiener measure on $(C, \mathcal{C})$.
- $(C, \mathcal{C}, \mu)$ will serve as canonical space for Brownian trajectories.
- Let V denote the space of nondecreasing continuous functions $[0, \infty) \to [0, \infty]$ starting from 0. By $A = (A_t)_{t \geq 0}$ we denote the canonical process on V.
- We introduce the σ-fields

$$\mathcal{V}_t = \sigma(\{\{A_s \leq u\} : s \in [0, \infty), u \in [0, t]\}), \quad \mathcal{V} = \bigvee_{t \geq 0} \mathcal{V}_t .$$

- Note that $\mathbb{V} = (\mathcal{V}_t)_{t \geq 0}$ is the smallest filtration with respect to which $A = (A_t)_{t \geq 0}$ is a time-change.

A Canonical Setting: Definitions

- The filtered space $(V, \mathcal{V}; \mathbb{V})$ will serve as canonical space for trajectories of a continuous time-change.
- We set $\Omega^* = C \times V$, $\mathcal{G}^* = \mathcal{C} \otimes \mathcal{V}$ and denote by $(\mathcal{G}_t^*)_{t\geq 0}$ the smallest right-continuous filtration containing $(\mathcal{C}_t \otimes \mathcal{V}_t)_{t\geq 0}$.
- W and A will be considered as defined on $(\Omega^*, \mathcal{G}^*)$.
- We introduce the process X and the filtration $\mathbb{F}$ by

$$X_t = W_{A_t}, \quad \mathcal{F}_t = \mathcal{G}^*_{A_t}, \quad t \geq 0\,.$$

  As always, $T = (T_t)_{t\geq 0}$ denotes the right inverse of $A = (A_t)_{t\geq 0}$: $T_t := \inf\{s \geq 0 : A_s > t\}$, $t \in [0, \infty]$. Note that the filtration $\mathbb{V} = (\mathcal{V}_t)_{t\geq 0}$ is just generated by the process T.
- The process X on $(\Omega^*, \mathcal{G}^*)$ will serve as a canonical representation for CLMs up to T_∞.

A Canonical Setting: Characterization of CLMs

Let us consider a probability kernel K from $(C, \mathcal{C})$ into $(V, \mathcal{V})$ satisfying the condition

$$(8) \qquad K(\cdot, E) \text{ is } \mathcal{C}_t\text{-measurable } \forall E \in \mathcal{V}_t, \quad t \geq 0 .$$

We say that K is nonanticipative.

Given a nonanticipative probability kernel K from $(C, \mathcal{C})$ into $(V, \mathcal{V})$, by $\mathbf{P}$ we denote the unique probability measure on $(\Omega^*, \mathcal{G}^*)$ which satisfies

$$(9) \qquad \mathbf{P}(D \times E) = \int_D K(w, E)\, d\mu(w), \quad D \times E \in \mathcal{C} \otimes \mathcal{V} .$$

A Canonical Setting: Characterization of CLMs

Theorem

(i) X *is a* CLM *on* $(\Omega^*, \mathcal{G}^*, \mathbf{P})$ *up to time* T_∞*, and* $\langle X \rangle = A$.

(ii) X *is a* CLM *if and only if* $\mathbf{P}(\{T_\infty = \infty\}) = 1$.

Theorem

For any CLM $\widetilde{\mathbf{X}}$ *up to time* $\widetilde{T}_\infty$ *starting from* 0 *defined on an arbitrary* $(\Omega, \mathcal{F}, \widetilde{\mathbf{P}})$*, there exists a nonanticipative probability kernel* K *such that*

$$\mathrm{Law}_{\mathbf{P}}(X) = \mathrm{Law}_{\widetilde{\mathbf{P}}}(\widetilde{X}),$$

where $\mathbf{P}$ *is defined through* K *as in* (9)*. If* K *and* K' *are two such kernels, then they are* μ*-indistinguishable.*

A Canonical Setting: Characterization of CLMs

Remark: The kernel K from the above theorem can be constructed as a regular conditional distribution

$$K(b,E) = \widetilde{\mathbf{P}}(\{\langle \widetilde{\mathbf{X}} \rangle \in E\} | B = b), \quad b \in C,\ E \in \mathcal{V},$$

where

$$\widetilde{X} = B_{\langle \widetilde{\mathbf{X}} \rangle} := (B_{\langle \widetilde{\mathbf{X}} \rangle_t})_{t < \widetilde{T}_\infty}$$

is any DAMBIS–DUBINS–SCHWARZ representation of the CLM $\widetilde{\mathbf{X}}$ up to $\widetilde{T}_\infty$ as a time-changed Brownian motion B, possibly, on an enlargement of $(\Omega, \mathcal{F}, \widetilde{\mathbf{P}})$.

- The two theorems state that the correspondence between nonanticipative kernels K and distributions of CLMs is one-to-one.
- The described canonical setting can be viewed as a converse to the DAMBIS–DUBINS–SCHWARZ theorem, which states that each CLM is a time-changed Brownian motion.

A Canonical Setting: References

The above theorems are slight generalizations of results from:

Walther, Mario:
Eindimensionale stochastische Differentialgleichungen mit verallgemeinerter Drift bezüglich stetiger lokaler Martingale.
PhD-thesis, Friedrich-Schiller-University of Jena (2007)

Engelbert,H.-J.; Urusov, M.A.; Walther, M.:
A Canonical Setting and Separating Times for Continuous Local Martingales.
Stoch. Proc. Appl. **119** (2009), 1039–1054

A Canonical Setting: Stochastic Exponentials

We now apply the canonical setting to stochastic exponentials of CLMs.

Similar as above, given the canonical CLM X on $(\Omega^*, \mathcal{G}^*, \mathbf{P})$, we define

$$d\mathbf{Q}_t = \mathcal{E}(W)_t \, d\mathbf{P} = \exp(W_t - \frac{1}{2}t) \, d\mathbf{P} \quad \text{on} \quad \mathcal{G}_t^* .$$

Theorem

The family $(\mathbf{Q}_t)_{t \geq 0}$ can uniquely be extended to a probability measure Q on $(\Omega^*, \mathcal{G}^*)$.

Proof. 1) $(\Omega^*, \mathcal{G}_t^*)$ are standard Borel spaces.

2) For every decreasing family G_n of atoms of $\mathcal{G}_n^*$ its intersection is non-empty.

3) It remains to apply Thm. V.4.1 of Parthasarathy (1967). □

A Canonical Setting: Stochastic Exponentials

Using this probability measure $\mathbf{Q}$, we now arrive at the following result:

Theorem

The following conditions are equivalent:

(i) $(\mathcal{E}(X), \mathbb{F})$ *is a martingale.*

(ii) $\mathbf{Q}(\{A_t < \infty\}) = 1, \quad \forall t \geq 0.$

(iii) $\mathbf{Q}(\{T_\infty = \infty\}) = 1$ *where* $T_\infty = \inf\{t \geq 0 : A_t = \infty\}$.

(iv) $(X - A, \mathbb{F})$ *is a proper* CLM *such that* $\langle X - A \rangle = A$ *on* $(\Omega^*, \mathcal{G}^*, \mathbf{Q})$.

A Canonical Setting: Stochastic Exponentials

Remark: After having the existence of **Q** on $(\Omega^*, \mathcal{G}^*)$, it can easily be verified that:

$$\mathbf{Q}(D \times E) = \int_D K(w, E)\, d\nu(w), \quad D \times E \in \mathcal{C} \otimes \mathcal{V},$$

where ν is the distribution of a Brownian motion with drift on $(C, \mathcal{C})$. Hence the kernel K disintegrates **P** w.r.t. μ as well as **Q** w.r.t. ν.

From this we obtain another characterization:

Corollary

Conditions (i)–(iv) are also equivalent to each of the following conditions:

(v) $K(w, \{A_t < \infty\}) = 1 \quad \nu$-a.s., $\forall t \geq 0$.

(vi) $K(w, \{T_\infty = \infty\}) = 1 \quad \nu$-a.s.

Changing the roles of **P** and **Q**, we obtain the following result where T_∞ again appears as exploding time.

Corollary

Suppose that $(X, \mathbb{F})$ is a CLM on $(\Omega^*, \mathcal{G}^*, \mathbf{P})$.

1) *Then, on $(\Omega^*, \mathcal{G}^*, \mathbf{Q})$, the process $(-X + A, \mathbb{F})$ is a CLM up to the stopping time T_∞ such that $\langle X - A \rangle = A$.*

2) *The stochastic exponential $(\mathcal{E}(-X + A), \mathbb{F})$ is always a* **Q***-martingale.*

Application: Some More Examples

Consider the SDE

$$Y_t = y_0 + \int_0^t a(Y_u)\,du + \int_0^t \sigma(Y_u)\,dB_u, \quad t \geq 0\,, \tag{10}$$

and the (possibly, exploding) CLM

$$X_t = \int_0^t b(Y_s)\,dB_u\,, \quad t \geq 0\,.$$

with Brownian motion B.

Problem: We ask for conditions on the coefficients b, a and σ for $\mathcal{E}(X)$ being a martingale.

Solution of the Problem: Investigate integral functionals of type

$$\int_0^t b^2(Y_u)\,du$$

of a solution Y of the SDE (10) and find analytical criteria for

$$\int_0^t b^2(Y_u)\,du < \infty \quad \forall t \geq 0 \quad \mathbf{Q}\text{-a.s.}$$

Note that w.r.t. $\mathbf{Q}$ the drift for the SDE of Y changes from a to $a + \sigma b$.

Under reasonable conditions on the coefficients b, a and σ, this plan can always be successfully realized.

In their paper

Mijatović, A.; Urusov, M.A.:
On the Martingale Property of Certain Local Martingales: Criteria and Applications.
Preprint 2009

MIJATOVIĆ and URUSOV followed another approach, using the concept of separating times of the second author and non-explosive criteria for solutions of SDEs given in:

Cherny, A.; Engelbert, H.-J.:
Singular Stochastic Differential Equations.
Lecture Notes in Mathematics **1858**, Springer, 2005

THANK YOU!