A NOTE ON HOMEOMORPHISM FOR BACKWARD DOUBLY SDES AND APPLICATIONS

A. AMAN

joint work with **M. N'zi and J.M. Owo**

Équipe de Probabilités et Statistiques

Université de Cocody, Abidjan

ITN Marie Curie Workshop: Stochastic control and Finance, Roscoff 18-23 March

Outline

INTRODUCTION

Presentation

- $$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s, Z_s) dB_s \\ &\quad - \int_t^T Z_s dW_s, \quad t \in [0, T], \end{aligned} \tag{1}$$
- dW is a forward Itô integral
- dB is a backward Itô integral
- ξ, a $\mathcal{F}_T^W$-measurable variable $\equiv$terminal condition
- The functions f and g $\equiv$ coefficients (also generators)
- (Y, Z) $\equiv$ unknowns

INTRODUCTION

Présentation

- BDSDE (1) $\Rightarrow$ by Pardoux and Peng (1994): Existence et uniqueness result under Lipschitz condition on the coefficients
- The Lipschitz constant α of g with respect z satisfies $0 < \alpha < 1$, which is a very natural condition for many particular situations, e.g., g linear on z and does not depend on y.
- The solution (Y, Z) is adapted to $\mathcal{F}_t^W \otimes \mathcal{F}_{t,T}^B$ which is not a filtration.

INTRODUCTION

Motivation

- As application BDSDE (1) give in the Markovian framework the representation to SPDE

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T [\mathcal{L}u(s,x) + f(s,x,u(s,x),(\partial_x u.\sigma)(s,x))]ds \\ &+ \int_t^T g(s,x,u(s,x))dB_s,\ 0 \leq t \leq T, \end{aligned} \tag{2}$$

- $$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}.$$

INTRODUCTION

Other results

- Bally and Matoussi (2001): Weak solutions of parabolic semilinear SPDEs in Sobolev spaces.
- Matoussi and Scheutzow (2002): SPDEs with nonlinear noise term give by Itô-Kunita stochastic integral.
- Buckdahn and Ma (2001,2002): Notion of stochastic viscosity solution for SPDE
- Boufoussi and Mrhardy (2007, 2008): Stochastic viscosity solution for SPDE with Neuman-Dirichlet boundary condition and multivalued SPDE.
- Aman and Mrhardy (2008) : Obstacle problem for SPDE with Neuman-Dirichlet boundary condition

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Other works about BDSDE

- M. N'zi and J. M. Owo (2008, 2009): BDSDE with non-Lipschitz and discontinuous condition
- A. Aman (2009): L^p-solution of BDSDE with non Lipschitz condition
- L. Hu and Y. Ren (2009): GBDSDE with Lévy process and stochastic PDIE Neuman-Dirichlet boundary condition
- A. Aman and Y. Ren (2010): Viscosity solution for SPDIEs with nonlinear Neuman-Dirichlet boundary condition

BDSDE with parameter

- $$Y_t = \xi(x) + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s) dB_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{3}$$

- Solution depends on x via $\xi(x) \implies (Y^x, Z^x)$.
- In the sequel g do not depend on z due to the use of strict comparison theorem for BDSDE, which works only in this case.

FORMULATION OF PROBLEM

Aim of topic

$x \mapsto \xi(x,\omega)$ homeomorphism on $\mathbb{R}$ for almost all ω;

$\Longrightarrow$

$x \mapsto Y^x(\omega)$ homeomorphism on $\mathbb{R}$ for almost all ω?

FORMULATION OF PROBLEM

Additional assumptions

$(H^2_{f,g})$ Let set $C_1 > 0$ and $\epsilon_1 > 0$.

For all $z \in \mathbb{R}^d$, $t \in [0, T]$ and any $y \in \mathbb{R}$ such that $|y| \leq \epsilon_1$,

(i) $y\, f(t, y, z) \geq -C_1\|z\|^2$,

(ii) $g(t, 0) = 0$.

FORMULATION OF PROBLEM

Additional assumptions

(H_ξ^1) For almost ω, $x \mapsto \xi(x, \omega)$ is increasing (or decreasing) and is an homeomorphism on $\mathbb{R}$,

(H_ξ^2) for any $R > 0$, there are δ_R, $C_R > 0$ such that

$$\mathbb{E}|\xi(x) - \xi(y)|^2 \leq C_R|x - y|^{1+\delta_R}, \quad \forall\, |x|, |y| \leq R.$$

(H_ξ^3) For some $\beta < \frac{1-2C_1}{2} \wedge 0$,

$$\liminf_{|x| \to \infty} \mathbb{E}|\xi(x)|^{4\beta} = 0.$$

FORMULATION OF PROBLEM

Example

Let denote by

$$f(y,z) = y + \arctan(y)(1+\sin(z)).$$

Then f satisfy $(H^2_{f,g}(i))$

Moreover H^3_ξ holds if $\xi(x)$ verify this condition

for some $R_0 > 0$ and $\varepsilon > 0$,

$$\inf_{\{|x|\geq R_0\}} \frac{\xi(x,\omega)}{g(x)} \geq \varepsilon$$

$g : \mathbb{R} \to \mathbb{R}$ continuous such that

$\lim_{x\to\mp\infty} g(x) = \pm\infty$ or $\lim_{x\to\mp\infty} g(x) = \pm\infty$.

Main result

Theorem

Under Lipschitz condition on f and g

Under additional assumptions,

for almost $\omega \in \Omega$, the map $x \mapsto Y_t^x(\omega)$ is a homeomorphism on $\mathbb{R}$, for every $t \in [0, T]$.

Preliminaries results

Comparison theorem

Let $(Y^1, Z^1) \longrightarrow$ solutions of BDSDEs (ξ^1, f^1, g) solution of BDSDEs

$(Y^2, Z^2) \longrightarrow$ solution of BDSDEs (ξ^2, f^2, g).

If $\xi^1 > \xi^2$, a.s.

$f^1(t, Y^1, Z^1) > f^2(t, Y^1, Z^1)$, a.s., $\forall t \in [0, T]$,

then, $Y^1_t > Y^2_t$, a.s.

Preliminaries results

Injection property

For any $R > 0$ and any $|x|, |y| \leq R$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t^x - Y_t^y|^2\right) \leq C_R|x-y|^{1+\delta_R}$$

In particular, $\{Y_t^x,\ (t,x) \in [0,T] \times \mathbb{R}\}$ admits a continuous modification on $[0,T] \times \mathbb{R}$. and

$$\mathbb{P}(\{\omega :\ Y_t^x(\omega) < Y_t^y(\omega),\ \forall\, x < y,\ t \in [0,T]\}) = 1.$$

Preliminaries results

Property at infinity

Under Lipschitz condition on f and g, additional assumptions $(H^2_{f,g})$ and (H^3_ξ), we have

$$\liminf_{|x|\to\infty} \mathbb{E}\left(\sup_{0\le t\le T} |Y^x_t|^{4\beta}\right) = 0. \tag{4}$$

Proof

Proof of main Theorem (Injection)

Assumption (H_ξ^1),

Comparison theorem

Injection property

$\Longrightarrow \quad x \mapsto Y_t^x(\omega)$ are continuous and injective a.s., for all $t \in [0, T]$.

Proof

Proof of main Theorem (Surjection)

Since $\beta < 0$, for any $n \in \mathbb{N}^*$, $x \in \mathbb{R}$ and $t \in [0, T]$, by Tchebychev's inequality,

$$\begin{aligned}
\mathbb{P}\left(|Y_t^x| > n\right) &= \mathbb{P}\left(|Y_t^x|^{4\beta} < n^{4\beta}\right) \\
&\geq \mathbb{P}\left(\sup_{0\leq t\leq T} |Y_t^x|^{4\beta} < n^{4\beta}\right) \\
&= 1 - \mathbb{P}\left(\sup_{0\leq t\leq T} |Y_t^x|^{4\beta} > n^{4\beta}\right) \\
&\geq 1 - \frac{\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t^x|^{4\beta}\right)}{n^{4\beta}}.
\end{aligned}$$

Proof

Proof of main Theorem (Surjection)

By virtue of Property at infinity, we have, for any $n \in \mathbb{N}^*$ and for all $t \in [0, T]$

$$\liminf_{|x|\to\infty} \mathbb{P}\Big(|Y_t^x| > n\Big) = 1.$$

Therefore

$$\lim_{|x|\to+\infty} |Y_t^x| = +\infty, \quad \forall t \in [0, T], \quad a.s.$$

Hence, by injectivity

$$\lim_{x\uparrow+\infty} Y_t^x = \pm\infty, \quad \lim_{x\downarrow-\infty} Y_t^x = \mp\infty, \quad \forall t \in [0, T], \quad a.s.$$

Consequently, $x \mapsto Y_t^x(\omega)$ is surjective.

Standing Assumptions

On coefficients of SDE

$(A^1_{\sigma,b})$ For all $t \in [0, T]$, $\sigma(t,.)$, $b(t,.) \in C^3_{l,b}(\mathbb{R})$;

$(A^2_{\sigma,b})$ For some $C_{\sigma,b} > 0$ and for all $(t,x) \in [0,T] \times \mathbb{R}$

$$|\sigma(t,x)| + |b(t,x)| \leq C_{\sigma,b}|x|;$$

Standing Assumptions

On coefficients of BDSDE

(A_f^1) for all $t \in [0,T], (x,y,z) \mapsto f(t,x,y,z)$ is of class C^3, and their derivatives of order one and two are bounded on $[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$;

(A_f^2) for every $t \in [0,T]$, the function $x \mapsto f(t,x,0,0)$ has polynomial growth at infinity together with all partial derivatives up order three;

(A_f^3) for some $C_1 > 0$ and $\epsilon_1 > 0$, it holds that $yf(\omega,t,x,y,z) \geq -C_1|z|^2$, for all $(\omega,t) \in \Omega \times [0,T]$ and $|y| \leq \epsilon_1$, $x, z \in \mathbb{R}$;

Standing Assumptions

On coefficients of BDSDE

(A_g^1) for all $t \in [0, T], (t, x, y) \longrightarrow g(t, x, y)$ is of class C^3, and all derivatives are bounded on $[0, T] \times \mathbb{R} \times \mathbb{R}$.

(A_g^2) for all $(t, x) \in [0, T] \times \mathbb{R}, \;\; g(t, x, 0) = 0,$

(A_h^1) $h \in C_p^3(\mathbb{R}),$

(A_h^2) $x \mapsto h(x)$ is increasing (or decreasing) and a homeomorphism on $\mathbb{R}$.

(A_h^3) there are constants $c_1, \delta > 0$ such that $\;|h(x)| \geq c_1|x|^{\delta}$.

Main result

Theorem

Under the assumptions (**A**) and for any $t \in [0, T]$,

the map $x \mapsto Y_s^{t,x}$ is a homeomorphism on $\mathbb{R}$ for all $s \in [t, T]$, a.s.

In particular, denoting $Y_t^{t,x} = u(t, x)$ the unique solution to SPDEs (2),

$x \mapsto u(\omega, t, x)$ is a homeomorphism on $\mathbb{R}$ for almost all ω and $t \in [0, T]$.

Proof of Preliminaries results

Proof of comparison theorem

Define

$$
\begin{aligned}
\bar{\xi} &= \xi^1 - \xi^2, \quad \bar{Y}_t = Y_t^1 - Y_t^2, \quad \bar{Z}_t^{(i)} = Z_t^{1(i)} - Z_t^{2(i)},\\
\bar{f}_t &= f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)
\end{aligned}
$$

and

$$
a_t = \frac{f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1)}{(Y_t^1 - Y_t^2)\mathbf{1}_{\{Y_t^1 \neq Y_t^2\}}} \quad b_t^i = \frac{f^1(s, Y_s^2, \tilde{Z}_s^{(i-1)}) - f^1(s, Y_s^2, \tilde{Z}_s^{(i)})}{(Z_t^{1(i)} - Z_t^{2(i)})\mathbf{1}_{\{Z_t^{1(i)} \neq Z_t^{2(i)}\}}}
$$

$$
c_t = \frac{g(s, Y_s^1) - g(s, Y_s^2)}{(Y_t^1 - Y_t^2)\mathbf{1}_{\{Y_t^1 \neq Y_t^2\}}},
$$

where $\widetilde{Z}^{(i)} = \left(Z^{2(1)}, Z^{2(2)}, ..., Z^{2(i)}, Z^{1(i+1)}, ..., Z^{1(d)}\right)$

Proof of Preliminaries results

Proof of comparison theorem

$(\bar{Y}, \bar{Z})$ is a unique solution of the following linear BDSDE

$$\begin{aligned} \bar{Y}_t &= \bar{\xi} + \int_t^T [a_s \bar{Y}_s + b_s \bar{Z}_s + \bar{f}_s] ds \\ &\quad + \int_t^T c_s \bar{Y}_s dB_s - \int_t^T \bar{Z}_s dW_s, \quad t \in [0, T]. \end{aligned}$$

Now, by the explicit expression for linear BDSDE

$$\bar{Y}_t = \widetilde{\mathbb{E}} \left(\Gamma_{t,T} \bar{\xi} + \int_t^T \Gamma_{t,r} \bar{f}_r dr \mid \mathcal{F}_t \right),$$

Proof of Preliminaries results

Proof of comparison theorem

where

$$\Gamma_{s,t} = \exp\left(\int_s^t a_r dr + \int_s^t c_r dB_r - \frac{1}{2}\int_s^t \|c_r\|^2 dr\right)$$

Therefore, since $\bar{\xi} > 0$ and $\bar{f}_t > 0, \forall\ t \in [0, T]$,

we have $\bar{Y}_t > 0$, a.s. i.e. $Y_t^1 > Y_t^2$, a.s. for each $t \in [0, T]$

Proof of injection property

(Y^x, Z^x) and (Y^y, Z^y) be respectively the unique solution of BDSDE associated to $\xi(x)$ and $\xi(y)$
Define

$$\bar{\xi} = \xi(x) - \xi(y), \ \bar{Y}_t = Y_t^x - Y_t^y, \ \ \bar{Z}_t = Z_t^x - Z_t^y$$

By virtue of Itô's formula, taking expectation, using $(H^1_{f,g})$, Young's inequality, Gronwall's inequality and (H^2_ξ)

$$\mathbb{E}\left(| \ \bar{Y}_t \ |^2 + \int_t^T \|\bar{Z}_s\|^2 ds \right) \leq C|x-y|^{1+\delta_R}. \tag{5}$$

Proof of injection property

Applying again Itô's formula, taking the sup and hence expectation, it follows by Young inequality, Burkhölder-Davis-Gundy inequality and (5), we get

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]} \mid \bar{Y}_t \mid^2\right) &\leq C\left(\mathbb{E}|\bar{\xi}|^2 + \mathbb{E}\int_0^T |\bar{Y}_s|^2 ds + \mathbb{E}\int_t^T \|\bar{Z}_s\|^2 ds\right) \\ &\leq C|x-y|^{1+\delta_R}. \end{aligned} \tag{6}$$

Proof of property at infinity

Assume that for all $x \in \mathbb{R}$ $\mathbb{E}|\xi(x)|^{4\beta} < +\infty$. For any $0 < \epsilon < \varepsilon_1$ and $\alpha < 0$,

$$\begin{aligned}
(|Y_t^x|^2 + \epsilon)^\alpha &= (|Y_T^x|^2 + \epsilon)^\alpha + 2\alpha \int_t^T (|Y_s^x|^2 + \epsilon)^{\alpha-1} Y_s^x f(s, Y_s^x, Z_s^x) ds \\
&+ 2\alpha \int_t^T (|Y_s^x|^2 + \epsilon)^{\alpha-1} Y_s^x g(s, Y_s^x) dB_s \\
&- 2\alpha \int_t^T (|Y_s^x|^2 + \epsilon)^{\alpha-1} Y_s^x Z_s^x dW_s \\
&+ 2\alpha(\alpha - 1) \int_t^T |Y_s^x|^2 (|Y_s^x|^2 + \epsilon)^{\alpha-2} \|g(s, Y_s^x)\|^2 ds \\
&+ A_\alpha.
\end{aligned} \tag{7}$$

Proof of property at infinity

$$
\begin{aligned}
A_\alpha &= \alpha \int_t^T (|\, Y_s^x|^2 + \epsilon)^{\alpha-1} \|g(s, Y_s^x)\|^2 ds \\
&\quad -2\alpha(\alpha-1) \int_t^T |Y_s^x|^2 (|Y_s^x|^2 + \epsilon)^{\alpha-2} \|Z_s^x\|^2 ds \\
&\quad -\alpha \int_t^T (|Y_s^x|^2 + \epsilon)^{\alpha-1} \|Z_s^x\|^2 ds
\end{aligned}
$$

First for each $x \in \mathbb{R}$

$$
\sup_{0 \le t \le T} \mathbb{E}|Y_t^x|^{4\beta} < +\infty. \tag{8}
$$

Proof of property at infinity

$(H^2_{f,g})$ implies $f(s,0,0) = 0$, which together with $(H^1_{f,g})$ gives

$$|f(t,y,z)| \leq C_f(|y| + \|z\|) \text{ and } \|g(t,y)\|^2 \leq C_g|y|^2.$$

For $0 < \epsilon < \epsilon_1$, putting $\alpha = 2\beta$, taking expectation, using (5) and letting $\epsilon \downarrow 0$,

$$\mathbb{E} \mid Y^x_t \mid^{4\beta} \quad \leq \quad \mathbb{E}|\xi(x)|^{4\beta} + C' \int_t^T \mathbb{E} \mid Y^x_s \mid^{4\beta} ds.$$

Therefore it follows from Gronwall's inequality

$$\mathbb{E}|Y^x_t|^{4\beta} \quad \leq \quad C\mathbb{E}|\xi(x)|^2,$$

which proves (8)

Proof of property at infinity

Now, starting from (7), using Young's inequality with property of β and Gronwall's inequality

$$\mathbb{E}\left(|Y_t^x|^{4\beta}+\int_t^T |Y_s^x|^{2(2\beta-1)}\|Z_s^x\|^2 ds\right) \quad \leq \quad C\mathbb{E}|\xi(x)|^{4\beta}$$

Similar to the above calculations, from (7) with Doob's maximal inequality

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq T}\mid Y_t^x\mid^{4\beta}\right) &\leq C\mathbb{E}\left(|\xi(x)|^{4\beta}+\int_0^T |Y_s^x|^{4\beta}ds\right.\\ &\quad \left.+\int_0^T |Y_s^x|^{4\beta-2}\|Z_s^x\|^2 ds\right)\\ &\leq C\mathbb{E}\mid\xi(x)\mid^{4\beta},\end{aligned}$$

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THANK FOR YOUR ATTENTION

GOD BLESS US