On stochastic 2D Navier Stokes equations and hydrodynamical models

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Outline

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D bounded domain of $\mathbb{R}^2$, $x = (x_1, x_2)$,
$u(x,t) = (u^1(x,t), u^2(x,t))$ fluid velocity, $p(x,t)$ pressure
divergence: $\mathrm{div}\, u = \sum_{i=1,2} \partial_i u_i$
Laplace operator: $\Delta u = (\sum_{i=1,2} \partial_i^2 u^k, k = 1, 2)$ (Stokes operator if one adds the incompressibility condition $\mathrm{div}\, u = 0$ on D)
Find a pair (u, p) (u velocity, p pressure) such that

$$\partial_t u - \nu \Delta u + u.\nabla u + \nabla p = f, \quad \mathrm{div}\, u = 0 \quad \text{in} \quad D, \quad u = 0 \quad \text{on} \quad \partial D$$

$\nu > 0$ viscosity, n outwards normal to ∂D and $f(x,t)$ external force

$$H = \{ f \in \left[L^2(D) \right]^2 : \mathrm{div}\, f = 0 \text{ in } D \text{ and } f \,.\, n = 0 \text{ on } \partial D \}$$

$V \equiv \left[H_0^1(D) \right]^2 \cap H$, $\quad V \subset H$, $(H, |\ |)$ and $(V, \|\ \|)$ Hilbert spaces
Project on divergence free fields (integration by parts: if, $\mathrm{div}\ u = 0$ then $(\nabla p, u) = 0$ if $\mathrm{div}\ u = 0$

$A : V \to V'$ et $B : V \times V \to V'$ defined by:

$$\langle Au\,,\, v\rangle = \nu \sum_{j=1,2} \int_D \nabla u^j \cdot \nabla v^j \,dx$$

$$\langle B(u,v), w\rangle = \int_D [u \cdot \nabla v]\, w\, dx \equiv \sum_{i,j=1,2} \int_D u^j \partial_j v^i w^i \,dx,\ \forall u, v, w \in V$$

Properties of A and B

$A = -\nu\Delta$ non-negative, unbounded, self-adjoint linear operator on H,
$B : V \times V \to V'$ bilinear continuous $\forall u_1, u_2, u_3 \in V$

$$\langle B(u_1, u_2), u_3\rangle = -\langle B(u_1, u_3), u_2\rangle$$

(Proof: $\sum_{i,j} \int_D u^j \partial_j v^i w^i \,dx = -\sum_{i,j} \int_D v^i \left[w^i \partial_j u^j + u^j \partial_j w^i\right] dx$
for fixed i, $\sum_j \partial_j u^j = \operatorname{div} u = 0$)

- $V \subset \mathcal{L}^4(D) \subset H$ and for $u \in V$, $\|u\|^2_{\mathcal{L}^4(D)} \leq |u|\,\|u\|$
  (Proof: for real-valued functions f, g $\|fg\|_1 \leq \frac{1}{4}\|\partial_1 f\|_1 \partial_2 g\|_1$
  with f^2 and g^2, $\|f^2 g^2\|_1 \leq \|f\partial_1 f\|_1 \|g\partial_2 g\|_1$
  Schwarz's inequality : $\|f^2 g^2\|_1 \leq \|f\|_2 \|\nabla f\|_2 \|g\|_2 \|\nabla g\|_2$; then $f = g$)
- For $\eta > 0$ there exists $C_\eta > 0$ such that

$$|\langle B(u_1, u_1)\,,\, u_3\rangle| \leq \eta\,\|u_1\|^2 + C_\eta\,|u_1|^2\,\|u_3\|^4_{\mathcal{L}^4(D)}$$

  (Proof: Hölder's inequality
  $|\int_D u_1 \nabla u_1 u_3| \leq \|u_1\|_{\mathcal{L}^4(D)} |\nabla u_1| \|u_3\|_{\mathcal{L}^4(D)} \leq \|u_1\|^{\frac{3}{2}} |u_1|^{\frac{1}{2}} \|u_3\|_{\mathcal{L}^4(D)}$
  Then Young's inequality with exponents $4/3$ and 4 yields
  $|\langle B(u_1, u_1)\,,\, u_3\rangle| \leq \frac{4\alpha}{3}\|u_1\|^2 + \frac{1}{4\alpha}|u_1|^2\,\|u_3\|^4_{\mathcal{L}^4(D)}$)
- If $B(u) := B(u, u)$ then
  $|\langle B(u_1) - B(u_2)\,,\, u_1 - u_2\rangle| \leq \eta \|u_1 - u_2\|^2 + C_\eta |u_1 - u_2|^2\,\|u_2\|^4_{\mathcal{L}^4(D)}$

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Project on divergence free functions, suppress the pressure $\mathrm{div}\,\nabla p = 0$ add a Coriolis term (replace the forcing term f by $f - Ru$ where $R(u^1, u^2) = c_0(-u^2, u^1)$, c_0 constant. $\partial_t u - \nu\Delta u + u.\nabla u + Ru = f$

Abstract setting $(H, |.|)$ Hilbert, R linear continuous operator on H

A non negative, self-adjoint operator (unbounded) operator on H

$V = Dom(A^{\frac{1}{2}})$; for $v \in V$ set $\|v\| = |A^{\frac{1}{2}} v|$

$\mathcal{H}$ Banach space such that $V \subset \mathcal{H} \subset H$ and $\|v\|^2_{\mathcal{H}} \leq K_0 |v|\,\|v\|$

$B : V \times V \to V'$ bilinear continuous such that $\forall u_1, u_2, u_3 \in V$

$$\langle B(u_1, u_2), u_3\rangle = -\langle B(u_1, u_3), u_2\rangle$$

for $\eta > 0$ there exists $C_\eta > 0$ such that

$$|\langle B(u_1, u_1)\,,\; u_3\rangle| \leq \eta\,\|u_1\|^2 + C_\eta\,|u_1|^2\,\|u_3\|^4_{\mathcal{H}}$$

$$d_t u(t) + \Big[Au(t) + B\big(u(t)\big) + Ru(t)\Big]dt = \sigma(u(t))dW_t\;,\;\; u(0) \in H$$

$D = (0, l) \times (0, 1)$, $x = (x_1, x_2)$ spatial variable
p pressure, $\phi = (u, \theta, \beta)$ satisfy the coupled non-linear equations
$u \in \mathbb{R}^2$ velocity field, $\theta \in \mathbb{R}$ temperature field, $\beta \in \mathbb{R}^2$ magnetic field
ν, κ, η and S physical constants,

$$
\begin{aligned}
&\frac{\partial}{\partial t} u + u \cdot \nabla u - \nu \Delta u + \nabla p \\
&\qquad + \nabla\big(\frac{1}{2}|\beta|^2\big) - S \beta \cdot \nabla \beta - \theta e_2 = \sigma_1(t, \phi)\, dW^1(t)\,, \\
&\qquad \frac{\partial}{\partial t}\theta + u \cdot \nabla \theta - \kappa \Delta \theta \; - u_2 = \sigma_2(t, \phi)\, dW^2(t)\,, \\
&\qquad \frac{\partial}{\partial t}\beta - \eta \Delta \beta + u \cdot \nabla \beta - \beta \cdot \nabla u = \sigma_3(t, \phi)\, dW^3(t)\,,
\end{aligned}
$$

where Δ is the Laplace operator (Stokes operator after Leray projection)

$$div(u) = div(\beta) = 0$$

$$u = \theta = \beta_2 = \frac{\partial}{\partial x_2}\beta_1 = 0 \quad \text{on } x_2 \in \{0,1\}$$

$$u, p, \theta, \beta, u_{x_1}, \theta_{x_1}, \beta_{x_1} \text{ period l in } x_1$$

$H = L^2(D)^5$ with divergence, periodicity and boundary conditions
$V = H^1(D)^5$ with the same conditions
$V \hookrightarrow H = H' \hookrightarrow V'$ $\quad \mathcal{H} = L^4(D)^5 \cap H$ and $\|u\|^2_{\mathcal{H}} \le K_0 |u| \, \|u\|$
$B(\phi) = (B_1(u,u) - SB_1(\beta,\beta)\,,\ B_2(u,\theta)\,,\ B_1(u,\beta) - B_1(\beta,u))$

$$\langle B_1(u,v), w\rangle = \int_D [u\cdot\nabla v] w dx \quad := \sum_{i,j=1,2} \int_D u_i\, \partial_i v_j\, w_j dx,$$

$$\langle B_2(u,\theta), \eta\rangle = \int_D [u\cdot\nabla\theta]\eta dx \quad := \sum_{i=1,2} \int_D u_i\, \partial_i\, \theta\ \eta\, dx.$$

Studied by A. Cheskidov, D. Holm, E. Olson & E. Titi
$D \subset \mathbb{R}^3$ bounded domain, A Stokes operator

$$\partial_t u - \nu \Delta u + v.\nabla u + \nabla p = f,$$
$$(1 - \alpha\Delta)v = u, \quad \mathrm{div}\, u = 0, \quad \mathrm{div}\, v = 0 \quad \text{in} \quad D,$$
$$v = u = 0 \quad \text{on} \quad \partial D.$$

$G_\alpha = (1 - \alpha\Delta)^{-1}$ Green operator
$H = \{u \in [L^2(D)]^3 : \mathrm{div} u = 0 \text{ in } D \text{ and } u\,.\,n = 0 \text{ on } \partial D\}$
As $H^{1/2}(D) \subset L^3(D)$, set $\mathcal{H} = [L^3(D)]^3 \cap H$. Since $H^1(D) \subset L^6(D)$ and $A^{\frac{1}{2}} G_\alpha$ is bounded on H, previous conditions fulfilled with
$B_\alpha(u_1, u_2) := B(G_\alpha u_1, u_2)$

Covariance operator Q symmetric non-negative on H, $Trace(Q) < +\infty$
$H_0 = Q^{\frac{1}{2}} H$ Hilbert space $(\phi, \psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi)$, $\forall \phi, \psi \in H_0$,
embedding $i : H_0 \to H$ is Hilbert-Schmidt (hence compact), and
$i\, i^* = Q$
$W(t)$ H-valued Wiener process with covariance operator Q.
W is Gaussian, has independent time increments, for $s, t \geq 0$, $f, g \in H$,

$$\mathbb{E}(W(s), f) = 0 \quad \text{and} \quad \mathbb{E}(W(s), f)(W(t), g) = (s \wedge t)\,(Qf, g).$$

W has the representation

$$W(t) = \lim_{n\to\infty} W_n(t) \;\text{ in }\; L^2(\Omega; H) \;\text{ with }\; W_n(t) = \sum_{j=1}^{n} q_j^{1/2} \beta_j(t) e_j,$$

where β_j are independent (real) Wiener processes, $\{e_j\}$ is ONB in H of eigen-elements of Q, with $Qe_j = q_j e_j$.

$L_Q = \{S \in L(H_0, H) \,:\, SQ^{\frac{1}{2}} \text{ Hilbert Schmidt from } H \text{ to } H\}$,
$\|S\|^2_{L_Q} = Trace(S\,Q\,S^*)$, where S^* is the adjoint of S.
For any BON $\{\psi_k\}$ in H, the L_Q-norm can be written

$$|S|^2_{L_Q} = tr([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k\geq 1} |SQ^{1/2}\psi_k|^2 = \sum_{k\geq 1} |[SQ^{1/2}]^*\psi_k|^2$$

$\sigma \in C([0,T] \times V) \to L_Q$
There exist constants K_i and L_i such that for $t \in [0,T]$, $\phi, \psi \in V$
$|\sigma(t,\phi)|^2_{L_Q} \leq K_0 + K_1|\phi|^2 + K_2\|\phi\|^2$,
$|\sigma(t,\phi) - \sigma(t,\psi)|^2_{L_Q} \leq L_1|\phi - \psi|^2 + L_2\|\phi - \psi\|^2$.
(in the above examples, σ may depend on the gradient of the solution)

With these notations, the above models (Bénard $\phi = (u, \theta, \beta)$, 2D Navier Stokes, "shell models" or 3D Leray Navier Stokes $\phi = u$) are written:

$$d\phi + [A\phi + B(\phi) + R\phi]dt = \sigma(\phi)dW(t) \;, \quad \phi(0) = \xi \in H \qquad (1)$$

ξ $\mathcal{F}_0$-measurable, independent of W.
Solution means: (ϕ_t) adapted for $(\mathcal{F}_t)$ and for $\psi \in D(A)$,

$$(\phi(t), \psi) - (\phi(0), \psi) + \int_0^t \big[(\phi(s), A\psi) + \langle B(\phi(s)), \psi\rangle + (R\phi(s), \psi)\big]\, ds = \int_0^t \big(\sigma(\phi(s))\, dW(s)\, ,\, \psi\big).$$

Weak solution for analysts and **strong solution** for probabilists

$\sigma(u)$ estimated in terms of u in V with constants K_2 and L_2

Theorem

Let $E|\xi|^4 < +\infty$. Then for K_2 small enough and $L_2 < 2$, there exists $C = C(K_i, L_i, T)$ such that the evolution equation has a unique solution $\phi \in X = C([0,T], H) \cap L^2(0,T;V)$. Furthermore,

$$E\Big(\sup_{0\leq t\leq T} |\phi(t)|^4 + \int_0^T \|\phi(t)\|^2\, dt + \int_0^T \|\phi(t)\|_{\mathcal{H}}^4\, dt\Big) \leq C\,(1 + E|\xi|^4)$$

- Ferrario 1997 (Boussinesq equation) $\phi = (u, \theta)$, Barbu-Da Prato 2007 (MHD equation) $\phi = (u, \beta)$ for additive noise
- Sritharan-Sundar (Navier Stokes) $\phi = u$, Duan-M (Boussinesq) $\phi = (u, \theta)$, Chueshov-M., Manna-Shritharan-Sundar (shell models)

Set of controls

$$\mathcal{S}_M = \Big\{ h \in L^2([0,T], H_0) : \int_0^T |h(s)|_0^2 \, ds \le M \Big\}$$

$$\mathcal{A}_M = \{ h \quad (\mathcal{F}_t) \, \text{predictable} \; : h(\omega) \in \mathcal{S}_M \text{ a.s.} \}$$

First idea: shift W a random element of $\mathcal{A}_M$ and use Girsanov
Need more general stochastic controls $\tilde\sigma \in C\big([0,T] \times V; L(H_0,H)\big)$
there exist constants $\tilde K_{\mathcal{H}}$, $\tilde K_i$, and $\tilde L_j$, for $i = 0, 1$ and $j = 1, 2$ such that for $u, v \in V$, $t \in [0,T]$,

$$|\tilde\sigma(t,u)|^2_{L(H_0,H)} \le \tilde K_0 + \tilde K_1 |u|^2 + \tilde K_{\mathcal{H}} \|u\|^2_{\mathcal{H}},$$

$$|\tilde\sigma(t,u) - \tilde\sigma(t,v)|^2_{L(H_0,H)} \le \tilde L_1 |u-v|^2 + \tilde L_2 \|u-v\|^2.$$

$\tilde R : [0,T] \times H \mapsto H$ is continuous global growth and Lipschitz

Recall that K_2 and L_2 are the growth and Lipshitz constants in front of the V norm.

Theorem

(Chueshov-M.) Suppose that either $\tilde\sigma = \sigma$, or that $\tilde\sigma$ satisfies the above conditions Let $M>0$, h such that $\int_0^T |h(s)|_0^2\, ds \le M$ a.s., suppose that $K_2 \le \kappa_2$ and $L_2 \le \lambda_2$ for small κ_2, λ_2. Let $u_h(0)=\xi$ be in $\mathcal{F}_0$ s.t. $E|\xi|^4 < +\infty$.

$$du_h + [Au_h + B(u_h,u_h) + \tilde R u_h]dt = \sigma(u_h)dW(t) + \tilde\sigma(u_h)\, h\, dt$$

has a unique solution in $X = C([0,T],H) \cap L^2(0,T;V)$ s.t.

$$\mathbb{E}\Big(\sup_{0\le t\le T} |u_h(t)|^4 + \int_0^T \|u_h(t)\|^2\, dt\Big) \le C(M,\kappa_2,\lambda_2)\,\big(1+E|\xi|^4\big).$$

$F : [0,T] \times V \to V'$ be defined by
$F(t,u) = -Au - B(u,u) - \tilde{R}(t,u)$, $\forall t \in [0,T]$, $\forall u \in V$.
$\langle F(u) - F(v)\,,\, u - v\rangle \leq -(1-\eta)\|u-v\|^2 + \big(R_1 + C_\eta \|v\|_{\mathcal{H}}^4\big)\,|u-v|^2.$

$\{\varphi_n\}_{n\geq 1}$ ONB of H such that $\varphi_n \in Dom(A)$. For $n \geq 1$, let
$H_n = span(\varphi_1, \cdots, \varphi_n) \subset Dom(A)$, $P_n : H \to H_n$ orthogonal projection from H onto H_n, $\sigma_n = P_n\sigma$ and $\tilde{\sigma}_n = P_n\tilde{\sigma}$
For $h \in \mathcal{A}_M$, evolution equation on (n-dim. space) H_n defined by $u_{n,h}(0) = P_n\xi$:

$$du_{n,h}(t) = \big[F(u_{n,h}(t)) + \tilde{\sigma}(u_{n,h}(t))h(t)\big]dt + \sigma(u_{n,h}(t))dW_n(t)$$

and $W_n = \sum_{1\leq j\leq n} \sqrt{q_j}\beta_j(t)e_j$. For $k = 1, \cdots, n$

$$\begin{aligned} d(u_{n,h}(t), \varphi_k) \;&=\; \big[\langle F(u_{n,h}(t)), \varphi_k\rangle + (\tilde{\sigma}(u_{n,h}(t))h(t), \varphi_k)\big]dt \\ &\quad + \sum_{j=1}^{n} q_j^{\frac{1}{2}} \big(\sigma(u_{n,h}(t))e_j\,,\, \varphi_k\big)\,d\beta_j(t). \end{aligned}$$

B is bilinear and F locally Lipschitz. There exists a maximal solution $u_{n,h}$ and a stopping time $\tau_{n,h}$ such that the evolution equation for $u_{n,h}$ holds for $t < \tau_{n,h}$ and as $t \uparrow \tau_{n,h} < T$, $|u_{n,h}(t)| \to \infty$.
Prove that $\tau_{n,h} = T$ a.s.

Proposition

Fix $M > 0$, $T > 0$, $h \in \mathcal{A}_M$, $0 \le K_2 \le \bar{K}_2$ and $\xi \in L^{2p}(\Omega, H)$.
$|\tilde{\sigma}(t,u)|^2_{L(H_0,H)} \le \tilde{K}_0 + \tilde{K}_1|u|^2 + \tilde{K}_2\|u\|^2, \quad \forall t \in [0,T], \ \forall u \in V,$
For $p \ge 1$ there exists $\bar{K}_2 = \bar{K}_2(p,T,M)$ and $C = C(p,K_2,M,T)$ such that for $0 \le K_2 < \bar{K}_2$: $\tau_{n,h} = T$ a.s. and a modification of the solution $u_{n,h} \in C([0,T], H_n)$

$$\sup_n \mathbb{E}\Big(\sup_{0\le t\le T} |u_{n,h}(t)|^{2p} + \int_0^T \|u_{n,h}(s)\|^2 \, |u_{n,h}(s)|^{2(p-1)} ds \Big) \le C\big(\mathbb{E}|\xi|^{2p} + 1\big)$$

Fix $N > 0$ and $\tau_N = \inf\{t : |u_{n,h}(t)| \geq N\} \wedge T$. Itô's formula for $|.|^2$ and $\langle B(u,u),u\rangle = 0$,

$$|u_{n,h}(t\wedge\tau_N)|^2 = |P_n\xi|^2 + 2\int_0^{t\wedge\tau_N}\big(\sigma_n(u_{n,h}(s))dW_n(s), u_{n,h}(s)\big)$$
$$-2\int_0^{t\wedge\tau_N}\|u_{n,h}(s)\|^2 ds + \int_0^{t\wedge\tau_N}|\sigma_n(u_{n,h}(s))\,\Pi_n|^2_{L_Q}\,ds$$
$$-2\int_0^{t\wedge\tau_N}\big(\tilde{R}(u_{n,h}(s)) - \tilde{\sigma}_n(u_{n,h}(s))h(s), u_{n,h}(s)\big)\,ds$$

Itô's formula for $z \to z^p$ with $p \geq 2$ and $z = |u_{n,h}(t\wedge\tau_N)|^2$ plus conditions on coefficients

Lemma

Let X, Y, I and φ be non-negative processes and Z be a non-negative integrable random variable. Assume that I is non-decreasing and there exist non-negative constants C, $\alpha, \beta, \gamma, \delta$ with $\int_0^T \varphi(s)\, ds \leq C$ a.s., $2\beta e^C \leq 1$, $2\delta e^C \leq \alpha$ and such that for $0 \leq t \leq T$,

$$X(t) + \alpha Y(t) \leq Z + \int_0^t \varphi(r)\, X(r)\, dr + I(t), \text{ a.s.},$$

$$\mathbb{E}(I(t)) \leq \beta\, \mathbb{E}(X(t)) + \gamma \int_0^t \mathbb{E}(X(s))\, ds + \delta\, \mathbb{E}(Y(t)) + \tilde{C},$$

where $\tilde{C} > 0$ is a constant. If $X \in L^\infty([0, T] \times \Omega)$, then we have

$$\mathbb{E}\big[X(t) + \alpha Y(t)\big] \leq 2 \exp\left(C + 2t\gamma e^C\right)\left(\mathbb{E}(Z) + \tilde{C}\right), \quad t \in [0, T].$$

Proof of the well posedeness and apriori estimates
using the generalized Gronwall lemma

Apply the generalized Gronwall lemma to
$X(t) = \sup_{s\le t\wedge\tau_N} |u_{n,h}(t\wedge\tau_N)|^{2p}$,
$Y(t) = \int_0^{t\wedge\tau_N} |u_{n,h}(r)|^{2(p-1)} \, \|u_{n,h}(r)\|^2 \, dr$, for $t\in[0,T]$

$$X(t) + pY(t) \le Z(|\xi|^{2p}, T, M) + \int_0^{t\wedge\tau_N} \varphi(r) X(r) dr + I(t)$$

where $\int_0^T \varphi(s)ds \le C(T,M)$ a.s.,
$I(t) = \sup_{0\le s\le t} |J(s)|$,

$$J(t) = 2p\Big| \int_0^{t\wedge\tau_N} \big(\sigma_n(u_{n,h}(r))\, dW_n(r), u_{n,h}(r)\, |u_{n,h}(r)|^{2(p-1)}\big)\Big|$$

Use the BDG inequality: For β small enough and then K_2 small enough, there exist $C, \tilde{C}$ such that for $t\in[0,T]$,

$$\mathbb{E}\big[X(t) + Y(t)\big] \le 2\,\exp\big(CM + \tilde{C} t e^{CM}\big)\,\big(\mathbb{E}Z(|\xi|^{2p}, T, M) + C(p,T)\big)$$

.

$N \to \infty$, $\tau_N \to \tau_{n,h}$ and upper estimate above independent of N and n.
Hence on $\tau_{n,h} < T$, $\sup_{s \le \tau_N} |u_{n,h}(s)| = +\infty$. Contradiction.
Let $\mathbb{E}|\xi|^4 < \infty$ and use interpolation $\|u\|_{\mathcal{H}}^4 \le C|u|^2\|u\|^2$.
Set $\Omega_T = [0,T] \times \Omega$. There exists a subsequence of $u_{n,h}$ and

$$u_h \in \mathcal{X} := L^2(\Omega_T, V) \cap L^4(\Omega_T, \mathcal{H}) \cap L^4(\Omega, L^\infty([0,T], H)),$$

$F_h \in L^2(\Omega_T, V')$ and $S_h, \tilde{S}_h \in L^2(\Omega_T, L_Q)$, and of r.v.
$\tilde{u}_h(T) \in L^2(\Omega, H)$ such that:
(i) $u_{n,h} \to u_h$ weakly in $L^2(\Omega_T, V)$,
(ii) $u_{n,h} \to u_h$ weakly in $L^4(\Omega_T, \mathcal{H})$,
(iii) $u_{n,h}$ is weak star converging to u_h in $L^4(\Omega, L^\infty([0,T], H))$,
(iv) $u_{n,h}(T) \to \tilde{u}_h(T)$ weakly in $L^2(\Omega, H)$,
(v) $F(u_{n,h}) \to F_h$ weakly in $L^2(\Omega_T, V')$,
(vi) $\sigma_n(u_{n,h})\Pi_n \to S_h$ weakly in $L^2(\Omega_T, L_Q)$,
(vii) $\tilde{\sigma}_n(u_{n,h})h \to \tilde{S}_h$ weakly in $L^{\frac{4}{3}}(\Omega_T, H)$

Pass evolution equation to the limit (inner product with $f_k(t)\varphi_j$)
$f_k \in H^1(-\delta, T+\delta)$ such that $\|f_k\|_\infty = 1$, $f_k = 1$ on $(-\delta, t-\frac{1}{k})$ and $f_k = 0$ on $\big(t, T+\delta\big)$ implies

$$0 = (\xi - u_h(t), \varphi_j) + \int_0^t (S_h(s)dW(s), \varphi_j) + \int_0^t \langle F_h(s) + \tilde{S}_h(s), \varphi_j\rangle ds$$

j arbitrary and for $f = 1_{(-\delta, T+\delta)}$ yields $u_h(T) = \tilde{u}_h(T)$ where

$$u_h(t) = \xi + \int_0^t S_h(s)dW(s) + \int_0^t F_h(s)ds + \int_0^t \tilde{S}_h(s)ds$$

Prove that $ds \otimes d\mathbb{P} a.s.$

$$S_h(s) = \sigma(u_h(s)),\ F_h(s) = F(u_h(s)) \text{ and } \tilde{S}_h(s) = \tilde{\sigma}(u_h(s))\ h(s)$$

Let $v \in \mathcal{X} = L^4(\Omega_T, \mathcal{H}) \cap L^4\big(\Omega, L^\infty([0,T], H)\big) \cap L^2(\Omega_T, V)$. Suppose that $L_2 < 2$ and let $0 < \eta < \frac{2-L_2}{3}$; set

$$r(t) = \int_0^t \Big[2\, R_1 + 2\, C_\eta\, \|v(s)\|_{\mathcal{H}}^4 + L_1 + 2\sqrt{\tilde{L}_1}|h(s)|_0 + \frac{\tilde{L}_2}{\eta}|h(s)|_0^2 \Big]\, ds,$$

Apply Itô's formula to $|u(t)|^2\, e^{-r(t)}$ for $u = u_h$ and $u = u_{n,h}$ leads to prove that upper estimate $\liminf_n X_n$, where

$$X_n = \mathbb{E} \int_0^T e^{-r(s)} \Big[- r'(s) \big\{ \big| u_{n,h}(s) - v(s) \big|^2 + 2 \big(u_{n,h}(s) - v(s)\,,\, v(s) \big) \big\}$$
$$+ 2\langle F(u_{n,h}(s)), u_{n,h}(s)\rangle + |\sigma_n(u_{n,h}(s))\Pi_n|_{L_Q}^2 + 2\big(\tilde{\sigma}(u_{n,h}(s))h(s), u_{n,h}(s)\big)\Big]\, ds$$

Some coercivity and monoticity properties (only valid in 2D) based on

$$\langle F(u) - F(v)\,,\, u - v\rangle \le -(1-\eta)\|u - v\|^2 + \big(R_1 + C_\eta \|v\|_{\mathcal{H}}^4\big)\, |u - v|^2$$

imply

For any $v \in \mathcal{X}$,

$$\mathbb{E}\int_0^T e^{-r(s)}\Big\{ - r'(s)|u_h(s)-v(s)|^2 + 2\langle F_h(s)-F(v(s)), u_h(s)-v(s)\rangle + |S_h(s)-\sigma(v(s))|^2_{L_Q} + 2\Big(\tilde S_h(s) - \tilde\sigma(v(s))h(s)\,,\ u_h(s)-v(s)\Big)\Big\} ds \leq 0.$$

$v = u_h \in \mathcal{X}$ implies $S_h(s) = \sigma(u_h(s))$
$\tilde v \in L^\infty([0,T]\times\Omega)$ and $v_\lambda = u_h - \lambda \tilde v$ previous result with v_λ and r_λ
Let $\lambda \to 0$ and divide the inequality for v_λ and r_λ by $\lambda > 0$ (resp. $\lambda < 0$) yields

$$\mathbb{E}\int_0^T e^{-r_0(s)}\Big[\langle F_h(s)-F(u_h(s)), \tilde v(s)\rangle + \big(\tilde S_h(s)-\tilde\sigma(u_h(s))h(s), \tilde v(s)\big)\Big] ds = 0$$

Hence $u_h(t) = \xi + \int_0^t \sigma(u_h(s))dW(s) + \int_0^t \big[F(u_h(s)) + \tilde\sigma(u_h(s))h(s)\big] ds$

- For $\delta > 0$, $e^{-\delta A}$ maps H to V and V' to H. For $\delta > 0$

$$e^{-\delta A} u_h \in C([0,T],H) \text{ a.s.}$$

Set $G_\delta = Id - e^{-\delta A}$, apply Itô's formula to $|G_\delta u_h(t)|^2$
As $\delta \to 0$,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |G_\delta(u_h(t))|^2\Big) = 0$$

- Let $v \in C([0,T],H)$ be another solution, set $U = u_h - v$ and $\tau_N = \inf\{t \geq 0 : |u_h(t)| \geq N\} \wedge \inf\{t \geq 0 : |v(t)| \geq N\} \wedge T$
Apply Itô's formula for

$$\exp\Big(- a \int_0^{s\wedge\tau_N} \|u_h(r)\|_{\mathcal{H}}^4 dr\Big) |U(s\wedge\tau_N)|^2$$

Apply the extended Gronwall lemma

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Apply the extended Gronwall lemma

Some "weak" control of time increments

Given $M>0$, $N>0$, $h\in\mathcal{A}_M$, let u_h denote the solution to

$$du_h(t)+[Au_h(t)+B(u_h(t))+\tilde{R}(t,u_h(t))]\,dt=$$
$$\sigma(t,u_h(t))\,dW(t)+\tilde{\sigma}(t,u_h(t))\,h(t)\,dt$$

$$G_N(t)=\Big\{\omega\,:\,\Big(\sup_{0\le s\le t}|u_h(s)(\omega)|^2\Big)\vee\Big(\int_0^t\|u_h(s)(\omega)\|^2ds\Big)\le N\Big\}.$$

Lemma

Under the conditions of the well-posedeness theorem, if the initial condition $\xi\in L^4(\Omega;H)$, there exists a positive constant C such that for any $h\in\mathcal{A}_M$, if $\psi_n:[0,T]\to[0,T]$ is a Borel function with $s\le\psi_n(s)\le s+c2^{-n}$ or $s-c2^{-n}\le\psi_n(s)\le s$

$$I_n(h):=\mathbb{E}\Big[1_{G_N(T)}\int_0^T|u_h(s)-u_h(\psi_n(s))|^2\,ds\Big]\le C\,2^{-\frac{n}{2}}.$$

- Evolution equation perturbed by a "small" parameter ε

$$d\phi^\varepsilon + [A\phi^\varepsilon + B(\phi^\varepsilon) + R\phi^\varepsilon]\,dt = \sqrt{\varepsilon}\,\sigma(\phi^\varepsilon)dW(t),$$

$\phi(0) = \xi \in H$. Solution exists if $\varepsilon \leq \varepsilon_0$ for all K_i

- Prove a LDP as $\varepsilon \to 0$ in $X := C\big([0,T];H\big) \cap L^2\big((0,T);V\big)$

$$\|\phi\|_X = \Big\{ \sup_{0\leq s\leq T} |\phi(s)|^2 + \int_0^T \|\phi(s)\|^2 ds \Big\}^{\frac{1}{2}}.$$

For every closed (resp. open) set F (resp. G) of X:

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\phi^\varepsilon \in F) \leq -\inf\{I(\psi), \psi \in F\}.$$

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\phi^\varepsilon \in G) \geq -\inf\{I(\psi), \psi \in G\}.$$

with a good rate function $I : X \to [0,+\infty]$, i.e., level sets $\{\psi \in X : I(\psi) \leq M\}$ are compact subsets of X.

- Evolution equation perturbed by a "small" parameter ε

$$d\phi^\varepsilon + [A\phi^\varepsilon + B(\phi^\varepsilon) + R\phi^\varepsilon]\, dt = \sqrt{\varepsilon}\, \sigma(\phi^\varepsilon) dW(t),$$

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with a good rate function $I : X \to [0, +\infty]$, i.e., level sets $\{\psi \in X : I(\psi) \le M\}$ are compact subsets of X.

Let $h \in L^2([0,T],H_0)$; let $\phi_h = G^0(\int_0^\cdot h(s)ds) = \mathcal{G}^0(h)$ denote the deterministic controlled equation

$$d\phi_h(t) + \big[A\phi_h(t) + B(\phi_h(t)) + R\phi_h(t)\big]dt = \sigma(\phi_h(t))h(t)dt, \quad \phi_h(0) = \xi$$

Theorem

(Chueshov-M.) Let $\xi \in H$ and $K_2 = L_2 = 0$. The solution ϕ^ε of

$$d\phi^\varepsilon + [A\phi^\varepsilon + B(\phi^\varepsilon) + R\phi^\varepsilon]dt = \sqrt{\varepsilon}\,\sigma(\phi^\varepsilon)dW(t), \quad \phi^\varepsilon(0) = \xi \in H.$$

satisfies a LDP in $X = C([0,T];H) \cap L^2(0,T;V)$ with good r.f.

$$I_\xi(\psi) = \inf\{\|h\|^2_{L^2([0,T],H_0)}/2 \,:\, h \in L^2(0,T;H_0),\ \psi = \mathcal{G}^0(h)\}$$

Proved for 2D NS (Shritharan-Sundar), Boussinesq (Duan-M.), small perturbed shell models (Manna, Shritharan & Sundar)

The main step is the following:

Proposition

Suppose $K_2 = L_2 = 0$, let ξ be $\mathcal{F}_0$-measurable such that $E|\xi|_H^4 < +\infty$. Let h_ε converge to h_0 in distribution as random elements taking values in $\mathcal{A}_M$ (predictable elements which a.s.¡belong to the ball S_M of the RKHS), and endowed with the weak topology of $L_2(0,T;H_0)$. Then as $\varepsilon \to 0$, the solution u_{h_ε} of the stochastic controlled equation converges in distribution to the solution u_{h_0} of the controlled equation in $X = C([0,T];H) \cap L^2((0,T);V)$, where for $\varepsilon \geq 0$: $u_{h_\varepsilon}(0) = \xi$ and

$$du_{h_\varepsilon} + [Au_{h_\varepsilon} + B(u_{h_\varepsilon}) + \tilde{R}(t, u_{h_\varepsilon})]dt = \sigma(u_{h_\varepsilon})h_\varepsilon(t)dt + \sqrt{\varepsilon}\,\sigma(u_{h_\varepsilon})dW(t)$$

Let the positive viscosity coefficient $\nu \to 0$ and

$$d_t u^\nu(t) + \big[\nu A u^\nu(t) + B(u^\nu(t))\big]\, dt = \sqrt{\nu}\,\sigma(t, u^\nu(t))\, dW(t), u^\nu(0) = \xi$$

Prove exponential decay of $P(u^\nu(.) \in \Gamma)$ as $\nu \to 0$ for $\Gamma \subset Y$ that is

$$\lim_{\nu \to 0} \nu \ln P(u^\nu \in \Gamma)$$

in terms of some rate function and interior (resp. closure) of Γ for some **topology which is not the "optimal"** one

Why ? The rate function is formulated in terms of the "irregular" inviscid case, for h in the RKHS of the noise,

$$du_h^0(t) + B(u_h^0(t))\, dt = \sigma(t, u_h^0(t))\, h(t)\, dt\,, \quad u_h^0(0) = \xi$$

Requires some more hypothesis on σ with Radonifying operators (extend trace-class operators for non Hilbert Sobolev spaces)

One can **extend the stochastic calculus** (Itô's formula and BDG inequality) to Radonifying operators

Theorem

(Bessaih-M.) Let $\xi \in V$ *satisfy* $curl\ \xi = \partial_1 \xi_2 - \partial_2 \xi_1 \in L^\infty(D)$, $\sigma \in C(V; L_Q(H_0, V))$ *be such that* $curl\ \sigma \in C(H^{1,q}; R(H_0, L^q(D)))$ *with* $q > 2$ *satisfies "growth and Lipschitz conditions". Then as* $\nu \to 0$, *the distribution of the solution* u^ν *to*
$du_t^\nu + [\nu A u_t^\nu + B(u_t^\nu, u_t^\nu)]\, dt = \sqrt{\nu}\sigma(u_t^\nu)\, dW(t)$
with the initial condition $u_0^\nu = \xi$ *satisfies in* $\mathcal{X} = C([0,T]; L^q(D) \cap H)$ *endowed with the norm* $\|u\|_{\mathcal{X}} := \sup_{0 \le t \le T} |u_t|_q$ *satisfies a LDP with the good rate function*

$$I(u) = \inf\{\|h\|^2_{L^2([0,T],H_0)}/2 \ :\ u = u_h^0,\ h \in L^2(0,T; H_0\} \quad \textit{and}$$

and u_h^0 *is the unique solution to the control equation*

$$du_h^0(t) + B(u_h^0(t), u_h^0(t))\, dt = \sigma(u_h^0(t))\, h(t)\, dt,\ u_h^0(0) = \xi$$

Prove a "Stroock-Varadhan" theorem to characterize the support in $X = C([0,T];H) \cap L^2([0,T],V)$ of the distribution of general 2D hydrodynamical models

$$du(t)+\big[Au(t)+B\big(u(t)\big)+Ru(t)\big]\,dt=\sigma(u(t))\,dW(t),\quad u(0)=\xi\in H,$$

SPDE setting, for hyperbolic, wave, parabolic, Burgers, "mild solutions" in Hilbert spaces, similar results proved by Bally-M.-Sanz Solé, M.-Sanz Solé, Twardowska-Zabczyck, Cardon-Weber-M, Nakayama

Condition (R) Recall that $V \subset \mathcal{H} \subset H$

(i) $t\in[0,T]\mapsto \|u(t)\|_{\mathcal{H}}$ is continuous a.s.

(ii) there exists $q>0$ such that for any constant $C>0$ and $\tau_C := \inf\{t : \sup_{s\le t}|u(s)|^2 + \int_0^t \|u(s)\|^2 ds \ge C\}\wedge T$

$$\mathbb{E}\Big(\sup_{[0,\tau_C]} \|u(t)\|_{\mathcal{H}}^q\Big) < \infty$$

$\sigma : H \to L_Q(H_0, H)$, $(e_j, j \geq 1)$ CONS of H such that $Qe_j = q_j\, e_j$ with $\sum_j q_j = Trace(Q) < +\infty$, $\sigma_j : H \to H$ defined by $\sigma_j(u) := \sigma(u)e_j$, $\forall u \in H$

- For any j, σ_j twice (Fréchet) differentiable, with bounded derivatives.
- Stratonovich correction $\rho(u) = \sum_{j\geq 1} D\sigma_j(u)\,\sigma_j(u)$

Then if $\xi \in H$ and condition **(R)** holds, the support of the distribution of the solution u to

$$du(t) + [Au(t) + B(u(t)) + Ru(t)]\,dt = \sigma(u(t))dW(t),\ u_0 = \xi,$$

is the closure in X of $S(L^2([0,T], H_0))$ where $S(h)_0 = \xi$ and

$$dS(h)_t + [AS(h)_t + B(S(h)_t) + RS(h)_t]\,dt = \sigma(S(h)_t)h(t)dt - \frac{1}{2}\rho(S(h)_t)dt$$

The support characterization follows from one result of convergence in probability of some general sequence of evolution equations driven by W, a finite-dimensional, linear adapted time interpolation W^n of W and an element h of the RKHS of W. (Mackevičius, Aida Kusuoka & Stroock and M. & Sanz-Solé for diffusion processes)
In general, condition (R) holds for $\mathcal{H} = Dom(A^{\frac{1}{4}})$ when:

$$|\sigma(u)|^2_{L_Q} \leq K_0 + K_1|u|^2, \; |\sigma(u) - \sigma(v)|^2_{L_Q} \leq L|u-v|^2$$
$$|A^{\frac{1}{4}}\sigma(t,u)|^2_{L_Q(H_0,H)} \leq K(1 + \|u\|^2_{\mathcal{H}}), \; |A^{\frac{1}{4}} R(u)| \leq \bar{R}_0(1 + \|u\|_{\mathcal{H}}) \,.$$

In first examples, condition (R) holds:
• for 2D Navier-Stokes equation on periodic domains with no restriction if $(B(u), Au) = 0$
• for Boussinesq or 2D MHD models for $\mathcal{H} = Dom(A^{\frac{1}{4}}) \subset L^4(D)$
• for GOY or Sabra shell models for $\mathcal{H} = Dom(A^s)$ and $0 \leq s \leq \frac{1}{4}$

$$d_t u(t) + [B(u(t), u(t)) + \nabla p]dt = f(t, u) + \sigma(t, u(t))dW_t$$

with $\mathrm{div}\ u = 0$ in D, $\langle u, n\rangle = 0$ on ∂D

Theorem

(Brzezniak-Peszat) Suppose that the noise $W(t,x)$ is space homogeneous with RKHS H_0. Let $u_0 \in H^{1,q}$ for some $q > 2$,
$f : [0,T] \times H^{1,a} \to W^{1,a}$ for $a = 2, q$,
$\sigma : [0,T] \times H^{1,2} \to L_{\mathcal{HS}}(H_0, W^{1,2})$ and
$\sigma : [0,T] \times H^{1,2} \to Radonifying(H_0, W^{1,q})$.
Then there exists a triple (Ω, W, u) such that W is a has the imposed spectral measure (related to the covariance structure) and $u(0) = u_0$, for every $p \in [1,\infty)$, $u \in L^p\big(\Omega; L^\infty(0,T; H^{1,2} \cap H^{1,q})\big)$ and $u(t)$ satisfies the stochastic Euler equation.

No smoothing effect of the Stokes operator: viscosity $\nu = 0$

Remarks:

• stronger conditions on initial condition and diffusion coefficient

• Weak probabilistic solution : prove tightness of approximations with a viscosity coefficient $\nu \to 0$

• Use again $\langle B(u,u), u\rangle = 0$ and the equation satisfied by $curl u = \partial_1 u_2 - \partial_2 u_1$ with

$$\langle \mathrm{curl}\ B(u,v), \mathrm{curl}\ v|\mathrm{curl}\ v|^{q-2}\rangle = 0$$

for $u, v \in H^{2,q}$

Stochastic 3D NS equations

The coercivity argument used in the identification fails.
Spatially homogeneous noise with diffusion coefficient on $\mathbb{R}^3$ and use some weighted $L^p(a)$ spaces (with $a > \frac{3}{2}$)

$$\|f\|_{L^p(a)}^p = \int_{\mathbb{R}^3} |f(x)|^p (1+|x|^2)^{-a} dx$$

• Capinsky-Peszat: get rid on the pressure by "testing solution on appropriate functions" involving the weight. Give an initial distribution on $L^2(a)$ and prove the existence of a triple (Ω, W, u) such that for $a > \frac{3}{2}$, there exists a solution $u \in L^p(\Omega, L^\infty(0,T; L^2(a))$
They approximate the solution by an auxilliary equation u^ϵ and prove tightness

$$d_t u^\epsilon + \big[-\Delta u^\epsilon + B(u^\epsilon, u^\epsilon) + \epsilon |u^\epsilon|^4 u^\epsilon - \frac{1}{\epsilon} \nabla \mathrm{div}\; u^\epsilon \big] dt$$
$$= f(t, u^\epsilon) dt + \sigma(t, u^\epsilon) dW(t)$$

Stochastic 3D NS equations

• Basson has an **additive divergence free noise homogeneous noise** on $L^2(r)$ with $r > \frac{3}{2}$ and an initial spatially homogeneous distribution on $L^2(r)$ with null divergence.

He approximates by periodic solutions (proving the tightness)

He solves first the equation $d_t z(t) = \Delta z(t)dt + W(t)$ (which is divergence free) and then uses deterministic estimates for

$$d_t v(t) - \Delta v(t)dt + B(v(t) + z(t), v(t) + z(t))dt + \nabla p = 0$$

He proves energy estimates involving the pressure for a solution $u \in \cap_{b>3} L^\infty(0, T; L^2(b)) \cap L^2(0, T; H^{1,2}(a))$ for $b > \frac{3}{2}$