Radonifying Operators and Stochastic Integration

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Lévy Processes

- U, V Banach spaces
- $(L(t) : t \in [0,T]\ )$ Lévy process with values in U
- $N(t,\Lambda) := \sum_{s\in[0,t]} \mathbb{1}_\Lambda(\Delta(L(s)))$ for $\Lambda \in \mathfrak{B}(U)$ with $0 \notin \bar{\Lambda}$;

  $=$ number of jumps of L of size in Λ
- $\nu(\Lambda) := E[N(1,\Lambda)]$ for $\Lambda \in \mathfrak{B}(U)$ with $0 \notin \Lambda$ (Lévy measure)
- $M(t,\Lambda) := N(t,\Lambda) - t\nu(\Lambda)$ (compensated Poisson random measure)
- $B := \{u \in U : \|u\| \leqslant 1\}$ (ball in U)

Integration

Definition A function

$$F : [0, T] \times B \to V$$

is called *stochastically Pettis integrable of order α* if

(1) $\int_{[0,T]\times B} |\langle F(s,u), v^* \rangle|^2 \, \nu(du)ds < \infty$ for all $v^* \in V^*$.

(2) there exists a V-valued random variable Y with $E\,\|Y\|^\alpha < \infty$ s.t.

$$\langle Y, v^* \rangle = \int_{[0,T]\times B} \langle F(s,u), v^* \rangle \, M(ds, du)$$

for all $v^* \in V^*$.

Application: Lévy-Itô Decomposition

Theorem (SPA 2009)

For every Lévy Process $(L(t) : t \in [0,T])$ there exists

- $b \in U$
- Wiener process $(W(t) : t \in [0,T])$ in U

such that P-a.s.

$$L(t) = bt + W(t) + \underbrace{\int_{[0,T]\times B} u\, M(dt,du)}_{\text{Pettis integral}} + \underbrace{\int_{[0,T]\times B^c} u\, N(dt,du)}_{\text{Poisson sum}}$$

for all $t \in [0,T]$.

Excursion: Cylindrical Measures I

A linear mapping $T : V^* \to L^0(\Omega, P)$ is called cylindrical random variable.

For $v_1^*, \dots, v_n^* \in V^*$ and $C \in \mathfrak{B}(\mathbb{R}^n)$ define

$$Z := \{v \in V : (\langle v, v_1^* \rangle, \dots, \langle v, v_n^* \rangle) \in C\}.$$

Define a set function by

$$\mu_T(Z) := P\Big((Tv_1^*, \dots, Tv_n^*) \in C\Big).$$

Then μ_T is

- a set function on the set of all sets of the form Z.
- called cylindrical distribution of T.
- finite additive.
- in general not a measure on $\mathfrak{B}(U)$ but maybe extendable.

Excursion: Cylindrical Measures II

Definition For a cylindrical distribution μ the function

$$\varphi_\mu : V^* \to \mathbb{C}, \qquad \varphi_\mu(v^*) := \int_V e^{i\langle v, v^* \rangle} \, \mu(dv)$$

is called *characteristic function*.

Theorem (Levy continuity theorem)
For two cylindrical measures μ and ϱ the following are equivalent:

(a) $\mu = \varrho$;

(b) $\varphi_\mu = \varphi_\varrho$.

Excursion: Cylindrical Measures III

Theorem: (Bochner's Theorem)

Let $\varphi : V^* \to \mathbb{C}$ be a function. Then the following are equivalent:

(a) there exists a cylindrical distribution with characteristic function φ;

(b) the function φ satisfies:

(i) $\varphi(0) = 1$;

(ii) φ is postive definite;

(iii) φ is continuous on every finite-dimensional subspace $G \subseteq V^*$.

Factorising

For $F:[0,T]\times B\to V$ define the cylindrical random variable

$$Z:V^*\to L^0(\Omega,P),\qquad Zv^*:=\int_{[0,T]\times B}\langle F(s,u),v^*\rangle\, M(ds,du)$$

and let Q be its covariance operator

$$Q:V^*\to V,\qquad (Qv^*)(w^*):=E\big[(Zv^*)(Zw^*)\big],$$

where $V\subseteq V^{**}$. It follows that $Q=RR^*$ for an operator R with

$$V^*\ \xrightarrow{R^*}\ L^2([0,T]\times B,\nu\otimes\text{leb})\ \xrightarrow{R}\ V$$

and there exists a cylindrical measure m on $L^2([0,T]\times B,\nu\otimes\text{leb})$ s.t.

$$P_Z=m\circ R^{-1}$$

Conclusion

The relation $P_Z = m \circ R^{-1}$ results in:

Theorem: For a function $F : [0,T] \times B \to V$ the following are equivalent:

(a) F is stochastically integrable of order α, i.e.

$$\langle Y, v^* \rangle = \int_{[0,T]\times B} \langle F(s,u), v^* \rangle \, M(ds, du) = Z v^*$$

for a V-valued random variable Y with $E \|Y\|^\alpha < \infty$.

(b) $m \circ R^{-1}$ extends to a genuine measure with α-th moment.

Review: Gaussian case

Let $(W(t) : t \in [0,T])$ be a real-valued Wiener process and define

$$Z : V^* \to L^0(\Omega, P), \qquad Zv^* := \int_{[0,T]} \langle F(s), v^* \rangle \, W(ds).$$

Then Z is a cylindrical r.v. and its covariance operator satisfies

$$Q : V^* \to V, \qquad (Qv^*)(w^*) := E[(Zv^*)(Zw^*)],$$

where $V \subseteq V^{**}$. It follows that $Q = RR^*$ where

$$V^* \xrightarrow{R^*} L^2([0,T], \text{leb}) \xrightarrow{R} V$$

and for the canonical Gaussian cylindrical measure γ on $L^2([0,T], \text{leb})$ it holds

$$P_Z = \gamma \circ R^{-1}$$

Review: Gaussian case

Theorem: For a function $F : [0,T] \to V$ the following are equivalent:

(a) F is stochastically integrable of order α, i.e.

$$\langle Y, v^* \rangle = \int_{[0,T]} \langle F(s,u), v^* \rangle \, W(ds) = Zv^*$$

for V-valued random variable Y.

(b) $\gamma \circ R^{-1}$ extends to a genuine measure with α-th moment.

Review: canonical Gaussian cylindrical measure γ

Definition: Let H be a Hilbert space. The cylindrical distribution γ with characteristic function

$$\varphi : H \to \mathbb{C}, \qquad \varphi(h) := e^{-\frac{1}{2}\|h\|^2}$$

is called *canonical Gaussian cylindrical distribution*.

Well considered:

$$\{R : L^2([0,T], \text{leb}) \to V : \ \gamma \circ R^{-1} \text{extends to a measure}\}$$

is a Banach space, left and right ideal property,......

Back to Lévy processes

For $F : [0, T] \times B \to V$ define the cylindrical random variable

$$Z : V^* \to L^0(\Omega, P), \qquad Zv^* := \int_{[0,T]\times B} \langle F(s,u), v^* \rangle \, M(ds, du)$$

and let Q be its covariance operator

$$Q : V^* \to V, \qquad (Qv^*)(w^*) := E\big[(Zv^*)(Zw^*)\big],$$

where $V \subseteq V^{**}$. It follows that $Q = RR^*$ for an operator R with

$$V^* \xrightarrow{R^*} L^2([0,T] \times B, \nu \otimes \text{leb}) \xrightarrow{R} V$$

and there exists a cylindrical distribution m on $L^2([0,T] \times B, \nu \otimes \text{leb})$ s.t.

$$P_Z = m \circ R^{-1}$$

The canonical infinitely divisible cylindrical measure m

Theorem: Properties of the cylindrical distribution m:

(a) the characteristic function $\varphi_m : L^2([0,T] \times B,\, \nu \otimes \text{leb}) \to \mathbb{C}$:

$$\varphi_m(f) = \exp\left(\int_{[0,T]\times B} \left(e^{if(s,u)} - 1 - if(s,u) \right) \nu(du)ds \right).$$

(b) For every Lévy process the cylindrical distribution m is not σ-additive.

(c) Some more properties...

m-radonifying

Define the linear space and norms

$$\mathscr{R}_m^\alpha := \{R : L^2([0,T] \times B, \nu \otimes \text{leb}) \to V : m\text{-radonifying of order } \alpha\}$$

$$\|R\|_1 := \left(\int_V \|v\|^\alpha \, (m \circ R^{-1})(dv) \right)^{1/\alpha}$$

$$\|R\|_2 := \sup_{\|v^*\| \leqslant 1} \left(\int_{[0,T]\times B} |R^* v^*(t,u)|^2 \, \nu(du)dt \right)^{1/2}$$

Theorem:

The space $\mathscr{R}_m^\alpha$ with $\|\cdot\|_1 + \|\cdot\|_2$ is a Banach space.

Hilbert spaces

Theorem If V is a Hilbert space the following are equivalent:

(a) $R : L^2([0,T] \times B, \nu \otimes \mathsf{leb}) \to V$ is m-radonifying of order $\alpha \in [1,2]$;

(b) $R : L^2([0,T] \times B, \nu \otimes \mathsf{leb}) \to V$ is Hilbert-Schmidt;

(c) $\displaystyle\int_{[0,T]\times B} \|F(s,u)\|^2 \, \nu(du)ds < \infty$.

p-type Banach spaces

Definition A Banach space V is of type $p \in [1,2]$ if there exists a constant $C_p > 0$ such that:

$$X_1, \dots, X_n \ V\text{-valued, independent random variables,}$$

$$E\|X_k\|^p < \infty \text{ and } E\|X_k\| = 0$$

$$\Longrightarrow E\left\|\sum_{k=1}^{n} X_k\right\|^p \leqslant C_p \sum_{k=1}^{n} E\|X_k\|^p .$$

Examples: Hilbert spaces are of type 2

Every Banach space is of type 1

L^p is of type p for $p \in [1,2]$

l^p is of type p for $p \in [1,2]$

q-cotype Banach spaces

Definition A Banach space V is of cotype $q \in [2, \infty]$ if there exists a constant $C_q > 0$ such that:

$$X_1, \ldots, X_n \ V\text{-valued, independent random variables,}$$

$$E \|X_k\|^q < \infty \text{ and } E \|X_k\| = 0$$

$$\implies E \left\| \sum_{k=1}^{n} X_k \right\|^q \geqslant C_q \sum_{k=1}^{n} E \|X_k\|^q .$$

Examples: Every Hilbert space is of cotype 2

L^q is of cotype q for $q \in [2, \infty)$

l^q for of cotype q for $q \in [2, \infty)$

type and cotype Banach spaces

Theorem

(a) If V is a Banach space of type $p \in [1, 2]$ then

$$\int_{[0,T]\times B} \|F(t,u)\|^p \; \nu(du)dt < \infty$$

implies that R is m-radonifying of order p.

(b) If V is a Banach space of cotype $q \in [2, \infty)$ then

$$\int_{[0,T]\times B} \|F(t,u)\|^q \; \nu(du)dt < \infty$$

is necessary that R is m-radonifying of order q.

Evolution equation on Banach spaces

$$dX(t) = AX(t)\,dt + F\,dW(t) + \int_{[0,T]\times B} G(u)\,M(dt,du) + \int_{[0,T]\times B^c} H(u)\,N(dt,du)$$

$$X(0) = x_0$$

- A is generator of C_0-semigroup $(T(t))_{t\geqslant 0}$;
- $F : U \to V$ linear bounded operator
- $G : U \to V$ with $\int_{[0,T]\times B} \langle G(u), v^*\rangle\,\nu(du) < \infty$ for all $v^* \in V^*$;
- $H : U \to V$ measurable;
- $x_0 \in V$.

Evolution equation on Banach spaces

Definition:

A V-valued process $(Y(t) : t \geqslant 0)$ is called weak solution if P-a.s.

$$\langle Y(t), v^* \rangle = \langle y_0, v^* \rangle + \int_0^t \langle Y(s), A^* v^* \rangle \, ds + \langle FW(t), v^* \rangle + \int_{[0,t]\times B} \langle G(u), v^* \rangle \, M(ds, du) + \int_{[0,t]\times B^c} \langle H(u), v^* \rangle \, N(ds, du)$$

for every $v^* \in \mathrm{D}(A^*)$ and $t \geqslant 0$.

Evolution equation on Banach spaces

Theorem: The following are equivalent:

(a) there exists a weak solution $(Y(t) : t \geqslant 0)$;

(b) (i) $t \mapsto T(t)F$ is stochastically Pettis integrable with respect to W

(ii) $(t,u) \mapsto T(t)G(u)$ is stochastically Pettis integrable with respect to M

In this situation, the solution Y is represented P-a.s. by

$$\begin{aligned} Y(t) = T(t)y_0 &+ \int_0^t T(t-s)F\,W(ds) \\ &+ \int_{[0,t]\times D} T(t-s)G(u)\,M(ds,du) \\ &+ \int_{[0,t]\times\{u:\ \|u\|\geqslant 1\}} T(t-s)H(u)\,N(ds,du). \end{aligned}$$