Quadratic and Superquadratic BSDEs and Related PDEs

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I. Utility maximization

The financial market consists of one bond with interest rate zero and $d \leq m$ stocks. In case $d < m$ we face an incomplete market. The price process of stock i evolves according to the equation

$$\frac{dS_t^i}{S_t^i} = b_t^i dt + \sigma_t^i dB_t, \quad i = 1, \ldots, d, \tag{1}$$

where b^i (resp. σ^i) is a $\mathbb{R}-$ valued (resp. $\mathbb{R}^{1\times m}$–valued) stochastic process. The lines of the $d \times m$–matrix σ are given by the vector σ_t^i, $i = 1, \ldots, d$. The volatility matrix $\sigma = (\sigma^i)_{i=1,\ldots,d}$ has full rank (i.e. $\sigma\sigma^{tr}$ is invertible $\mathbb{P}$-a.s.) The predictable $\mathbb{R}^m$–valued process (called the risk premium) is defined by:

$$\theta_t = \sigma_t^{tr}(\sigma_t\sigma_t^{tr})^{-1}b_t, \qquad t \in [0, T].$$

A d–dimensional $\mathcal{F}_t$–predictable process $\pi = (\pi_t)_{0\leq t\leq T}$ is called trading strategy if $\int \pi \frac{dS}{S}$ is well defined, e.g. $\int_0^T \|\pi_t\sigma_t\|^2 dt < \infty$ P–a.s. For $1 \leq i \leq d$, the process π_t^i describes the amount of money invested in stock i at time t. The number of shares is $\frac{\pi_t^i}{S_t^i}$.

The wealth process X^π of a trading strategy π with initial capital x satisfies the equation

$$X_t^\pi = x + \sum_{i=1}^{d} \int_0^t \frac{\pi_{i,u}}{S_{i,u}} dS_{i,u} = x + \int_0^t \pi_u \sigma_u (dB_u + \theta_u du).$$

Suppose our investor has a liability F at time T.

Let us recall that for $\alpha > 0$ the exponential utility function is defined as

$$U(x) = -\exp(-\alpha x), \quad x \in \mathbb{R}.$$

We allow constraints on the trading strategies. Formally, they are supposed to take their values in a closed set, i.e. $\pi_t(\omega) \in \mathcal{C}$, with $\mathcal{C} \subseteq \mathbb{R}^{1\times d}$, and $0 \in \mathcal{C}$.

Definition

[**Admissible Strategies with constraints**] Let $\mathcal{C}$ be a closed set in $\mathbb{R}^{1\times d}$ and $0 \in \mathcal{C}$. The set of admissible trading strategies $\mathcal{A}_D$ consists of all d–dimensional predictable processes $\pi = (\pi_t)_{0\leq t\leq T}$ which satisfy $\int_0^T |\pi_t\sigma_t|^2\,dt < \infty$ and $\pi_t \in \mathcal{C}$ $\mathbb{P}$-a.s., as well as

$$\{\exp(-\alpha X^{\pi}_{\tau}) : \quad \tau \quad \text{stopping time with values in} \quad [0,T]\}$$

is a uniformly integrable family.

So the investor wants to solve the maximization problem

$$V(x) := \sup_{\pi\in\mathcal{A}_D} E\left[-\exp\left(-\alpha\left(x + \int_0^T \pi_t \frac{dS_t}{S_t} - F\right)\right)\right], \tag{2}$$

where x is the initial wealth. V is called value function.

This problem has been studied by many authors, but they suppose that the constraint is convex in order to apply convex duality. Our starting point of the work is the paper [Hu-Imkeller-Muller, AAP 2005] where both the risk premium θ and the liability F are bounded. The main method can be described as follows.

In order to find the value function and an optimal strategy one constructs a family of stochastic processes $R^{(\pi)}$ with the following properties:

- $R_T^{(\pi)} = -\exp(-\alpha(X_T^\pi - F))$ for all $\pi \in \mathcal{A}_D$;
- $R_0^{(\pi)} = R_0$ is constant for all $\pi \in \mathcal{A}_D$;
- $R^{(\pi)}$ is a supermartingale for all $\pi \in \mathcal{A}_D$ and there exists a $\pi^* \in \mathcal{A}_D$ such that $R^{(\pi^*)}$ is a martingale.

The process $R^{(\pi)}$ and its initial value R_0 depend of course on the initial capital x.

Given processes possessing these properties we can compare the expected utilities of the strategies $\pi \in \mathcal{A}_D$ and $\pi^* \in \mathcal{A}_D$ by

$$E[-\exp(-\alpha(X_T^{\pi} - F))] \leq R_0(x) = E[-\exp(-\alpha(X_T^{\pi^*} - F))] = V(x), \qquad (3)$$

whence π^* is the desired optimal strategy.

Construction of $R^{(\pi)}$:

$$R_t^{(\pi)} := -\exp(-\alpha(X_t^{(\pi)} - Y_t)), \qquad t \in [0, T],\ \pi \in \mathcal{A}_D,$$

where (Y, Z) is a solution of the BSDE

$$Y_t = F - \int_t^T Z_s dB_s + \int_t^T f(s, Z_s) ds, \quad t \in [0, T].$$

In these terms one is bound to choose a function f for which $R^{(\pi)}$ is a supermartingale for all $\pi \in \mathcal{A}_D$ and there exists a $\pi^* \in \mathcal{A}_D$ such that $R^{(\pi^*)}$ is a martingale. This function f also depends on the constraint set $(\mathcal{C})$ where (π_t) takes its values. One gets then

$$V(x) = R_0^{(\pi, x)} = -\exp(-\alpha(x - Y_0)), \qquad \text{for all } \pi \in \mathcal{A}_D.$$

In order to satisfy the supermartingale and the martingale properties, one finds

$$f(t,z)=\frac{\alpha}{2}\min_{\pi\in\mathcal{C}}|\pi\sigma-(z+\frac{1}{\alpha}\theta_t)|^2-z\theta_t-\frac{1}{2\alpha}|\theta_t|^2.$$

The function f is well defined because it only depends on the distance between a point and a closed set.

Important: the generator f is of quadratic growth with respect to z!

Lemma

Suppose that both the risk premium θ and the liability F are bounded. Then, the value function of the optimization problem (2) is given by

$$V(x) = -\exp(-\alpha(x - Y_0)),$$

where Y_0 is defined by the unique solution (Y, Z) of the BSDE

$$Y_t = F - \int_t^T Z_s dB_s + \int_t^T f(s, Z_s) ds, \quad t \in [0, T], \tag{4}$$

with

$$f(\cdot, z) = \frac{\alpha}{2} \min_{\pi \in \mathcal{C}} |\pi\sigma - (z + \frac{1}{\alpha}\theta)|^2 - z\theta - \frac{1}{2\alpha}|\theta|^2.$$

There exists an optimal trading strategy $\pi^* \in \mathcal{A}_D$ with

$$\pi_t^* \in argmin\{|\pi\sigma - (Z_t + \frac{1}{\alpha}\theta_t)|, \pi \in \mathcal{C}\}, \quad t \in [0, T],\ P - a.s. \tag{5}$$

II. Dynamic g Risk Measures (Barrieu-El Karoui, arXiv 2007)

Definition

Assume that (ξ_T, g) satisfies: (1) g is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$-measurable generator satisfying $z \to g(t,z)$ is convex, and

$$|g(t,z)| \leq |g(t,0)| + \frac{k}{2}|z|^2, \quad |g(t,0)|^{\frac{1}{2}} \in M;$$

(2) ξ_T is an $\mathcal{F}_{\mathcal{T}}$-measurable bounded random variable.

Define $\mathcal{R}^g(\xi_T)$ as the unique solution of the BSDE $(-\xi_T, g)$.

Proposition

(Barrieu-El Karoui) $\mathcal{R}^g$ is a dynamic risk measure.

Inf-convolution

In the case of bounded ξ, B-E established

Proposition

$$\mathcal{R}^{g^A \square g^B}(\xi_T)_t = \mathcal{R}^{g^A} \square \mathcal{R}^{g^B}(\xi_T)_t.$$

Backward Stochastic Differential Equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s \qquad (\mathrm{E}_{\xi,f})$$

- ξ is the terminal value : $\mathcal{F}_T$–measurable
- f is the generator

- (Y, Z) is the unknown
- (Y, Z) has to be adapted to $\mathcal{F}$

Pardoux–Peng, '90

If f is Lipschitz w.r.t. (y, z) and

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(s,0,0)|^2\, ds\right] < +\infty$$

$(\mathrm{E}_{\xi,f})$ has a unique square integrable solution.

Nonlinear Feynman-Kac Formula

Semilinear PDE (P)

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),(\nabla_x u)^{tr}\sigma(t,x)) = 0, \quad u(T,.) = g,$$
$$\mathcal{L}u(t,x) = \frac{1}{2}\mathrm{trace}(\sigma\sigma^*\nabla_x^2 u(t,x)) + b(t,x)\cdot\nabla_x u(t,x).$$

Linear part $\Longrightarrow$ SDE

$$X_t^{t_0,x_0} = x_0 + \int_{t_0}^t b\left(s, X_s^{t_0,x_0}\right)\, ds + \int_{t_0}^t \sigma\left(s, X_s^{t_0,x_0}\right)\, dB_s$$

Nonlinear part $\Longrightarrow$ BSDE (B)

$$Y_t^{t_0,x_0} = g\left(X_T^{t_0,x_0}\right) + \int_t^T f\left(s, X_s^{t_0,x_0}, Y_s^{t_0,x_0}, Z_s^{t_0,x_0}\right)\, ds - \int_t^T Z_s^{t_0,x_0}\, dB_s$$

Nonlinear Feynman-Kac Formula

If u is smooth solution to (P)

$$\left(u\left(t, X_t^{t_0,x_0}\right), (\nabla_x u)^{tr}\sigma\left(t, X_t^{t_0,x_0}\right)\right) \quad \text{solves the BSDE (B)}$$

Feynman-Kac's Formula

$u(t,x) := Y_t^{t,x}$ is a (viscosity) solution to (P).

Quadratic BSDEs

A real valued BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s \qquad (\mathrm{E}_{\xi,f})$$

- B is a Brownian motion in $\mathbf{R}^d$;
- ξ is $\mathcal{F}_T$–measurable;
- the generator $f : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R}^d \longrightarrow \mathbf{R}$ is measurable and
  - $(y, z) \longmapsto f(t, y, z)$ is continuous
  - f is quadratic with respect to z:

$$|f(t, y, z)| \leq \alpha(t) + \beta|y| + \frac{\gamma}{2}|z|^2$$

where $\beta \geq 0$, $\gamma > 0$ and α is a nonnegative process.

The bounded case

If ξ and α – or more generally $|\alpha|_1 := \int_0^T \alpha(s)\, ds$ – are bounded

- Existence
- Uniqueness, Comparison Theorem
- Stability

References:

- M. Kobylanski (AP 2000);
- M.-A. Morlais (non Brownian setting, Ph. D 2007)

These results yield

The Nonlinear Feynman-Kac Formula

Applications of bounded case

- Utility maximization: El Karoui & Rouge (MF 2000), Hu, Imkeller & Muller (AAP 2005) (with closed constraint), Mania & Schweizer (AAP 2005), Becherer (AAP 2006), Morlais (Ph D 2007)
- Stochastic linear quadratic control: Bismut (1970-1979), Peng, Kohlman & Tang, Hu & Zhou (with cone constraint SICON 2005), Schweizer et al.
- Quadratic g risk measure: Barrieu & El Karoui

The unbounded case

- Boundedness of ξ and α is not necessary to construct a solution;
- Exponential moment is enough !

Theorem

Existence of Solution [Briand & Hu, PTRF 2006] Let $\zeta := |\xi| + \int_0^T \alpha(s)\,ds$ *and let us assume that* $\mathbb{E}\left[\exp\left(\gamma e^{\beta T}\zeta\right)\right] < +\infty$.

Then, $(\mathrm{E}_{\xi,f})$ *has at least a solution such that*

$$|Y_t| \leq \frac{1}{\gamma} \log \mathbb{E}\left(\exp\left(\gamma e^{\beta T}\zeta\right) \mid \mathcal{F}_t\right). \quad (6)$$

Construction : $f \geq 0$, $\xi \geq 0$

(Y^n, Z^n) minimal solution

$$Y_t^n = \xi \wedge n + \int_t^T \mathbf{1}_{s \leq \sigma_n} f(s, Y_s^n, Z_s^n)\, ds - \int_t^T Z_s^n\, dB_s$$

$$\sigma_n = \inf \left\{ t \geq 0 : \int_0^t \alpha(s)\, ds \geq n \right\}$$

Step 1: a priori estimate

$$0 \leq Y_t^n \leq \frac{1}{\gamma} \mathbb{E} \left(\exp \left[\gamma e^{\beta T} \left(\xi + \int_0^T \alpha(s)\, ds \right) \right] \,\middle|\, \mathcal{F}_t \right)$$

Step 2: taking the limit in n: Difficult step: Localization procedure

The localization procedure

Main Idea: Work on the interval $[0, \tau_k]$ where

$$\tau_k = \inf \left\{ t \geq 0 : \frac{1}{\gamma} \mathbb{E} \left(\exp \left[\gamma e^{\beta T} \left(\xi + \int_0^T \alpha(s)\, ds \right) \right] \Big| \mathcal{F}_t \right) \geq k \right\} \wedge T$$

Set $\boldsymbol{Y_k^n(t) = Y_{t\wedge\tau_k}^n,\ Z_k^n(t) = 1_{t\leq\tau_k} Z_t^n}$

$$Y_k^n(t) = Y_{\tau_k}^n + \int_{t\wedge\tau_k}^{\tau_k} \mathbf{1}_{s\leq\sigma_n} f\left(s,\, Y_k^n(s), Z_k^n(s)\right) ds - \int_{t\wedge\tau_k}^{\tau_k} Z_k^n(t)\, dB_s$$

For fixed $\boldsymbol{k}$, $\quad (Y_k^n)_{n\in\mathbf{N}}$ is nondecreasing, $\qquad 0 \leq Y_k^n(t) \leq k$

The localization procedure

k fixed, $\lim_{n\to+\infty}$

$$Y_k(t) = \xi_k + \int_{t\wedge\tau_k}^{\tau_k} f(s, Y_k(s), Z_k(s))\, ds - \int_{t\wedge\tau_k}^{\tau_k} Z_k(s)\, dB_s, \quad \xi_k = \sup_{n\geq 1} Y^n_{\tau_k}$$

- By construction

$$Y_k(t) = Y_{k+1}(t\wedge\tau_k), \qquad Z_k(t) = \mathbf{1}_{t\leq\tau_k} Z_{k+1}(t)$$

- Define (Y, Z) by

$$Y_t = Y_k(t), \qquad Z_t = Z_k(t) \qquad \text{if } t \leq \tau_k$$

$$Y_t = \xi_k + \int_{t\wedge\tau_k}^{\tau_k} f(s, Y_s, Z_s)\, ds - \int_{t\wedge\tau_k}^{\tau_k} Z_s\, dB_s$$

- $k \longrightarrow +\infty$ gives a solution

Remarks

Questions

Uniqueness ?   Stability ?   Feynman-Kac formula ?

Answer

When f is convex (or concave) w.r.t. z

Motivation: Stochastic Control Problem (Fuhrman, Hu & Tessitore SICON 2006)

Controlled diffusion process

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)[dW_s + r(u_s)\, ds]$$

where u takes its values in a nonempty closed set C.

Minimize the cost functional

$$J(u) = \mathbb{E}\left[g(X_T) + \int_0^T G(t, X_t, u_t)\, dt\right]$$

over all the admissible controls u.

Motivation

Associated BSDE

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Z_s)\, ds - \int_t^T Z_s\, dB_s$$

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$$

$$f(t, x, z) = \inf\{G(t, x, u) + r(u)z : u \in C\}$$

Important feature of the generator

$$z \longmapsto f(t, x, z) \text{ is } \textbf{concave}$$

Assumptions (H)

There exist $\beta \geq 0$, $\gamma \geq 0$ and a nonnegative process α s.t. $\mathbb{P}$–a.s.

- f is Lipschitz w.r.t. y: for any t, z,

$$|f(t,y,z) - f(t,y',z)| \leq \beta\,|y - y'|;$$

- quadratic growth in z:

$$|f(t,y,z)| \leq \alpha(t) + \beta|y| + \frac{\gamma}{2}|z|^2;$$

- for any t, y, $z \longmapsto f(t,y,z)$ is a convex function;
- ξ is $\mathcal{F}_T$–measurable and

$$\forall \lambda > 0, \qquad \mathbb{E}\left[\exp\left(\lambda\left[|\xi| + \int_0^T \alpha(s)\,ds\right]\right)\right] < +\infty.$$

Some estimates

Proposition

$(\mathrm{E}_{\xi,f})$ *has a solution* (Y, Z) *s.t.*

$$\forall p \geq 1, \qquad \mathbb{E}\left[\sup_{t\in[0,T]} e^{p|Y_t|} + \left(\int_0^T |Z_s|^2\, ds\right)^{p/2}\right] \leq C$$

where C *depends only on* p, T *and the exponential moments of* $|\xi| + |\alpha|_1$.

- The estimate for Y comes directly from

$$|Y_t| \leq \frac{1}{\gamma} \log \mathbb{E}\left(\exp\left(\gamma e^{\beta T}(|\xi| + |\alpha|_1)\right) \;\middle|\; \mathcal{F}_t\right)$$

- For Z, standard computation starting from Itô's formula to

$$\frac{1}{\gamma^2}\left(e^{\gamma|Y_t|} - 1 - \gamma|Y_t|\right)$$

Comparison theorem

Theorem

Uniqueness of Solution [Briand & Hu, PTRF 2008] Let (Y,Z) *and* (Y',Z') *be solution to* $(\mathrm{E}_{\xi,f})$ *and* $(\mathrm{E}_{\xi',f'})$ *where* (ξ,f) *satisfies (H) and* Y, Y' *belongs to* $\mathcal{E}$ *(*$\mathcal{E}$ *:= exponential moment of all order).*

If $\xi \leq \xi'$ *and* $f \leq f'$ *then*

$$\forall t \in [0,T], \qquad Y_t \leq Y'_t$$

If moreover, $Y_t = Y'_t$ *then*

$$\mathbb{P}\left(\xi = \xi', \int_t^T f(s, Y'_s, Z'_s)\, ds = \int_t^T f'(s, Y'_s, Z'_s)\, ds\right) > 0.$$

In particular, $(\mathrm{E}_{\xi,f})$ *has a unique solution in the class* $\mathcal{E}$*.*

Main idea: Estimate of $Y_t - \mu Y'_t$ for $\mu \in (0,1)$.

Proof: f independent of y

Set, for $\mu \in (0,1)$, $U_t = Y_t - \mu Y'_t$, $V_t = Z_t - \mu Z'_t$.

$$U_t = U_T + \int_t^T F_s\,ds - \int_t^T V_s\,dB_s, \qquad F_s = f(s, Z_s) - \mu f'(s, Z'_s)$$

$$F_t = [f(t, Z_t) - \mu f(t, Z'_t)] + \mu [f(t, Z'_t) - f'(t, Z'_t)]$$

and $\delta f(t) := f(t, Z'_t) - f'(t, Z'_t) \leq 0$.

$$Z_t = \mu Z'_t + (1-\mu)\frac{Z_t - \mu Z'_t}{1-\mu}$$

$$\begin{aligned} f(t, Z_t) &= f\left(t, \mu Z'_t + (1-\mu)\frac{Z_t - \mu Z'_t}{1-\mu}\right) \\ \textbf{Convexity} \quad &\leq \mu f(t, Z'_t) + (1-\mu) f\left(t, \frac{Z_t - \mu Z'_t}{1-\mu}\right) \end{aligned}$$

$$f(t, Z_t) - \mu f(t, Z'_t) \le (1-\mu) f\left(t, \frac{V_t}{1-\mu}\right) \le (1-\mu)\alpha(t) + \frac{\gamma}{2(1-\mu)}|V_t|^2$$

$$F_t \le \mu \delta f(t) + (1-\mu)\alpha(t) + \frac{\gamma}{2(1-\mu)}|V_t|^2 \qquad (7)$$

Second step

An exponential change of variable to remove the quadratic term

$$P_t = e^{cU_t}, \qquad Q_t = cP_t V_t, \qquad c \ge 0$$

$$P_t = P_T + c\int_t^T P_s\left(F_s - \frac{c}{2}|V_s|^2\right) ds - \int_t^T Q_s\, dB_s$$

$c = \frac{\gamma}{1-\mu}$ yields

$$P_t \leq \mathbb{E}\left(\exp\left[\gamma \int_t^T \left(\alpha(s) + (1-\mu)^{-1}\mu\delta f(s)\right)\, ds\right] P_T \,\Big|\, \mathcal{F}_t\right)$$

$$P_T = \exp\left(\frac{\gamma}{1-\mu}(\xi - \mu\xi')\right) = \exp\left(\gamma\left(\xi + \frac{\mu}{1-\mu}\delta\xi\right)\right)$$

$$P_t \leq \mathbb{E}\left(\exp\left[\gamma\left(\xi + \int_0^T \alpha(s)\, ds\right) + \gamma\frac{\mu}{1-\mu}\left(\delta\xi + \int_t^T \delta f(s)\, ds\right)\right] \,\Big|\, \mathcal{F}_t\right)$$

In particular,

$$Y_t - \mu Y_t' \leq \frac{1-\mu}{\gamma}\log\mathbb{E}\left(\exp\left[\gamma\left(\xi + \int_0^T \alpha(s)\, ds\right)\right] \,\Big|\, \mathcal{F}_t\right)$$

and sending μ to 1, we get

$$Y_t - Y_t' \leq 0.$$

Strict Comparison

If $Y_t = Y'_t$, then $P_t = e^{\gamma Y_t}$ and

$$0 < \mathbb{E}[P_t] \leq \mathbb{E}\left[\exp\left(\gamma\left(\xi + \int_0^T \alpha(s)\,ds\right) + \gamma\frac{\mu}{1-\mu}\left(\delta\xi + \int_t^T \delta f(s)\,ds\right)\right]\right)$$

Sending μ to 1,

$$0 < \mathbb{E}\left[\exp\left(\gamma\left[\xi + \int_0^T \alpha(s)\,ds\right]\right)\mathbf{1}_{\delta\xi + \int_t^T \delta f(s)\,ds = 0}\right]$$

Press if late

The general case

$$F_t = f(t, Y_t, Z_t) - \mu f'(t, Y_t', Z_t') = f(t, Y_t, Z_t) - \mu f(t, Y_t', Z_t') + \mu \delta f(t)$$

$$\begin{aligned} & f(t, Y_t, Z_t) - \mu f\left(t, Y_t', Z_t'\right) \\ & \quad = \quad f(t, Y_t, Z_t) - \mu f\left(t, Y_t, Z_t'\right) + \mu\left(f\left(t, Y_t, Z_t'\right) - f\left(t, Y_t', Z_t'\right)\right). \end{aligned}$$

Convexity

$$f(t, Y_t, Z_t) - \mu f\left(t, Y_t, Z_t'\right) \leq (1-\mu)(\alpha(t) + \beta|Y_t|) + \frac{\gamma}{2(1-\mu)}|V_t|^2$$

Linearization: $a(t) = \left(Y_t - Y_t'\right)^{-1}\left(f\left(t, Y_t, Z_t'\right) - f\left(t, Y_t', Z_t'\right)\right) \mathbf{1}_{Y_t - Y_t' \neq 0}$

$$\mu\left(f\left(t, Y_t, Z_t'\right) - f\left(t, Y_t', Z_t'\right)\right) = \mu a(t)\left(Y_t - Y_t'\right) \leq a(t) U_t + (1-\mu)\beta|Y_t|$$

$$F_t \leq \mu \delta f(t) + (1-\mu)(\alpha(t) + 2\beta|Y_t|) + \frac{\gamma}{2(1-\mu)}|V_t|^2 + a(t) U_t.$$

The general case

Set $E_t = \exp\left(\int_0^t a(s)\, ds\right)$, $\widetilde{U}_t = E_t\, U_t$ and $\widetilde{V}_t = E_t\, V_t$. Then,

$$\widetilde{U}_t = \widetilde{U}_T + \int_t^T \widetilde{F}_s\, ds - \int_t^T \widetilde{V}_s\, dB_s$$

with, since $|a(t)| \leq \beta$,

$$\widetilde{F}_t \leq \mu E_t\, \delta f(t) + (1-\mu) E_t(\alpha(t) + 2\beta |Y_t|) + \frac{\gamma e^{\beta T}}{2(1-\mu)} \left|\widetilde{V}_t\right|^2$$

This is the same inequality as before.

Stability

Assume that (ξ_n, f_n) satisfies (H) with α_n, β, γ and

$$\forall \lambda > 0, \qquad \sup_{n\geq 1} \mathbb{E}\left[\exp\left\{\lambda\left(|\xi_n| + |\alpha_n|_1\right)\right\}\right] < +\infty.$$

Theorem

If $\xi_n \longrightarrow \xi$ $\mathbb{P}$–*p.s. and* $dt \otimes d\mathbb{P}$–*a.e.*, $\forall (y, z)$, $f_n(t, y, z) \longrightarrow f(t, y, z)$, *then*

$$\forall p \geq 1, \qquad \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t - Y_t^n|^p + \left(\int_0^T |Z_s - Z_s^n|^2 \, ds\right)^{p/2}\right] \longrightarrow 0.$$

Proof.

Same method as in the proof of comparison theorem to

$$Y_t - \mu Y_t^n, \qquad Y_t^n - \mu Y_t$$

□

Application to PDEs

- Probabilistic representation for

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),(\nabla_x u)^{tr}\sigma(t,x)) = 0, \quad u(T,.) = g,$$

$$\mathcal{L}u(t,x) = \frac{1}{2}\mathrm{trace}(\sigma\sigma^*\nabla_x^2 u(t,x)) + b(t,x)\cdot\nabla_x u(t,x).$$

- The SDE: X^{t_0,x_0} solution to

$$X_t = x_0 + \int_{t_0}^t b(s,X_s)\,ds + \int_{t_0}^t \sigma(s,X_s)\,dB_s$$

- The BSDE: $(Y^{t_0,x_0}, Z^{t_0,x_0})$ solution to

$$Y_t = g\left(X_T^{t_0,x_0}\right) + \int_t^T f\left(s, X_s^{t_0,x_0}, Y_s, Z_s\right)\,ds - \int_t^T Z_s\,dB_s$$

- Nonlinear Feynman-Kac formula: $u(t,x) := Y_t^{t,x}$ is a viscosity solution

Assumptions

- b, σ, f and g are continuous;
- b, σ Lipschitz w.r.t. x

$$|b(t,x) - b(t,x')| + |\sigma(t,x) - \sigma(t,x')| \leq \beta|x - x'|;$$

- restriction: σ is bounded;
- f is Lipschitz w.r.t. y

$$|f(t,x,y,z) - f(t,x,y',z)| \leq \beta|y - y'|;$$

- $z \longmapsto f(t,x,y,z)$ is convex;
- $\exists p < 2$ s.t.

$$|g(x)| + |f(t,x,y,z)| \leq C\left(1 + |x|^p + |y| + |z|^2\right).$$

First properties

Proposition

$u(t,x) := Y_t^{t,x}$ is continuous and

$$|u(t,x)| \le C\left(1+|x|^p\right).$$

Proof.

- Since σ is bounded,

$$\forall \lambda > 0, \qquad \mathbb{E}\left[\exp\left(\lambda \sup_{t\in[0,T]} |X_t^{t_0,x_0}|^p\right)\right] \le e^{C(1+|x|^p)}$$

- $$\text{a priori estimate } |u(t,x)| \le C\left(1+|x|^p\right)$$

- Stability $\Longrightarrow$ Continuity

□

u is a viscosity solution

Definition

A continuous function u s.t. $u(T,\cdot) = g$ is a viscosity subsolution (**supersolution**) if, whenever $u - \varphi$ has a local maximum (**minimum**) at (t_0, x_0) where φ is $\mathcal{C}^{1,2}$,

$$\partial_t \varphi(t_0, x_0) + \mathcal{L}\varphi(t_0, x_0) + f(t_0, x_0, u(t_0, x_0), (\nabla_x u)^{tr} \sigma(t_0, x_0)) \geq 0, \quad (\leq 0)$$

Solution = Subsolution + Supersolution

Proposition

$u(t,x) := Y_t^{t,x}$ is a viscosity solution to the PDE.

Proof.

Markov property : $u(t, X_t^{t_0,x_0}) = Y_t^{t_0,x_0}$, and Comparison theorem

□

Extension and Open Questions

- Weaken the integrability assumptions Delbaen, Hu and Richou (Arxiv 2009): Uniqueness holds among solutions which admit some given exponential moments. These exponential moments are natural as they are given by the existence theorem.
- Open Question 1: Prove uniqueness and stability without convexity

$$|f(t, y, z) - f(t, y, z')| \leq C|z - z'|(1 + |z| + |z'|)$$

- Open Question 2: Multi-dimensional quadratic BSDEs and system of quadratic PDEs.

Superquadratic BSDEs (joint work with Delbaen and Bao)

(Arxiv 2009, to appear in PTRF).

Let us consider the following BSDE:

$$Y_t = \xi - \int_t^T g(Z_s)\,ds + \int_t^T Z_s\,dB_s, \tag{8}$$

where g is convex with $g(0) = 0$, and is superquadratic, i.e.

$$\limsup_{z\to\infty} \frac{g(z)}{|z|^2} = \infty;$$

and ξ is a bounded $\mathcal{F}_T$-measurable random variable.

The goal here is to look for a solution (Y, Z) such that Y is a bounded process.

Non-existence of solution

Different from BSDEs with quadratic growth, the bounded solution to the BSDE with superquadratic growth does not always exist.

Theorem

(Non-existence) There exists $\eta \in L^\infty(\mathcal{F}_T)$ such that BSDE (8) with sup-quadratic growth has no bounded solution.

Non-uniqueness of solution

Even if the BSDE has a bounded solution, the solutions are not unique. The main reason is that the generator g is superquadratic which makes $\int_0^t g(Z_r)dr$ grow much faster that $\int_0^t Z_r dB_r$ with respect to Z. Following this observation, we can construct other solutions.

Theorem

(Non-uniqueness) If the BSDE (g, ξ) with superquadratic growth has a bounded solution Y for some $\xi \in L^\infty(\mathcal{F}_T)$, then for each $y < Y_0$, there are infinitely many bounded solutions X with $X_0 = y$.

Existence of solution to BSDE in Markovian case

The BSDE with superquadratic growth is ill-posed. However, in the particular Markovian case, solutions to BSDE exist.

Define the diffusion process $X^{t,x}$ be the solution to the SDE:

$$X_s = x + \int_t^s b(r, X_r)\,dr + \int_t^s \sigma\,dB_r, t \le s \le T, \tag{9}$$

where b is Lipschitz with respect to x, and σ is a constant (matrix).

Let us consider the BSDE (8) with $\xi = \Phi(X_T^{t,x})$:

$$Y_s = \Phi(X_T^{t,x}) - \int_s^T g(Z_r)\,dr + \int_s^T Z_r\,dB_r.$$

Existence of Solution

Theorem

Let us suppose that Φ is bounded and continuous. Then there exists a solution (Y, Z) to Markovian BSDE.

Main tool: if Φ is smooth, we can get an estimate in the spirit of Gilding et al. by use of some martingale method:

$$|Z_t| \leq C||\Phi||_\infty (T-t)^{-\frac{1}{2}}.$$

Finally, we can prove that $u(t,x) = Y_t^{t,x}$ is a viscosity solution of the corresponding PDE.

Remark: Cheridito and Stadje (Arxiv 2010): No discrete convergence for quadratic BSDEs.

Richou (Ph D 2010): numerical simulation applying the above estimate.