Utility maximization problem under model uncertainty including jumps

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Plan de l'exposé

Plan

1. Bordigoni G., M. A., Schweizer, M. : A Stochastic control approach to a robust utility maximization problem. *Stochastic Analysis and Applications. Proceedings of the Second Abel Symposium, Oslo, 2005*, *Springer*, *125-151* *(2007)*.
2. Faidi, W., M.,A., Mnif, M. : Maximization of recursive utilities : A Dynamic Programming Principle Approach. Preprint (2010).
3. Jeanblanc, M., M. A., Ngoupeyou, A. : Robust utility maximization in a discontinuous filtration. Preprint (2010).

Problem

We present a problem of *utility maximization* under *model uncertainty* :

$$\sup_{\pi} \inf_{\mathbb{Q}} \mathbf{U}(\pi, \mathbb{Q}),$$

where

- π runs through a set of strategies (portfolios, investment decisions, ...)
- $\mathbb{Q}$ runs through a set of models $\mathcal{Q}$.

ONE KNOWN MODEL CASE

- If we have a one known model $\mathbb{P}$: in this case, $\mathcal{Q} = \{\mathbb{P}\}$ for $\mathbb{P}$ a given reference probability measure and $\mathbf{U}(\pi, \mathbb{P})$ has the form of a $\mathbb{P}$-expected utility from terminal wealth and/or consumption, namely

$$\mathbf{U}(\pi, \mathbb{P}) = \mathbb{E}\big(U(X_T^\pi)\big)$$

  where
- X^π is the wealth process

  and
- U is some utility function.

REFERENCES : DUAL APPROACH

- Schachermayer (2001) (one single model)
- Becherer (2007) (one single model)
- Schied (2007), Schied and Wu (2005)
- Föllmer and Gundel, Gundel (2005)

References : BSDE approach

- El Karoui, Quenez and Peng (2001) : Dynamic maximum principle (one single model)
- Hu, Imkeller and Mueller (2001) (one single model)
- Barrieu and El Karoui (2007) : Pricing, Hedging and Designing Derivatives with Risk Measures (one single model)
- Lazrak-Quenez (2003), Quenez (2004), $\mathcal{Q} \neq \{\mathbb{P}\}$ but one keep $\mathbf{U}(\pi, \mathbb{Q})$ as an expected utility
- Duffie and Epstein (1992), Duffie and Skiadas (1994), Skiadas (2003), Schroder & Skiadas (1999, 2003, 2005) : Stochastic Differential Utility and BSDE.
- Hansen & Sargent : they discuss the problem of robust utility maximization when model uncertainty is penalized by a relative entropy term.

Example : robust control without maximization

- Let us consider an agent with time-additive expected utility over consumptions paths :

$$\mathbb{E}[\int_0^T e^{-\delta t} u(c_t) dt].$$

with respect to some model $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, (B_t)_{t \geq 0})$ where $(B_t)_{t \geq 0}$ is Brownian motion under $\mathbb{P}$.

- Suppose that the agent has some preference to use another model $\mathbb{P}^\theta$ under which :

$$B_t^\theta = B_t - \int_0^t \theta_s ds$$

is a Brownian motion.

EXAMPLE

- The agent evaluate the distance between the two models in term of the relative entropy of $\mathbb{P}^\theta$ with respect to the reference measure $\mathbb{P}$:

$$\mathcal{R}^\theta = \mathbb{E}^\theta \Big[\int_0^T e^{-\delta t} |\theta_t|^2 dt \Big]$$

- In this example, our robust control problem will take the form :

$$V_0 := \inf_\theta \Big[\mathbb{E}^\theta \Big[\int_0^T e^{-\delta t} u(c_t) dt \Big] + \beta \mathcal{R}^\theta \Big].$$

- The answer of this problem will be that : $V_0 = Y_0$ where Y is solution of BSDE or recursion equation :

$$Y_t = \mathbb{E} \Big[\int_t^T e^{-\delta(s-t)} \big(u(c_s) ds - \frac{1}{2\beta} d\langle Y \rangle_s \big) \Big| \mathcal{F}_t \Big],$$

- This an example of Stochatic differential utility (SDU) introduced by Duffie and Epstein (1992).

Plan

Preliminary and Assumptions

Let us given :

- Final horizon : $T < \infty$
- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space where $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is a filtration satisfying the usual conditions of right-continuity and $\mathbb{P}$-completness.
- Possible scenarios given by

  $$\mathcal{Q} := \{\mathbb{Q} \text{ probability measure on } \Omega \text{ such that } \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{F}_T\}$$

- the density process of $\mathbb{Q} \in \mathcal{Q}$ is the càdlàg P-martingale

  $$Z_t^{\mathbb{Q}} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathbb{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}} \Big| \mathcal{F}_t\Big]$$

- we may identify $Z^{\mathbb{Q}}$ with $\mathbb{Q}$.
- Discounting process : $S_t^\delta := \exp(-\int_0^t \delta_s \, ds)$ with a discount rate process $\delta = \{\delta_t\}_{0 \leq t \leq T}$.

PRELIMINARY

- Let $\mathcal{U}^{\delta}_{t,T}(\mathbb{Q})$ be a quantity given by

$$\mathcal{U}^{\delta}_{t,T}(\mathbb{Q}) = \int_t^T e^{-\int_t^s \delta_r\,dr} U_s\,ds + e^{-\int_t^T \delta_r\,dr}\overline{U}_T$$

- where $U = (U_t)_{t\in[0,T]}$ is a utility rate process which comes from consumption and $\overline{U}_T$ is the terminal utility at time T which corresponds to final wealth.
- Let $\mathcal{R}^{\delta}_{t,T}(\mathbb{Q})$ be a penalty term

$$\mathcal{R}^{\delta}_{t,T}(\mathbb{Q}) = \int_t^T \delta_s e^{-\int_t^s \delta_r\,dr} \log\frac{Z_s^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}}\,ds + e^{-\int_t^T \delta_r\,dr} \log\frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}}.$$

for $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_T$.

COST FUNCTIONAL

- We consider the cost functional

$$c(\omega, \mathbb{Q}) := \mathcal{U}^{\delta}_{0,T}(\mathbb{Q}) + \beta \mathcal{R}^{\delta}_{0,T}(\mathbb{Q}) \; .$$

  with $\beta > 0$ is a constant which determines the strength of this penalty term.
- Our first goal is to

$$\text{minimize the functional} \quad \mathbb{Q} \longmapsto \Gamma(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}\left[c(., \mathbb{Q})\right]$$

  over a suitable class of probability measures $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_T$.

Relative entropy

- Under the reference probability $\mathbb{P}$ the cost functional $\Gamma(\mathbb{Q})$ can be written :

$$\Gamma(\mathbb{Q}) = \mathbb{E}^{\mathbb{P}}\left[Z_T^{\mathbb{Q}}\Big(\int_0^T S_s^{\delta} U_s\, ds + S_T^{\delta}\overline{U}_T\Big)\right]$$
$$+\beta\mathbb{E}^{\mathbb{P}}\left[\int_0^T \delta_s S_s^{\delta} Z_s^{\delta} \log Z_s^{\mathbb{Q}} ds + S_T^{\delta} Z_T^{\mathbb{Q}} \log Z_T^{\mathbb{Q}}\right].$$

- The second term is a discounted relative entropy with both an entropy rate as well a terminal entropy :

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}\left[\log Z_T^{\mathbb{Q}}\right], & \text{if } \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{F}_T \\ +\infty, & \text{if not} \end{cases}$$

FUNCTIONAL SPACES

- $L^{\exp}$ is the space of all $\mathcal{G}_T$-measurable random variables X with

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\gamma|X|\right)\right] < \infty \qquad \text{for all } \gamma > 0$$

- D_0^{exp} is the space of progressively measurable processes $y = (y_t)$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\exp\left(\gamma \operatorname{ess\,sup}_{0\le t\le T}|y_t|\right)\Big] < \infty, \quad \text{for all } \gamma > 0\,.$$

- D_1^{exp} is the space of progressively measurable processes $y = (y_t)$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\exp\left(\gamma \int_0^T |y_s|\, ds\right)\Big] < \infty \quad \text{for all } \gamma > 0\,.$$

FUNCTIONAL SPACES AND HYPOTHESES (I)

- $\mathcal{M}^p(\mathbb{P})$ is the space of all $\mathbb{P}$-martingales $M = (M_t)_{0\leq t\leq T}$ such that $\mathbb{E}^{\mathbb{P}}(\sup_{0\leq t\leq T} |M_t|^p) < \infty$.

- **Assumption (A)** : $0 \leq \delta \leq \|\delta\|_\infty < \infty$, $U \in D_1^{exp}$ and $\overline{U}_T \in L^{\text{exp}}$.
- Denote by $\mathcal{Q}_f$ is the space of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{G}_T)$ with $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{G}_T$ and $H(\mathbb{Q}|\mathbb{P}) < +\infty$, then :
- For simplicity we will take $\beta = 1$.

THEOREM (BORDIGONI G., M. A., SCHWEIZER, M.)

There exists a unique $\mathbb{Q}^*$ which minimizes $\Gamma(\mathbb{Q})$ over all $\mathbb{Q} \in \mathcal{Q}_f$:

$$\Gamma(\mathbb{Q}^*) = \inf_{\mathbb{Q}\in\mathcal{Q}_f} \Gamma(\mathbb{Q})$$

Furthermore, $\mathbb{Q}^*$ is equivalent to $\mathbb{P}$.

THE CASE : $\delta = 0$

- The spacial case $\delta = 0$ corresponds to the cost functional

$$\Gamma(Q) = \mathbb{E}^{\mathbb{Q}}\left[\mathcal{U}^0_{0,T}\right] + \beta H(\mathbb{Q}|\mathbb{P}) = \beta H(\mathbb{Q}|\mathbb{P}_{\mathcal{U}}) - \beta \log \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\frac{1}{\beta}\mathcal{U}^0_{0,T}\right)\right].$$

where $\mathbb{P}_{\mathcal{U}} \approx \mathbb{P}$ and $\dfrac{d\mathbb{P}_{\mathcal{U}}}{d\mathbb{P}} = c \exp\left(-\dfrac{1}{\beta}\mathcal{U}^0_{0,T}\right)$.

- Csiszar (1997) have proved the existence and uniqueness of the optimal measure $\mathbb{Q}^* \approx \mathbb{P}_{\mathcal{U}}$ which minimize the relative entropy $H(\mathbb{Q}|\mathbb{P}_{\mathcal{U}})$.

- I. Csiszár : *I*-divergence geometry of probability distributions and minimization problems. *Annals of Probability* **3**, p. 146-158 (1975).

PLAN

DYNAMIC STOCHASTIC CONTROL PROBLEM

We embed the minimization of $\Gamma(Q)$ in a stochastic control problem :

- The minimal conditional cost

$$J(\tau,\mathbb{Q}) := \mathbb{Q} - \text{ess inf}_{\mathbb{Q}'\in\mathcal{D}(\mathbb{Q},\tau)}\Gamma(\tau,\mathbb{Q}')$$

  with $\Gamma(\tau,\mathbb{Q}) := \mathbb{E}_Q\left[c(\cdot,\mathbb{Q})\,|\,\mathcal{F}_\tau\right]$,

- $\mathcal{D}(\mathbb{Q},\tau) = \{Z^{\mathbb{Q}'}\,|\,\mathbb{Q}'\in\mathcal{Q}_f \text{ et } \mathbb{Q}'=\mathbb{Q} \text{ sur } \mathcal{F}_\tau\}$ and $\tau\in\mathcal{S}$.

- So, we can write our optimization problem as

$$\inf_{\mathbb{Q}\in\mathcal{Q}_f}\Gamma(\mathbb{Q}) = \inf_{\mathbb{Q}\in\mathcal{Q}_f}\mathbb{E}^{\mathbb{Q}}\left[c(\cdot,\mathbb{Q})\right] = \mathbb{E}^{\mathbb{P}}\left[J(0,\mathbb{Q})\right].$$

- We obtain the following martingale optimality principle from stochastic control :

We have obtained by following El Karoui (1981) :

Proposition (Bordigoni G., M. A., Schweizer, M.)

1. *The family* $\{J(\tau, \mathbb{Q}) \,|\, \tau \in \mathcal{S}, \mathbb{Q} \in \mathcal{Q}_f\}$ *is a submartingale system ;*
2. $\tilde{\mathbb{Q}} \in \mathcal{Q}_f$ *is optimal if and only if* $\{J(\tau, \tilde{\mathbb{Q}}) \,|\, \tau \in \mathcal{S}\}$ *is a* $\tilde{\mathbb{Q}}$*-martingale system ;*
3. *For each* $\mathbb{Q} \in \mathcal{Q}_f$*, there exists an adapted RCLL process* $J^{\mathbb{Q}} = (J_t^{\mathbb{Q}})_{0 \le t \le T}$ *which is a right closed* $\mathbb{Q}$*-submartingale such that*

$$J_\tau^{\mathbb{Q}} = J(\tau, \mathbb{Q})$$

SEMIMARTINGALE DECOMPOSITION OF THE VALUE PROCESS

- We define for all $\mathbb{Q}' \in \mathcal{Q}_f^e$ and $\tau \in \mathcal{S}$:

$$\tilde{V}(\tau, \mathbb{Q}') := \mathbb{E}^{\mathbb{Q}'}\left[\mathcal{U}^{\delta}_{\tau,T} \,|\mathcal{F}_\tau\right] + \beta\mathbb{E}_{\mathbb{Q}'}\left[\mathcal{R}^{\delta}_{\tau,T}(\mathbb{Q}') \,|\mathcal{F}_\tau\right]$$

- The value of the control problem started at time τ instead of 0 is :

$$V(\tau, \mathbb{Q}) := \mathbb{Q} - \text{ess inf}_{\mathbb{Q}' \in \mathcal{D}(\mathbb{Q},\tau)} \tilde{V}(\tau, \mathbb{Q}')$$

- So we can equally well take the ess inf under $\mathbb{P} \approx \mathbb{Q}$ and over all $\mathbb{Q}' \in \mathcal{Q}_f$ and $V(\tau) \equiv V(\tau, Q')$ and one proves that V is $\mathbb{P}$-special semimartingale with canonical decomposition

$$V = V_0 + M^V + A^V$$

SEMIMARTINGALE BSDE : CONTINUOUS FILTRATION CASE

- We assume tha $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ is continuous.
- Let first consider the following quadratic semimartingale BSDE with :

DEFINITION (BORDIGONI G., M. A., SCHWEIZER, M.)

A solution of the BSDE is a pair of processes (Y, M) such that Y is a $\mathbb{P}$-semimartingale and M is a locally square-integrable locally martingale with $M_0 = 0$ such that :

$$\begin{cases} -dY_t = (U_t - \delta_t Y_t)dt - \dfrac{1}{2\beta} d < M >_t \ - dM_t \\ Y_T = \overline{U}_T \end{cases}$$

- *Note that Y is then automatically $\mathbb{P}$-special, and that if M is continuous, so is Y.*

BSDE : Brownian filtration

Remark

- *If $\mathbb{F} = \mathbb{F}^W$, for a given Brownian mtotion, then the semimartingale BSDE takes the standard form of quadratique BSDE :*

$$\begin{cases} -dY_t = \left(U_t - \delta_t Y_t - \frac{1}{2\beta}|Z_t|^2\right)dt \ - \ Z_t \cdot dW_t \\ Y_T = \overline{U}_T \end{cases}$$

- *Kobylanski (2000), Lepeltier et San Martin (1998), El Karoui and Hamadène (2003), Briand and Hu (2005, 2007).*
- *Hu, Imkeler and Mueler (06), Morlais (2008), Mania and Tevzadze (2006), Trevzadze (SPA, 2009)*

A^V AND M^V : THE CONTINUOUS FILTRATION CASE

THEOREM (BORDIGONI G., M. A., SCHWEIZER, M.)

Assume that $\mathbb{F}$ is continuous. Then the couple (V, M^V) is the unique solution in $D_0^{\exp} \times \mathcal{M}_{0,loc}(\mathbb{P})$ of the BSDE

$$\begin{cases} -dY_t = (U_t - \delta_t Y_t)dt - \frac{1}{2\beta} d < M >_t \; - dM_t \\ Y_T = U'_T \end{cases}$$

- *Moreover, $\mathcal{E}\left(-\frac{1}{\beta} M^V\right) = Z^{\mathbb{Q}^*}$ is a $\mathbb{P}-$martingale such that it's supremum belongs to $L^1(\mathbb{P})$ where $\mathbb{Q}^*$ is the optimal probability.*
- *We have also that $M^V \in \mathcal{M}_0^p(\mathbb{P})$ for every $p \in [0, +\infty[$*

Recursive relation

Lemma

Let (Y, M) be a solution of BSDE with M continuous. Assume that $Y \in D_0^{\exp}$ or $\mathcal{E}\left(-\frac{1}{\beta}M\right)$ is $\mathbb{P}-$martingale.
For any pair of stopping times $\sigma \leq \tau$, then we have the recursive relation

$$Y_\sigma = -\beta \log \mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{\beta} \int_\sigma^\tau (\delta_s Y_s - \alpha U_s)\, ds - \frac{1}{\beta} Y_\tau \right) \,\middle|\, \mathcal{F}_\sigma \right]$$

- As a consequence one gets the uniqueness result for the semimartingale BSDE.
- In the case where $\delta = 0$, then this yields to the entropic dynamic risk measure.

Plan

The model (I)

- We consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$. All the processes are taken $\mathbb{G}$-adapted, and are defined on the time interval $[0, T]$.
- Any special $\mathbb{G}$-semimartingale Y admits a canonical decomposition $Y = Y_0 + A + M^{Y,c} + Y^{Y,d}$ where A is a predictable finite variation process, Y^c is a continuous martingale and $M^{Y,d}$ is a pure discontinuous martingale.
- For each $i = 1, \ldots, n$, H^i is a counting process and there exist a positive adapted process λ^i, called the $\mathbb{P}$ intensity of H^i, such that the process N^i with $N^i_t := H^i_t - \int_0^t \lambda^i_s ds$ is a martingale.
- We assume that the processes $H^i, i = 1, \ldots, d$ have no common jumps.

The model

- Any discontinuous martingale admits a representation of the

$$dM_t^{Y,d} = \sum_{i=1}^{d} \hat{Y}_t^i dN_t^i$$

where $\hat{Y}^i, i = 1, \ldots, d$ are predictable processes.

The model :example from Credit Risk

Example (under immersion property)

- *We assume that* $\mathbb{G}$ *is the filtration generated by a continuous reference filtration* $\mathbb{F}$ *and* d *positive random times* $\tau_1, \cdots, \tau_d$ *which are the default times of* d *firms :* $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ *where*

$$\mathcal{G}_t = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau_1 \wedge t + \epsilon) \vee \sigma(\tau_2 \wedge t + \epsilon) \cdots \vee \sigma(\tau_d \wedge t + \epsilon)$$

  where $\sigma(\tau_i \wedge t + \epsilon)$ *is the generated* σ*-fields which is non random before the default times* τ_i *for each* $i = 1, \cdots, d$.
- *we note* $H_t^i = \mathbf{1}_{\{\tau_i \leq t\}}$.
- *We assume that each* τ_i *is* $\mathbb{G}$*-totaly inaccessible and there exists a positive* $\mathbb{G}$*-adapted process* λ^i *such that, the process* N^i *with* $N_t^i := H_t^i - \int_0^t \lambda_s^i ds$ *is a* $\mathbb{G}$*-martingale.*
- *Obviously, the process* λ^i *is null after the default time* τ_i.

The model :example from Credit Risk

Example

- *From Kusuoka, the representation of the discontinuous martingale $M^{Y,d}$ with respect to N^i holds true when the filtration $\mathbb{G}$ is generated by a Brownian motion and the default processes.*

Semimartingale BSDE with jumps

- Let first consider the following quadratic semimartingale BSDE with jumps :

Definition

A solution of the BSDE is a triple of processes $(Y, M^{Y,c}, \widehat{Y})$ such that Y is a P-semimartingale, M is a locally square-integrable locally martingale with $M_0 = 0$ and $\widehat{Y} = (\widehat{Y}^1, \cdots, \widehat{Y}^d)$ a $\mathbb{R}^d$-valued predictable locally bounded process such that :

$$\begin{cases} dY_t = [\sum_{i=1}^{d} g(\widehat{Y}_t^i)\lambda_t^i - U_t + \delta_t Y_t]dt + \frac{1}{2}d\langle M^{Y,c}\rangle_t + dM_t^{Y,c} + \sum_{i=1}^{d} \widehat{Y}_t^i dN_t^i \\ Y_T = \bar{U}_T \end{cases} \quad (1)$$

where $g(x) = e^{-x} + x - 1$.

EXISTENCE RESULT

THEOREM (JEANBLANC, M., M. A., NGOUPEYOU A.)

- *There exists a unique triple of process* $(Y, M^{Y,c}, \widehat{Y}) \in D_0^{\exp} \times \mathcal{M}_{0,loc}(P) \times \mathcal{L}^2(\lambda)$ *solution of the semartingale BSDE with jumps.*
- *Furthermore, the optimal measure* $\mathbb{Q}^*$ *solution of our minimization problem is given :*

$$dZ_t^{\mathbb{Q}^*} = Z_{t-}^{\mathbb{Q}^*} dL_t^{\mathbb{Q}^*}, \quad Z_0^{\mathbb{Q}^*} = 1$$

where

$$dL_t^{\mathbb{Q}^*} = -dM_t^{Y,c} + \sum_{i=1}^{d} \left(e^{-\widehat{Y}_t^i} - 1 \right) dN_t^i.$$

Plan

Comparison theorem for our BSDE

Theorem (Jeanblanc, M., M. A., Ngoupeyou A.)

Assume that for $k = 1, 2$, $(Y^k, M^{Y^k,c}, \widehat{Y}^k)$ is solution of the BSDE associated to $(\widetilde{U}^k, \bar{U}^k)$. Then one have

$$Y_t^1 - Y_t^2 \leq \mathbb{E}^{\mathbb{Q}^{*,2}} \left[\int_t^T \frac{S_s^\delta}{S_t^\delta} \left(U_s^1 - U_s^2 \right) ds + \frac{S_T^\delta}{S_t^\delta} \left(\bar{U}_T^1 - \bar{U}_T^2 \right) \middle| \mathcal{G}_t \right]$$

where $\mathbb{Q}^{*,2}$ the probability measure equivalent to $\mathbb{P}$ given by

$$\frac{dZ_t^{\mathbb{Q}^{*,2}}}{Z_{t^-}^{\mathbb{Q}^{*,2}}} = -dM_t^{Y^2,c} + \sum_{i=1}^d \left(e^{-\widehat{Y}_t^{i,2}} - 1 \right) dN_t^i.$$

In particular, if $U^1 \leq U^2$ and $\bar{U}_T^1 \leq \bar{U}_T^2$, one obtains

$$Y_t^1 \leq Y_t^2, \quad d\mathbb{P} \otimes dt\text{-}a.e.$$

Idea of the Proof (I)

Proof

We denote $\widehat{Y}^{i,12} := \widehat{Y}^{i,1} - \widehat{Y}^{i,2}$ *and* $M^{12,c} = M^{1,c} - M^{2,c}$*. Then :*

$$\begin{aligned} Y_t^{12} &= \bar{U}_T^{12} + \int_t^T \left(\widetilde{U}_s^{12} - \delta_s Y_s^{12} \right) ds - \sum_{i=1}^{d} \int_t^T \widehat{Y}_s^{i,12} dN_s^i \\ &- \sum_{i=1}^{d} \int_t^T \left[g(\widehat{Y}_s^{i,1}) - g(\widehat{Y}_s^{i,2}) \right] \lambda_s^i ds \\ &+ \frac{1}{2} \int_t^T \left(d\langle M^{2,c} \rangle_s - d\langle M^{1,c} \rangle_s \right) - \int_t^T dM_s^{12,c} \end{aligned} \tag{2}$$

Idea of the Proof (II)

Proof

Note that, for any pair of continuous martingales M^1, M^2, denoting $M^{12} = M^1 - M^2$:

$$-\langle M^2, M^{12}\rangle - \frac{1}{2}\langle M^2\rangle + \frac{1}{2}\langle M^1\rangle = \frac{1}{2}\langle M^{12}\rangle$$

Using the fact that the process $\langle M^{12}\rangle$ is increasing and that the function g is convex we get :

$$\begin{aligned} Y_t^{12} \leq \bar{U}_T^{12} &+ \int_t^T \left(\widetilde{U}_s^{12} - \delta_s Y_s^{12}\right) ds \\ &+ \sum_{i=1}^{d} \int_t^T (e^{-\widehat{Y}_s^{i,2}} - 1)\widehat{Y}_s^{i,12}\lambda_s^i ds - \int_t^T d\langle M^{2,c}, M^{12,c}\rangle_s \\ &- \int_t^T dM_s^{12,c} - \sum_{i=1}^{d} \int_t^T \widehat{Y}_s^{i,12} dN_s^i. \end{aligned}$$

Idea of the Proof (III)

Proof

- *Let N^* and $M^{*,c}$ be the $\mathbb{Q}^{*,2}$-martingales obtained by Girsanov's transformation from N and M^c, where $d\mathbb{Q}^{*,2} = Z^{\mathbb{Q}^{*,2}} d\mathbb{P}$.*
- *Then :*

$$Y_t^{12} \leq \bar{U}_T^{12} + \int_t^T \left(\widetilde{U}_s^{12} - \delta_s Y_s^{12}\right) ds - \sum_{i=1}^{d} \int_t^T \widehat{Y}_s^{i,12} dN_s^{i*} - \int_t^T dM_s^{*,c}$$

which implies that

$$Y_t^{12} \leq \mathbb{E}^{\mathbb{Q}^{*,2}}\left[\int_t^T e^{-\int_t^s \delta_r dr} \widetilde{U}_s^{12} ds + e^{-\int_t^T \delta_r dr} \bar{U}_T^{12} \middle| \mathcal{G}_t \right]$$

Concavity property for the semimartingale BSDE

Theorem

Let define the map $F : D_1^{\exp} \times D_0^{\exp} \longrightarrow D_0^{\exp}$ *such that for all* $(U, \bar{U}) \in D_1^{\exp} \times D_0^{\exp}$*, we have*

$$F(U, \bar{U}) = V$$

where $(V, M^{V,c}, \hat{V})$ *is the solution of BSDE associated to* $(U, \bar{U})$*. Then* F *is concave ,namely,*

$$F\left(\theta U^1 + (1-\theta)\widetilde{U}^2, \theta \bar{U}_T^1 + (1-\theta)\bar{U}_T^2\right) \geq \theta F(U^1, \bar{U}_T^1) + (1-\theta) F(U^2, \bar{U}_T^2).$$

Plan

Problem : recursive utility problem

- we assume that $U_s = \widetilde{U}(c_s)$ and $\bar{U}_T = \bar{U}(\psi)$ where $\widetilde{U}$ and $\bar{U}$ are given utility functions, c is a non-negative $\mathbb{G}$-adapted process and ψ a $\mathcal{G}_T$-measurable non-negative random variable.
- We study the following optimization problem :

$$\sup_{(c,\psi)\in\mathcal{A}(x)} \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T S_s^\delta U(c_s)ds + S_T^\delta \bar{U}(\psi)\right] + \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \delta_s S_s^\delta \ln Z_s^{\mathbb{Q}^*} ds + S_T^\delta \ln Z_T^{\mathbb{Q}^*}\right] := \sup_{(c,\psi)\in\mathcal{A}(x)} V_0^{x,\psi,c}$$

where V_0 is the value at initial time of the value process V, part of the solution $(V, M^V, \widehat{V})$ of our BSDE, in the case $U_s = U(c_s)$ and $\bar{U}_T = \bar{U}(\psi)$.

PROBLEM : RECURSIVE UTILITY PROBLEM

- The set $\mathcal{A}(x)$ is the convex set of controls parameters $(c, \psi) \in \mathcal{H}^2([0, T]) \times \mathbf{L}^2(\Omega, \mathcal{G}_T)$ such that :

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\int_0^T c_t dt + \psi\Big] \leq x,$$

where $\widetilde{\mathbb{P}}$ is a fixed pricing measure, i.e. a probability $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ with a Radon-Nikodym density $\widetilde{Z}$ with respect to $\mathbb{P}$ given by :

$$d\widetilde{Z}_t = \widetilde{Z}_{t-}(\theta_t dM_t^c + \sum_{i=1}^{n}(e^{-z_t^i} - 1)dN_t^i),\ \widetilde{Z}_0 = 1\ .$$

- Here, $\mathbb{Q}^*$ is the optimal model measure depends on c, ψ.
- In a complete market setting, the process c can be interpreted as a consumption, ψ as a terminal wealth, with the pricing measure $\widetilde{\mathbb{P}}$ is the risk neutral probability.

Assumptions on the utility functions

- The utility functions U and $\bar{U}$ satisfy the usual regular conditions :

1. Strictly increasing and concave.
2. Continuous differentiable on the set $\{U > -\infty\}$ and $\{\bar{U} > -\infty\}$, respectively,
3. $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$ and $\bar{U}'(\infty) := \lim_{x\to\infty} \bar{U}'(x) = 0$,
4. $U'(0) := \lim_{x\to 0} U'(x) = +\infty$ and $U'(0) := \lim_{x\to 0} \bar{U}'(x) = +\infty$,
5. Asymptotic elasticity $AE(U) := \lim \sup_{x\to+\infty} \dfrac{xU'(x)}{U(x)} < 1$.

Properties of the value function

Proposition

Let $G : \mathcal{A}(x) \longrightarrow D_0^{\text{exp}}$, *as* $G(c, \psi) = V$ *where* $(V, M^{V,c}, \widehat{V})$ *is the solution of the BSDE associated with* $(U(c), \bar{U}(\psi))$. *Then*

1. G *is strictly concave with respect to* (c, ψ),
2. *Let* $G_0(c, \psi)$ *be the value at initial time of* $G(c, \psi)$, *i.e.,* $G_0(c, \psi) = V_0$. *Then* $G_0(c, \psi)$ *is continuous from above with respect to* (c, ψ),
3. G_0 *is upper continuous with respect to* (c, ψ).

Regularity result on the value function

Theorem

- $(V^1, M^{1,c}, \widehat{V}^1)$ *the solution associated with* $(U(c^1), \bar{U}(\psi^1))$ *for a given* (c^1, ψ^1).
- *Let* $(V^\epsilon, M^{\epsilon,c}, \widehat{V}^\epsilon)$ *be the solution of the BSDE associated with* $(U(c^1 + \epsilon(c^2 - c^1)), \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)))$ *for a given* (c^2, ψ^2).
- *Then* V^ϵ *is right differentiable in* 0 *with respect to* ϵ *and the triple* $(\partial_\epsilon V, \partial_\epsilon \widetilde{M}^{V,c}, \partial_\epsilon \widehat{V})$ *is the solution of the following BSDE :*

$$\begin{cases} d\partial_\epsilon V_t = \left(\delta_t \partial_\epsilon V_t - U'(c_t^1)(c_t^2 - c_t^1)\right) dt + d\partial_\epsilon \widetilde{M}_t^{V,c} + \sum_{i=1}^{d} \partial_\epsilon \widehat{V}_t^i d\widetilde{N}_t^i. \\ \partial_\epsilon V_T = \bar{U}'(\psi^1)(\psi^2 - \psi^1) \end{cases}$$

where $\widetilde{N}^i = N^i - \int_0^\cdot (e^{-v_t^{1,i}} - 1)\lambda_t^i dt$

Regularity result on the value function

Theorem

Moreover, we obtain

$$\partial_\epsilon V_t = \mathbb{E}^{\mathbb{P}}\Big[\frac{Z_T^{\mathbb{Q}^*,1}}{Z_t^{\mathbb{Q}^*,1}}\frac{S_T^\delta}{S_t^\delta}\bar{U}'(\psi^1)(\psi^2-\psi^1)+\int_t^T\frac{Z_s^{\mathbb{Q}^*,1}}{Z_t^{\mathbb{Q}^*,1}}\frac{S_s^\delta}{S_t^\delta}U'(c_s^1)(c_s^2-c_s^1)ds\Big|\mathcal{G}_t\Big]$$

UNCONSTRAINTED OPTIMIZATION PROBLEM

- we solve first an equivalent unconstrained problem to the optimization problem : we associate with a pair $(c, \psi) \in \mathcal{A}(x)$ the quantity

$$X_0^{c,\psi} = \mathbb{E}^{\tilde{\mathbb{P}}} \left(\int_0^T c_s ds + \psi \right)$$

- In a complete market setting, $X^{c,\psi}$ is the initial value of the associated wealth.
- Define by

$$u(x) := \sup_{X_0^{c,\psi} \leq x} V_0^{(c,\psi)} \tag{3}$$

where $V_0^{(c,\psi)} = V_0$, $(V, M^{V,c}, \widehat{V})$ is the solution of the BSDE associated with $(U(c), \bar{U}(\psi))$.

UNCONSTRAINTED OPTIMIZATION PROBLEM

PROPOSITION

There exists an unique optimal pair (c^0, ψ^0) *which solves the unconstrainted optimization problem.*

PROOF

- *The uniqueness is a consequence of the strictly concavity property of* V_0.
- *We shall prove the existence by using Komlòs theorem.*
- *We first Step prove that* $\sup_{(c,\phi)\in\mathcal{A}(x)} V_0^{c,\phi} < +\infty$ *:*

  Because $\mathbb{P} \in \mathcal{Q}_f^e$, *we have :*

$$\sup_{(c,\phi)\in\mathcal{A}(x)} V_0^{c,\phi} \leq \sup_{(c,\phi)\in\mathcal{A}(x)} \mathbb{E}^{\mathbb{P}}\left[\bar{U}(\phi) + \int_0^T U(c_s)ds\right] := \widetilde{u}(x).$$

PROOF (2)

PROOF

- *Using the elasticity assumption on U and $\bar{U}$, we can prove that $AE(\widetilde{u}) < 1$, which permits to conclude that, for any $x > 0$, $\widetilde{u}(x) < +\infty$.*
- *Let $(c^n, \phi^n) \in \mathcal{A}(x)$ be a maximizing sequence such that :*

$$\nearrow \lim_{n \to +\infty} V_0^{c^n, \phi^n} = \sup_{(c,\phi) \in \mathcal{A}(x)} V_0^{c,\phi} < +\infty,$$

where the RHS is finite.

- *Then conclude by Using Komlòs theorem.*

OPTIMIZATION PROBLEM

THEOREM

- *There exists a constant* $\nu^* > 0$ *such that :*

$$u(x) = \sup_{(c,\psi)} \left\{ V_0^{(c,\psi)} + \nu^* \left(x - X^{(c,\psi)} \right) \right\}$$

and if the maximum is attained in the above constraint problem by (c^*, ψ^*) *then it is attained in the unconstraint problem by* (c^*, ψ^*) *with* $X^{(c,\psi)} = x$.

- *Conversely if there exists* $\nu^0 > 0$ *and* (c^0, ψ^0) *such that the maximum is attained in*

$$\sup_{(c,\psi)} \left\{ V_0^{(c,\psi)} + \nu^0 \left(x - X_0^{(c,\psi)} \right) \right\}$$

with $X_0^{(c,\psi)} = x$, *then the maximum is attained in our constraint problem by* (c^0, ψ^0).

The maximum principle (1)

- We now study for a fixed $\nu > 0$ the following optimization problem :

$$\sup_{(c,\psi)} L(c,\psi) \tag{4}$$

where the functional L is given by $L(c,\psi) = V_0^{(c,\psi)} - \nu X_0^{(c,\psi)}$

Proposition (Jeanblanc, M., M. A., Ngoupeyou A.)

The optimal consumption plan (c^0, ψ^0) *which solves* (4) *satisfies the following equations :*

$$U'(c_t^0) = \frac{Z_t^{\widetilde{\mathbb{P}}}}{Z_t^{\mathbb{Q}^*}} \frac{\nu}{\alpha S_t^\delta} \qquad \bar{U}'(\psi^0) = \frac{Z_T^{\widetilde{\mathbb{P}}}}{Z_T^{\mathbb{Q}^*}} \frac{\nu}{\bar{\alpha} S_T^\delta} \ \ a.s \tag{5}$$

where $\mathbb{Q}^*$ *is the model measure associated to the optimal consumption* (c^0, ψ^0).

THE MAIN STEPS OF THE PROOF OF THE PROPOSITION (I)

- Let consider the optimal consumption plan (c^0, ψ^0) which solve (4) and another consumption plan (c, ψ). Consider $\epsilon \in (0, 1)$ then :

$$L(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(c - c^0)) \leq L(c^0, \psi^0)$$

Then

$$\frac{1}{\epsilon}\left[V_0^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(\psi-\psi^0))} - V_0^{(c^0,\psi^0)}\right]$$
$$- \nu\frac{1}{\epsilon}\left[X_0^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(\psi-\psi^0)} - X_0^{(c^0,\psi^0)}\right] \leq 0$$

Because $\left(X_t^{(c,\psi)} + \int_0^t c_s ds\right)_{t\geq 0}$ is a $\widetilde{\mathbb{P}}$ martinagle we obtain :

$$\frac{1}{\epsilon}\left[X_t^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(\psi-\psi^0)} - X_t^{(c^0,\psi^0)}\right]$$
$$= \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_t^T (c_s - c_s^0)ds + (\psi - \psi^0)\Big|\mathcal{F}_t\right]$$

THE MAIN STEPS OF THE PROOF (II)

- Then the wealth process is right differential in 0 with respect to ϵ we define

$$\partial_\epsilon X_t^{(c^0,\psi^0)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(X_t^{(c^0+\epsilon(c-c^0),\psi^0+\epsilon(c-c^0))} - X_t^{(c^0,\psi^0)}\right)$$

- We take $\lim_{\epsilon \to 0}$ above, we obtain :

$$\partial_\epsilon V_0^{(c^0,\psi^0)} - \nu \partial_\epsilon X_0^{(c^0,\psi^0)} \leq 0.$$

The main steps of the Proof (III)

- Consider the optimal density $(Z^{\mathbb{Q}_t^{*,1}})_{t\geq 0}$ where its dynamics is given by

$$\frac{dZ_t^{\mathbb{Q}^{*,1}}}{Z_{t^-}^{\mathbb{Q}^{*,1}}} = -dM^{V,c} + \sum_{i=1}^{d} \left(e^{-\widehat{Y}^{1,i}} - 1\right) dN_t^i$$

then :

$$\partial_\epsilon V_t = \mathbb{E}^{\mathbb{Q}^{*,1}}\Big[\frac{S_T^\delta}{S_t^\delta}\bar{U}'(X_T^1)(X_T^2 - X_T^1) + \int_t^T \frac{S_s^\delta}{S_t^\delta} U'(c_s^1)(c_s^2 - c_s^1)ds \Big| \mathcal{G}_t\Big].$$

The main steps of the Proof (IV)

- From the last result and the explicitly expression of $(\partial_\epsilon X_t^{(c^0,\psi^0)})_{t\geq 0}$ we get :

$$
\begin{aligned}
&\partial_\epsilon V_0^{(c^0,\psi^0)} - \nu \partial_\epsilon X_0^{(c^0,\psi^0)} \\
&= \mathbb{E}^{\mathbb{P}}\big[S_T^\delta Z_T^{\mathbb{Q}^*,1}\bar{U}'(\psi^0)(\psi-\psi^0) + \int_0^T S_s^\delta Z_s^{\mathbb{Q}^*} U'(c_s^0)(c_s - c_s^0)ds\big] \\
&- \nu \mathbb{E}^{\mathbb{P}}\big[Z^{\widetilde{\mathbb{P}}}(\psi-\psi^0) + \int_0^T Z_s^{\widetilde{\mathbb{P}}}(c_s - c_s^0)ds\big]
\end{aligned}
$$

- Using the equality above we get :

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[\big(S_T^\delta Z_T^{\mathbb{Q}^*,1}\bar{U}'(\psi^0) - \nu Z^{\widetilde{\mathbb{P}}}\big)(\psi-\psi^0) \\
&+ \int_0^T \big(S_s^\delta Z_s^{\mathbb{Q}^*,1} U'(c_s^0) - \nu Z_s^{\widetilde{\mathbb{P}}}\big)(c_s - c_s^0)ds\big] \leq 0
\end{aligned}
$$

The main steps of the Proof (V)

- Let define the set $A := \{(Z^{\mathbb{Q}^*}\bar{U}'(\psi^0) - \nu Z^{\widetilde{\mathbb{P}}})(\psi - \psi^0) > 0\}$ taking $c = c^0$ and $\psi = \psi^0 + \mathbf{1_A}$ then $\mathbb{P}(A) = 0$ and we get :

$$(Z^{\mathbb{Q}^*}\bar{U}'(\psi^0) - \nu Z^{\widetilde{\mathbb{P}}}) \leq 0 \qquad a.s$$

- Let define for each $\epsilon > 0$

$$B := \{(Z^{\mathbb{Q}^*}\bar{U}'(\psi^0) - \nu Z^{\widetilde{\mathbb{P}}})(\psi - \psi^0) < 0, \psi^0 > \epsilon\}$$

- because $\{\psi^0 > 0\}$ due to Inada assumption, we can define $\psi = \psi^0 - \mathbf{1_B}$ then $\mathbb{P}(B) = 0$ and we get

$$(Z^{\mathbb{Q}^*}\bar{U}'(\psi^0) - \nu Z^{\widetilde{\mathbb{P}}}) \geq 0 \qquad a.s$$

We find the optimal consumption with similar arguments.

The maximum principle (2)

- we have also :

Theorem

Let I and $\bar{I}$ the inverse of the functions U' and $\bar{U}'$. The optimal consumption (c^0, ψ^0) which solve the unconstrained problem is given by :

$$c_t^0 = I\left(\frac{\nu^0}{S_t^\delta}\frac{Z_t^{\widetilde{\mathbb{P}}}}{Z_t^{\mathbb{Q}^*}}\right), \quad dt \otimes d\mathbb{P} \ a.s \ , \qquad \psi^0 = \bar{I}\left(\frac{\nu^0}{S_T^\delta}\frac{Z_T^{\widetilde{\mathbb{P}}}}{Z_T^{\mathbb{Q}^*}}\right) \ a.s. \ .$$

where $\nu^0 > 0$ satisfies :

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\int_0^T I\left(\frac{\nu^0}{S_t^\delta}\frac{Z_t^{\widetilde{\mathbb{P}}}}{Z_t^{\mathbb{Q}^*}}\right)dt + \bar{I}\left(\frac{\nu^0}{S_T^\delta}\frac{Z_T^{\widetilde{\mathbb{P}}}}{Z_T^{\mathbb{Q}^*}}\right)\right] = x.$$

THE MAIN STEPS OF THE PROOF (1)

- For any initial wealth $x \in (0, +\infty)$, there exists a unique ν^0 such that $f(\nu^0) = x$.
- Let $(c, \psi) \in \mathcal{A}(x)$ and $(V^{(c,\psi)}, M^{V,c}, v)$ $\big($resp. $(V^{(c^0,\psi^0)}, M^{V^0,c}, v^0)\big)$ the solution of the BSDE associated with $(U(c^0), \bar{U}(\psi^0))$ $\big($resp. $(U(c), \bar{U}(\psi))\big)$ then from comparison theorem, we get :

$$
\begin{aligned}
& V_0^{(c,\psi)} - V_0^{(c^0,\psi^0)} \\
& \leq \mathbb{E}^{\mathbb{Q}^*}\left[S_T^{\delta}\big(\bar{U}(\psi) - \bar{U}(\psi^0)\big) + \int_0^T S_s^{\delta}\big(U(c_s) - U(c_s^0)\big)ds\right] \\
& \leq \mathbb{E}^{\mathbb{Q}^*}\left[S_T^{\delta}\bar{U}'(\psi^0)(\psi - \psi^0) + \int_0^T S_s^{\delta}U'(c_s^0)(c_s - c_s^0)ds\right].
\end{aligned}
$$

THE MAIN STEPS OF THE PROOF (2)

- It follows from the maximum principle that :

$$\begin{aligned} V_0^{(c,\psi)} - V_0^{(c^0,\psi^0)} &\leq \nu^0 \mathbb{E}^{\mathbb{Q}^*}\left(\frac{Z_T^{\widetilde{\mathbb{P}}}}{Z_T^{\mathbb{Q}^*}}(\psi - \psi^0) + \int_0^T \frac{Z_s^{\widetilde{\mathbb{P}}}}{Z_s^{\mathbb{Q}^*}}(c_s - c_s^0)ds \right) \\ &\leq \nu^0 \left(\mathbb{E}^{\widetilde{\mathbb{P}}}\left(\psi + \int_0^T c_s ds\right) - \mathbb{E}^{\widetilde{\mathbb{P}}}\left(\psi^0 + \int_0^T c_s^0 ds\right)\right) \end{aligned}$$

- Since $(c,\psi) \in \mathcal{A}(x)$, then $\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\psi + \int_0^T c_s ds\right] \leq x$.
- Using that $\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\psi^0 + \int_0^T c_s^0 ds\right] = x$, we conclude :

$$V_0^{(c,\psi)} \leq V_0^{(c^0,\psi^0)}.$$

Plan

LOGARITHMIC CASE (1)

- We assume that δ is deterministic and $U(x) = \ln(x)$ and $\bar{U}(x) = 0$ (hence $I(x) = \frac{1}{x}$ for all $x \in (0, +\infty)$).
- The optimal process $c_t^* = I\left(\frac{\nu}{S_t^\delta}\frac{\widetilde{Z}_t}{Z_t^*}\right) = \frac{S_t^\delta}{\nu}\frac{Z_t^*}{\widetilde{Z}_t}$.
- For any deterministic function α such that $\alpha(T) = 0$, V admits a decomposition as

$$V_t = \alpha(t)\ln(c_t^*) + \gamma_t$$

- where γ is a process such that $\gamma_T = 0$.
- Recall that the Radon-Nikodym density $\widetilde{Z}$, and the Radon-Nikodym density of the optimal probability measure Z^* satisfy

$$d\widetilde{Z}_t = \widetilde{Z}_{t-}(\theta_t dM_t^c + \sum_{i=1}^{n}(e^{-z_t^i} - 1)dN_t^i),\ \widetilde{Z}_0 = 1$$

$$dZ_t^* = Z_{t-}^*(-dM_t^{V,c} + \sum_{i=1}^{n}(e^{-y_t^i} - 1)dN_t^i),\ Z_0^* = 1$$

Logarithmic Case (2)

- In order to obtain a BSDE, we introduce $J_t = \frac{1}{1+\alpha(t)}\beta_t$.

Proposition

(i) The value function V has the form

$$V_t = \alpha(t)\ln(c_t^*) + (1+\alpha(t))J_t$$

where

$$\alpha(t) = -\int_t^T e^{\int_t^s \delta(u)du} ds$$

and $(J, \bar{M}^{J,c}, \hat{J})$ is the unique solution of the following Backward Stochastic Differential Equation, where $k(t) = -\frac{\alpha(t)}{1+\alpha(t)}$:

Logarithmic Case (3)

Proposition

$$dJ_t = \Big((1+\delta(t))(1+k(t))J_t - k(t)\delta(t) \Big) dt + d\bar{M}_t^{J,c} + \frac{1}{2} d\langle \bar{M}^{J,c}\rangle_t$$
$$+ \frac{1}{2} k(t)(1+k(t))\theta_t^2 d\langle M^c \rangle_t$$
$$+ \sum_{i=1}^{n} j_t^i d\bar{N}_t^i + \sum_{i=1}^{n} \Big(g(j_t^i)\bar{\lambda}_t^i + \Big(k(t)(e^{-z_t^i} - 1) + e^{k(t)z_t^i} - 1 \Big) \lambda_t^i \Big) dt$$

- *The processes $\bar{M}^{J,c}$ and $d\bar{N}_t^i = dH_t^i - \bar{\lambda}_t^i dt$ are $\bar{\mathbb{P}}$-martingales where $\frac{d\bar{\mathbb{P}}}{d\bar{\mathbb{P}}}|_{\mathcal{G}_t} = Z_t^{\bar{\mathbb{P}}}$ and $\bar{\lambda}_t^i = e^{k(t)z_t^i}\lambda_t^i$ where*

$$dZ_t^{\bar{\mathbb{P}}} = -Z_{t-}^{\bar{\mathbb{P}}} \left(k(t)\theta_t dM_t^c - \sum_{i=1}^{d} (e^{k(t)z_t^i} - 1) dN_t^i \right)$$

Logarithmic Case (3)

Proposition

ii)

$$dc_t^* = c_{t-}^* \Big(-\delta_t dt - dM_t^{V,c} + \theta_t dM_t^c - \theta_t d\langle M^c, M^{V,c}\rangle_t + \sum_{i=1}^{d} (e^{(y_t^i - z_t^i)} - 1) dN_t^i - \sum_{i=1}^{d} (g(y_t^i) - g(z_t^i) - g(y_t^i - z_t^i)) \lambda_t^i dt \Big)$$

DISCUSSION

- study more explicit "models" in incomplete market
- Numerical scheme
- replace the entropic penalization by other convex term ! !
- consider robustness in the non-dominated case