On strong invariance for semilinear differential inclusions

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Let X be a real separable Banach space, $A : D(A) \subseteq X \mapsto X$ the infinitesimal generator of a C_0-semigroup of contractions, $\{S(t) : X \mapsto X |\ t \geq 0\}$, D a nonempty subset in X and $F : D \rightsquigarrow X$ a given multi-function.

Definition 1. By a *mild solution* to the autonomous multi-valued semilinear Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + F(u(t)) \\ u(0) = \xi, \end{cases}$$

on $[0, T]$, we mean a continuous function $u : [0, T] \mapsto D$, such that $\exists\ g \in L^1(0, T; X)$, $g(s) \in F(u(s))$ *a.e.* $s \in [0, T]$ and

$$u(t) = S(t)\xi + \int_0^t S(t-s)g(s)ds, \quad \forall t \in [0, T].$$

Definition 2. Let D an open set in X. The subset $K \subset D$ is *strong invariant* with respect to $A + F$ if for each $\xi \in K$ and each mild solution $u : [0, T] \mapsto D$, there exists $T^0 \in (0, T]$ such that $u(t) \in K$ for each $t \in [0, T^0]$.

Definition The subset $K \subset D$ is *weak invariant, or viable* with respect to $A + F$ if for each $\xi \in K$, there exists a mild solution $u : [0, T] \mapsto K$.

Previous results

Case $A = 0$. F is Lipschitz, has convex compact values, X-Hilbert space, K closed.

Theorem

The following are equivalent

(i) $F(x) \subseteq T_K(x), \forall x \in K;$

(ii) *K is strong invariant.*

Clarke, Ledyaev, Rădulescu (1997)[2].

About tangency

$v \in T_K(x) \Leftrightarrow \liminf_{h \downarrow 0} \frac{1}{h} \mathrm{dist}(x + hv, K) = 0$. Equivalently, $\exists h_n \downarrow 0, p_n \to 0$, $x + h_n v + h_n p_n \in K$.
Since F has compact convex values, (i) $F(x) \subseteq T_K(x)$ is equivalent to:

$\forall f \in L^1_{loc}(R_+, X), f(s) \in F(x)$ a.e. we have $\exists h_n \downarrow 0$, $p_n \to 0$,

$$x + \int_0^{h_n} f(s) ds + h_n p_n \in K. \tag{1}$$

Denote $\mathcal{E} = \{f \in L^1_{loc}; f(s) \in E.a.e.\}$ and $\mathcal{T}_K(x) = \{f(\cdot) \in L^1_{loc}; (1)\,\mathrm{holds}\}$.
So, the following are equivalent

(i) $F(x) \subseteq T_K(x), \forall x \in K$.

(ii) $\mathcal{F}(x) \subseteq \mathcal{T}_K(x), \forall x \in K$.

The semilinear case

Define $T_K^A(x)$ as follows:
$v \in T_K^A(x)$ iff $\liminf_{h \downarrow 0} \frac{1}{h}\mathrm{dist}(S(h)x + hv; K) = 0$. Equivalently, $\liminf_{h \downarrow 0} \frac{1}{h}\mathrm{dist}(S(h)x + \int_0^h S(h-s)vds; K) = 0$.
Again, when $F(x)$ has compact convex values, the following are equivalent:

(a) $F(x) \subseteq T_K^A(x), \forall x \in K$.

(b) $\mathcal{F}(x) \subseteq \mathcal{T}_K^A(x), \forall x \in K$

where $\mathcal{T}_K^A(x)$ is defined as follows:
$f(\cdot) \in \mathcal{T}_K^A(x)$ iff $\exists h_n \downarrow 0$, $p_n \to 0$ such that
$S(h_n)x + \int_0^{h_n} S(h_n - s)f(s)ds + h_n p_n \in K$.

Recall that $\mathcal{F}(x) = \{f \in L^1_{loc}; f(s) \in F(x).a.e.\}$

Uniqueness function

Definition 4. A function $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ which is continuous, nondecreasing and the only C^1-solution to the Cauchy problem

$$\begin{cases} x'(t) = \omega(x(t)) \\ x(0) = 0, \end{cases}$$

is $x \equiv 0$ is called a *uniqueness function*.

Lemma 2. [1] *Let $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a uniqueness function and let $(\gamma_k)_k$ be strictly decreasing to 0. Let $(x_k)_k$ be a bounded sequence of measurable functions, from $[0, T]$ to $\mathbb{R}_+$, such that*

$$x_k(t) \leq \gamma_k + \int_0^t \omega(x_k(s))ds$$

for $k = 1, 2, ...$ and for each $t \in [0, T]$. Then there exists $\tilde{T} \in (0, T]$ such that $\lim_k x_k(t) = 0$ uniformly for $t \in [0, \tilde{T}]$.

Main results

Theorem

Let X be a separable Banach space, D an open subset in X, K a nonempty and closed subset of D and $F : D \rightsquigarrow X$ a nonempty, closed and bounded valued multi-function. Assume that
(a) there exist an open neighborhood $V \subseteq D$ of K and one uniqueness function $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+,\ \omega(0) = 0$, such that

$$F(x) \subset F(y) + \omega(\|x - y\|)B(0,1), \quad \forall x \in V \setminus K, \forall y \in K;$$

(b) for every $x \in K$ we have $\mathcal{F}(x) \subseteq \mathcal{T}_K^A(x)$.

Then K is strong invariant with respect to $A + F$.

Theorem

Let X be a separable Banach space, D an open subset in X, K a nonempty and locally closed subset of D and $F : D \rightsquigarrow X$ a nonempty, closed and bounded valued multi-function. Assume that:
(a) $\exists L > 0$ *such that* $F(x) \subset F(y) + L\|x - y\|B(0,1),\ \forall x, y \in D$;

(b) K is strong invariant with respect to $A + F$.

Then for every $x \in K$ *we have* $\mathcal{F}(x) \subseteq \mathcal{T}_K^A(x)$.

- O. Cârjă, M. Necula, I. I. Vrabie, Viability, Invariance and Applications, North-Holland Mathematics Studies, 2007.
- F.H. Clarke, Yu. Ledyaev, M. Radulescu, Aproximate invariance and differential inclusions, J. Dynam. Control Systems, 3 (1997), 493–518.
- F.H. Clarke, Yu. Ledyaev, R. Stern, P. Wolenski, Nonsmooth Analysis and Control Theory, in: Graduate Texts in Mathematics, Vol. 178, Springer-Verlag, New York, 1998.
- H. Frankowska, H., A priori estimates for operational differential inclusions, *J. Differential Equations* 84 (1990), no. 1, 100–128.