Switching problems and related BSDE approximation

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Outline of the talk

- Starting and stopping problem (d=2)
- Numerical resolution of BSDE
- Numerical resolution of BSDE with oblique reflections
- An alternative approach : Constrained BSDEs with jumps

Starting and Stopping problem

Hamadene & Jeanblanc 05 :

- Consider e.g. a power station producing electricity whose price is given by a diffusion process X : $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$
- Two modes for the power station :
  mode **1** : operating, with running profit $f_1(X_t)dt$ and terminal one $g_1(X_T)$
  mode **0** : closed, with running profit $f_0(X_t)dt$ and terminal one $g_0(X_T)$

  $\hookrightarrow$ switching from one mode to another has a cost : $c > 0$
- Management decides to produce electricity only when it is profitable enough.
- The management strategy is (θ_j, α_j) : θ_j is a sequence of stopping times representing switching times from mode α_{j-1} to α_j.

  $(a_t)_{0 \le t \le T}$ is the state process, i.e. the management strategy.

Value processes

- Following a strategy a from t up to T, gives

$$J(a,t) = g_{a_T}(X_T) + \int_t^T f_{a_s}(X_s)ds - \sum_{j\geq 0} c\mathbf{1}_{\{t\leq\theta_j\leq T\}}$$

- The value processes starting respectively at time 0 in mode 1 and 2 are

$$Y_0^0 := \sup_{\{a\in\mathcal{A}\ s.t.\ a_0=0\}} \mathbb{E}\left[J(a,0)\right] \quad \text{and} \quad Y_0^1 := \sup_{\{a\in\mathcal{A}\ s.t.\ a_0=1\}} \mathbb{E}\left[J(a,0)\right]$$

- Y is solution of a coupled optimal stopping problem

$$Y_t^0 = \mathrm{ess}\sup_{t\leq\tau\leq T} \mathbb{E}\left[\int_t^\tau f_0(X_s)ds + (Y_\tau^1 - c)\mathbf{1}_{\{\tau<T\}} \mid \mathcal{F}_t\right]$$

$$Y_t^1 = \mathrm{ess}\sup_{t\leq\tau\leq T} \mathbb{E}\left[\int_t^\tau f_1(X_s)ds + (Y_\tau^0 - c)\mathbf{1}_{\{\tau<T\}} \mid \mathcal{F}_t\right]$$

with terminal conditions : $Y_T^0 = g_0(X_T)$ and $Y_T^1 = g_1(X_T)$

- The optimal strategy (θ_j^*, α_j^*) is given by

$$\alpha_{j+1}^* := 1 - \alpha_j^* \quad \text{and} \quad \theta_{j+1}^* := \inf\{s \geq \theta_j^* \mid Y_s^{\alpha_j^*} = Y_s^{\alpha_{j+1}^*} - c\}$$

System of reflected BSDEs

Y is the solution of the following system of reflected BSDEs :

$$Y_t^i = g_i(X_T) + \int_t^T f_i(X_s)ds - \int_t^T Z_s^i \cdot dW_s + \int_t^T dK_s^i \ , \ i \in \{0,1\} \ ,$$

with (the coupling...)

$$Y_t^1 \geq Y_t^0 - c \text{ and } Y_t^0 \geq Y_t^1 - c \ , \ \forall t \in [0,T]$$

and ('optimality' of K)

$$\int_0^T \Big(Y_s^1 - (Y_s^0 - c) \Big) dK_s^1 = 0 \text{ and } \int_0^T \Big(Y_s^0 - (Y_s^1 - c) \Big) dK_s^0 = 0$$

- Problem : Oblique reflections.
- Idea : Interpret $Y^1 - Y^0$ as the solution of a doubly reflected BSDE.

Related PDE

Associated coupled system of PDE

- on $\mathbb{R} \times [0, T)$

$$\min\Big(-\partial_t u_0 - \mathcal{L}u_0 - f_0, u_0 - u_1 + c\Big) = 0$$

$$\min\Big(-\partial_t u_1 - \mathcal{L}u_1 - f_1, u_1 - u_0 + c\Big) = 0$$

$$\text{with} \quad \mathcal{L} : u \mapsto \frac{\sigma^2}{2}\partial_{xx}u + b\,\partial_x u$$

- Terminal conditions

$$u_0(T,.) = g_0(.) \qquad \text{and} \qquad u_1(T,.) = g_1(.)$$

- Link via

$$Y_t^0 = u_0(t, X_t) \qquad \text{and} \qquad Y_t^1 = u_1(t, X_t)$$

Non exhaustive Literature

Literature on optimal switching :

- Hamadène & Jeanblanc 05 : starting and stopping problem ($d = 2$).
- Djehiche, Hamadène & Popier 07 : studied the multidimentional case.
- Carmona & Ludkovski 06 or Porchet, Touzi & Warin 07 : Additional constraints and numerical results.

Link with non linear Backward SDE :

- Hu & Tang 07 "multi-dimentional BSDEs with oblique reflection" BSDE representation for optimal switching in the case where X uncontrolled or at most partially controlled : $dX_t^a = \sigma(X_t^a)\Big[\mu_a(X_t^a)dt + dW_t\Big]$.
- Hamadène & Zhang 08 Generalization of Hu & Tang's BSDEs but still with an uncontrolled underlying diffusion.

Literature on control :

- Bouchard 09 : Relation with stochastic target problems with jumps.

Multi-dimensional reflected BSDE

- Multi-dimensional reflected BSDE (see Hamadène & Zhang 08) :
  Find m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{A}^2)^{\mathcal{I}}$ satisfying

$$\begin{cases} Y^i_t = g_i(X_T) + \int_t^T f_i(s, X_s, Y^1_s, \ldots, Y^m_s, Z^i_s)ds - \int_t^T Z^i_s dW_s + K^i_T - K^i_t \\ Y^i_t \geq h_{i,j}(t, Y^j_t) \\ \int_0^T [Y^i_t - \max_{j\in\mathcal{I}}\{h_{i,j}(t, Y^j_t)\}]dK^i_t = 0 \end{cases}$$

- Conditions on the constraint h in order to avoid instantaneous gain via circle switching.
- For any $i \neq j$, $h_{i,j}$ and f_i are increasing in y_j.
  $\Longrightarrow$ 'Interpretation' in terms of cooperative game options
- The reflections are oblique with respect to the domain of definition of Y.

FBSDE system

- **FSDE**
- **BSDE**

$$\begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ \mathbf{Y}_t = g(X_T) + \int_t^T f(s, X_s, \mathbf{Y}_s, \mathbf{Z}_s)\, ds - \int_t^T \mathbf{Z}_s\, dW_s \end{cases}$$

- **Solution and link with PDE** (Pardoux & Peng, 90 & 92);

$$\|\mathbf{Y}\|_{\mathcal{S}^2} := \mathbb{E}\left[\sup_{0\le r\le 1} |Y_r|^2\right]^{\frac{1}{2}} < \infty, \qquad \|\mathbf{Z}\|_{\mathcal{L}^2} := \mathbb{E}\left[\int_0^1 |Z_r|^2 dr\right]^{\frac{1}{2}} < \infty,$$

- **PDE** $\quad \mathcal{L}^X[y] + f(., y, \sigma\nabla y) = 0 \qquad y(T, .) = g(.)$

- **Approximation of the BM** (Chevance 97, Briand 01, Ma 02);

- **Discrete time scheme based on the path regularity of Z** (Zhang);

Discrete time scheme (Zhang 02)

- **FSDE**
- **BSDE**

$$\begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ \mathbf{Y_t} = g(X_T) + \int_t^T f(s, X_s, \mathbf{Y_s}, \mathbf{Z_s})\, ds - \int_t^T \mathbf{Z_s}\, dW_s \end{cases}$$

- Regular time grid $\pi := (t_i)_{i \leq n}$ on $[0, T]$
- **Forward Euler approximation X^π of X**

Initial value : $\mathbf{X_0^\pi} := x$

From t_i to t_{i+1} : $\mathbf{X^\pi_{t_i+1}} := X^\pi_{t_i} + \frac{1}{n} b(t_i, X^\pi_{t_i}) + \sigma(t_i, X^\pi_{t_i})(W_{t_{i+1}} - W_{t_i})$

- **Backward approximation $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ of (Y, Z)**

Terminal value : $\mathbf{Y^\pi_T} := g(X^\pi_T)$

From t_{i+1} to t_i :

$$\begin{cases} \mathbf{Z^\pi_{t_i}} := n\,\mathbb{E}\left[Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \mathbf{Y^\pi_{t_i}} := \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(t_i, X^\pi_{t_i}, \mathbf{Y^\pi_{t_i}}, \mathbf{Z^\pi_{t_i}}\right) \end{cases}$$

Intuition of the scheme

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r)\, dr - \int_{t_i}^{t_{i+1}} Z_r \cdot dW_r$$

Step 1 : Constant step driver ($\widetilde{Z}^\pi$ given by the representation of $Y^\pi_{t_{i+1}}$)

$$\mathbf{Y}^\pi_{\mathbf{t_i}} = \mathbf{Y}^\pi_{\mathbf{t_{i+1}}} + \frac{1}{n} f\left(t_i, X^\pi_{t_i}, \mathbf{Y}^\pi_{\mathbf{t_i}}, \mathbf{Z}^\pi_{\mathbf{t_i}}\right) - \int_{t_i}^{t_{i+1}} \widetilde{Z}^\pi_r \cdot dW_r$$

Step 2 : Best $\mathcal{L}^2(\Omega \times [t_i, t_{i+1}])$ approximation of $\widetilde{Z}^\pi$ by $\mathcal{F}_{t_i}$-meas. r.v.

$$\mathbf{Z}^\pi_{\mathbf{t_i}} := n\, \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \widetilde{Z}^\pi_r\, dr \mid \mathcal{F}_{t_i}\right] = \mathbf{n}\, \mathbb{E}\left[\mathbf{Y}^\pi_{\mathbf{t_{i+1}}}(\mathbf{W}_{\mathbf{t_{i+1}}} - \mathbf{W}_{\mathbf{t_i}}) \mid \mathcal{F}_{\mathbf{t_i}}\right]$$

Step 3 : Conditioning the first expression

$$\mathbf{Y}^\pi_{\mathbf{t_i}} = \mathbb{E}\left[\mathbf{Y}^\pi_{\mathbf{t_{i+1}}} \mid \mathcal{F}_{\mathbf{t_i}}\right] + \frac{\mathbf{1}}{\mathbf{n}} \mathbf{f}\left(\mathbf{t_i}, \mathbf{X}^\pi_{\mathbf{t_i}}, \mathbf{Y}^\pi_{\mathbf{t_i}}, \mathbf{Z}^\pi_{\mathbf{t_i}}\right).$$

Approximation Error (Zhang 02) $\quad Y_t = y(t, X_t)$

- **PDE** $\qquad \mathcal{L}^X[y] + f(., y, \sigma\nabla y) = 0 \qquad y(1, .) = g(.)$

- **Forward Euler approximation X^π of X**

$$\mathbf{X}_0^\pi := x \quad \text{and} \quad \mathbf{X}_{t_i+1}^\pi := X_{t_i}^\pi + \tfrac{1}{n} b(t_i, X_{t_i}^\pi) + \sigma(t_i, X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i})$$

- **Backward approximation $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ of (Y, Z)**

$$\mathbf{Y}_T^\pi := g(X_T^\pi) \;\; \& \;\; \begin{cases} \mathbf{Z}_{t_i}^\pi := n\,\mathbb{E}\left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \mathbf{Y}_{t_i}^\pi := \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + \tfrac{1}{n} f\left(t_i, X_{t_i}^\pi, \mathbf{Y}_{t_i}^\pi, Z_{t_i}^\pi\right) \end{cases}$$

- **Approximation Error**

$$\mathcal{E}\mathbf{rr}(\mathbf{Y}, \mathbf{Y}^\pi) := \sup_{t_i} \mathbb{E}\left[|Y_{t_i} - Y_{t_i}^\pi|^2\right] \qquad \mathcal{E}\mathbf{rr}(\mathbf{Z}, \mathbf{Z}^\pi) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[|Z_{t_i} - Z_{t_i}^\pi|^2\right]$$

$$\boxed{\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, Z^\pi) \;\leq\; C\,|\pi|}$$

Approximation Error (Gobet 05) $\mathbf{Y_t = y(t, X_t)}$

- **PDE** $\mathcal{L}^X[y] + f(., y, \sigma\nabla y) = 0 \qquad y(1,.) = g(.)$

- **Forward Euler approximation X^π of X**

$$\mathbf{X}_0^\pi := x \quad \text{and} \quad \mathbf{X}_{t_i+1}^\pi := X_{t_i}^\pi + \tfrac{1}{n} b(t_i, X_{t_i}^\pi) + \sigma(t_i, X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i})$$

- **Backward approximation $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ of (Y, Z)**

$$\mathbf{Y}_1^\pi := g(X_1^\pi) \;\; \& \begin{cases} \mathbf{Z}_{t_i}^\pi & := \; n\,\mathbb{E}\left[Y_{t_{i+1}}^\pi (W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \mathbf{Y}_{t_i}^\pi & := \; \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + \tfrac{1}{n}\mathbb{E}\left[f\left(t_i, X_{t_i}^\pi, \mathbf{Y}_{t_{i+1}}^\pi, Z_{t_i}^\pi\right) \mid \mathcal{F}_{t_i}\right] \end{cases}$$

- **Approximation Error**

$$\mathcal{E}\mathbf{rr}(\mathbf{Y}, \mathbf{Y}^\pi) := \sup_{t_i} \mathbb{E}\left[|Y_{t_i} - Y_{t_i}^\pi|^2\right] \qquad \mathcal{E}\mathbf{rr}(\mathbf{Z}, \mathbf{Z}^\pi) := \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[|Z_{t_i} - Z_{t_i}^\pi|^2\right]$$

$$\boxed{\mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, Z^\pi) \;\leq\; C\,|\pi|}$$

Addition of normal reflections (Bouchard Chassagneux 08)

- **Reflected BSDE on a boundary** $\ell(X_t)$

$$Y_t = g(X_T) + \int_t^T f(t, X_t, Y_t, Z_t)\mathrm{d}t - \int_t^T (Z_t)'\mathrm{d}W_t + \int_t^T \mathrm{d}K_t$$

$$Y_t \geq \ell(X_t) \text{ and } \int_0^T \Big(Y_t - \ell(X_t)\Big)\mathrm{d}K_t = 0$$

- **Forward Euler approximation** X^π **of** X
- **Backward approximation** $(\mathbf{Y}^\pi, \mathbf{Z}^\pi)$ **of** (Y, Z)

$$\mathbf{Y}^\pi_\mathrm{T} := g(X^\pi_1) \;\; \& \;\; \begin{cases} \mathbf{Z}^\pi_{\mathrm{t_i}} & := & n\,\mathbb{E}\left[Y^\pi_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) \mid \mathcal{F}_{t_i}\right] \\ \widetilde{\mathbf{Y}}^\pi_{\mathrm{t_i}} & := & \mathbb{E}\left[Y^\pi_{t_{i+1}} \mid \mathcal{F}_{t_i}\right] + \frac{1}{n} f\left(t_i, X^\pi_{t_i}, \mathbf{Y}^\pi_{\mathrm{t_i}}, Z^\pi_{t_i}\right) \\ \mathbf{Y}^\pi_{\mathrm{t_i}} & := & \max[\widetilde{Y}^\pi_{t_i} \,;\, \ell(X^\pi_{t_i})]\; \mathbf{1}_{\{t_i \in \Re\}} \end{cases}$$

with $\Re \subset \pi$ the **reflection grid** to be chosen properly.

- **Approximation Error** $\quad \mathcal{E}rr(Y, Y^\pi) + \mathcal{E}rr(Z, Z^\pi) \;\leq\; C\,|\pi|^{1/2}$

Obliquely reflected BSDEs

- Multidimensional system of reflected BSDEs

$$Y_t^i = g_i(X_T) + \int_t^T f_i(u, X_u, \mathbf{Y_u^i}, Z_u^i)\mathrm{d}u - \int_t^T Z_u^i \cdot \mathrm{d}W_u + \mathbf{K_T^i} - \mathbf{K_t^i}$$

$Y_t \in \mathcal{C}(X_t)$ (constrained by K) $\qquad \int_0^T \left(Y_t^i - \mathcal{P}^{\mathbf{i}}(\mathbf{X_t}, \mathbf{Y_t})\right)\mathrm{d}K_t^i = 0$

- The domain $\mathcal{C}(x)$ is given by $(m \geq 2)$

$$\mathcal{C}(x) := \{y \in \mathbb{R}^m | y^i \geq \mathcal{P}^{\mathbf{i}}(\mathbf{x}, \mathbf{y}) := \max_{\mathbf{j}}(\mathbf{y_j} - \mathbf{c_{ij}}(\mathbf{x}))\}$$

$\Longrightarrow$ $\mathcal{P}(x, .)$ is an oblique projection

- Non linear switching problems with cost matrix $c(X_t)$ at time t

Goal and method

Goal : Approximation scheme for **Continuously** Obliquely Reflected BSDE **(COR)** and convergence results...

Method :

(i) **Discretize the reflections** along a grid $\Re$

$\Longrightarrow$ Discretely Obliquely Reflected BSDE **(DOR)** $(\widetilde{Y}^{d\Re}, Z^{d\Re}, \tilde{K}^{d\Re})$

(ii) **Approximation scheme for the DOR** along a grid $\pi \supset \Re$

$\Longrightarrow$ Convergence of the scheme, via regularity of the DOR

(iii) **Convergence of the DOR to the COR** when $\Re$ is refined.

$\Longrightarrow$ The scheme converges to the COR ($\Re$ and π well chosen)

Discretely obliquely reflected BSDEs

- **Grid** $\Re := \{0 = r_0 < ... < r_k < ... < r_\kappa = T\}$ given.
- A **DOR** is a triplet $(\widetilde{Y}^{d\Re}, Z^{d\Re}, \tilde{K}^{d\Re})$ satisfying $\widetilde{Y}^{d\Re}_T := g(X_T)$ and

$$\widetilde{Y}^{d\Re}_t = g(X_T) + \int_t^T f(X_s, \widetilde{Y}^{d\Re}_s, Z^{d\Re}_s)\mathrm{d}s - \int_t^T Z^{d\Re}_s \cdot \mathrm{d}W_s + \tilde{\mathbf{K}}^{d\Re}_{\mathbf{T}} - \tilde{\mathbf{K}}^{d\Re}_{\mathbf{T}}$$

$$\tilde{K}^{d\Re}_t = \sum_{r \in \Re \setminus \{0\}} \Delta \tilde{K}^{d\Re}_r 1_{t \geq r} , \qquad \Delta \tilde{K}^{d\Re}_r = \mathcal{P}(X^\pi_r, \widetilde{Y}^{d\Re}_r) - \widetilde{Y}^{d\Re}_r$$

- To any strategy a and related cumulative cost process A^a, we associate the one-dimensional **'switched BSDE'**

$$U^a_t = g_{a_T}(X_T) + \int_t^T f_{a_s}(s, X_s, U^a_s, V^a_s)\mathrm{d}s - \int_t^T V^a_s \mathrm{d}W_s - A^a_T + A^a_t$$

- Same representation property as COR with switching times restricted to $\Re$

$$(Y^{d\Re}_t)^i = \operatorname{ess} \sup_{\{a \ / \ a_t = i\}} U^a_t =: U^{a*}_t$$

Regularity results for the DOR

$$\widetilde{Y}_t^{d\Re} = g(X_T) + \int_t^T f(X_s, \widetilde{Y}_s^{d\Re}, Z_s^{d\Re})\mathrm{d}s - \int_t^T Z_s^{d\Re} \cdot \mathrm{d}W_s + \tilde{K}_T^{d\Re} - \tilde{K}_T^{d\Re}$$

$$\tilde{K}_t^{d\Re} = \sum_{r \in \Re \setminus \{0\}} \Delta \tilde{K}_r^{d\Re} 1_{t \geq r} \;, \qquad \Delta \tilde{K}_r^{d\Re} = \mathcal{P}(X_r^\pi, \widetilde{Y}_r^{d\Re}) - \widetilde{Y}_r^{d\Re}$$

- Stability with respect to parameters f,b,σ... allows for regularization.
- Switching representation allows to work with one-dimensional BSDE.

$$\mathcal{R}eg(\widetilde{Y}^{d\Re}) \;:=\; \sup_{i \leq n} \sup_{t_i \leq t \leq t_{i+1}} \mathbb{E}\left[|\widetilde{Y}_s^{d\Re} - \widetilde{Y}_{t_i}^{d\Re}|^2 \right] \;\leq\; \frac{C}{n}$$

- $Z^{d\Re}$ representation using the optimal strategy a^* (here $f = f(x)$)

$$(Z_t^{d\Re})^i = \mathbb{E}\left[\nabla g^{a_T^*}(X_T) D_t X_T + \int_t^T \nabla f^{a_s^*}(X_s) D_t X_s \mathrm{d}s \mid \mathcal{F}_t \right]$$

$$\Longrightarrow \quad \mathcal{R}eg(Z^{d\Re}) \;:=\; \sum_{i=1}^n \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z_s^{d\Re} - Z_{t_i}^{d\Re}|^2 ds \right] \;\leq\; C\left(\frac{\kappa}{n} + n^{-\frac{1}{2}} \right)$$

Approximation Scheme

- Discretization grid $\pi \supset \Re$
- Start from the terminal condition $Y_T^\pi := g(X_T^\pi) \in \mathcal{C}(X_T^\pi)$
- Compute at each step

$$\begin{cases} \bar{Z}_{t_i}^\pi &= (t_{i+1}-t_i)^{-1}\,\mathbb{E}\left[(W_{t_{i+1}}-W_{t_i})\cdot Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] \\ \widetilde{Y}_{t_i}^\pi &= \mathbb{E}\left[Y_{t_{i+1}}^\pi \mid \mathcal{F}_{t_i}\right] + (t_{i+1}-t_i) f(t_i, X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi, \bar{Z}_{t_i}^\pi) \\ Y_{t_i}^\pi &= \widetilde{Y}_{t_i}^\pi \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi)\mathbf{1}_{\{t_i \in \Re\}} \end{cases}$$

- **Problem :** The projection operator is L-lipschitz with $L > 1$

$$\mathcal{E}rr(\widetilde{Y}^{d\Re}, \widetilde{Y}^\pi) + \mathcal{E}rr(Z^{d\Re}, \bar{Z}^\pi) \le L^\kappa \left[\mathcal{E}rr(X, X^\pi) + \mathcal{R}eg(\widetilde{Y}^{d\Re}) + \mathcal{R}eg(Z^{d\Re})\right]$$

- **Idea :** Monotonicity arguments and well chosen dominating BSDE
- **Drawback :** Requires f independent of z

Sketch of proof

1. Observe that $(Y^\pi, \widetilde{Y}^\pi, \bar{Z}^\pi)$ interprets as a **DOR**
   $\Longrightarrow$ Representation in terms of 'switched BSDEs' $(U^{\pi,a})_a$

2. Introduce another **DOR** $(\check{Y}, \check{Z}, \check{K})$ with terminal value $g(X_T) \vee g(X_T^\pi)$,

   driver $\quad f(t, X_t, \widetilde{Y}_t) \vee f(t_i, X_{t_i}^\pi, \widetilde{Y}_{t_i}^\pi) \quad$ and costs $\quad c(X_t) \wedge c(X_{t_i}^\pi)\ , \quad t_i \leq t < t_{i+1}\ .$

   $\Longrightarrow$ Representation in terms of 'switched BSDEs' $(\check{U}^a)_a$
   $\Longrightarrow$ Existence of an optimal strategy $\check{a}$ such that $\check{Y}_t^i = \check{U}_t^{\check{a}}$

3. Via comparison arguments, $\quad (\widetilde{Y}_t^{d\Re})^i \leq \check{Y}_t^i \quad$ and $\quad (\widetilde{Y}_t^\pi)^i \leq \check{Y}_t^i.$

4. From the switched representations,

$$U_t^{\check{a}} \leq (\widetilde{Y}_t^{d\Re})^i \leq \check{U}_t^{\check{a}} \quad \text{and} \quad U_t^{\pi,\check{a}} \leq (\widetilde{Y}_t^\pi)^i \leq \check{U}_t^{\check{a}}$$

5. We deduce

$$|(\widetilde{Y}_t^{d\Re})^i - (\widetilde{Y}_t^\pi)^i|^2 \leq 2\,(|\check{U}_t^{\check{a}} - U_t^{\pi,\check{a}}|^2 + |\check{U}_t^{\check{a}} - U_t^{\check{a}}|^2)$$

$\Longrightarrow$ Work with one-dimensional BSDEs switching simultaneously

Convergence results

- **Always convergence of the scheme**
- Distance between the scheme to the DOR (f independent of z)

$$\mathcal{E}rr(Y^{d\Re}, Y^{\pi}) \leq \frac{C}{n} \quad \text{and} \quad \mathcal{E}rr(Z^{d\Re}, \bar{Z}^{\pi}) \;\leq\; C\left(\frac{\kappa}{n} + n^{-\frac{1}{2}}\right)$$

- Distance between the DOR and the COR (f bounded in z)

$$\mathcal{E}rr(Y, Y^{d\Re}) \leq C\,\kappa^{-1-\varepsilon} \quad \text{and} \quad \mathcal{E}rr(Z, Z^{d\Re}) \leq C\,\kappa^{-\frac{1}{2}-\varepsilon}$$

- If f independent of z , we have

$$\Re = \pi \quad \Longrightarrow \quad \mathcal{E}rr(Y, Y^{\pi}) \leq C\,|\pi|^{1-\varepsilon}$$
$$|\Re| = |\pi|^{2/3} \quad \Longrightarrow \quad \mathcal{E}rr(Z, \bar{Z}^{\pi}) \leq C\,|\pi|^{\frac{1}{3}-\varepsilon}$$

General Multi-dimensional reflected BSDE

• Multi-dimensional reflected BSDE (see Hamadène & Zhang 08) :
Find m triplets $(Y^i, Z^i, K^i)_{i\in\mathcal{I}} \in (\mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{A}^2)^{\mathcal{I}}$ satisfying

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, Y_s^1, \ldots, Y_s^m, Z_s^i)ds - \int_t^T Z_s^i dW_s + K_T^i - K_t^i \\ Y_t^i \geq \max_{j\in\mathcal{I}} h_{i,j}(t, Y_t^j) \\ \int_0^T [Y_t^i - \max_{j\in\mathcal{I}}\{h_{i,j}(t, Y_t^j)\}] dK_t^i = 0 \end{cases}$$

where

- $(\xi^i)_{i\in\mathcal{I}} \in (\mathbf{L}^2(\Omega, \mathcal{F}_T, \mathbf{P}))^{\mathcal{I}}$,
- $h_{i,j} : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are a given constraint functions,
- $f_i : \ \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is an $\mathbb{F}$-progressively measurable map.

• Reinterpretation of the solution in terms of **constrained BSDE with jumps**

Idea : Introduce an independent random switching regime,
allowing to jump between the components of the solution !

Alternative BSDE representation

- Introduce the random switching regime I defined by

$$I_t = I_0 + \int_0^t \int_{\mathcal{I}} (i - I_{s-})\mu(ds, di) \quad t \leq T,$$

where μ is an independent Poisson measure on $\mathcal{I} := \{1, \ldots, m\}$.

- Consider the one-dimensional constrained BSDE with jumps :

$$\tilde{Y}_t = \xi^{I_T} + \int_t^T f_{I_s}(s, \tilde{Y}_s + \tilde{U}_s(1), \ldots, \tilde{Y}_s + \tilde{U}_s(m), \tilde{Z}_s)ds + \tilde{K}_T - \tilde{K}_t$$
$$- \int_t^T \tilde{Z}_s.dW_s - \int_t^T \int_{\mathcal{I}} \tilde{U}_s(i)\mu(ds, di), \quad 0 \leq t \leq T, \ a.s.$$

constrained by : $\tilde{Y}_{t-} - h_{I_{t-},j}(t, \tilde{Y}_{t-} + \tilde{U}_t(j)) \geq 0, d\mathbb{P} \otimes dt \otimes \lambda(dj) \ a.e.$

- Unique minimal solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ of the constrained BSDE with jumps relates to the solution $(Y^i, Z^i, K^i)_{i \in \mathcal{I}}$ of the multidimensional reflected BSDE

via $\tilde{Y}_t = Y_t^{I_{t-}}$, $\tilde{Z}_t = Z_t^{I_{t-}}$ and $\tilde{U}_t = \left[Y_t^j - Y_{t-}^{I_{t-}} \right]_{j \in \mathcal{I}}$.

- Use of probabilistic arguments valid in an eventually non Markovian context

Intuition when $m = 2$

- Multi-dimensional reflected BSDE :

Find (Y^0, Z^0, K^0) and (Y^1, Z^1, K^1) such that

$$\begin{cases} Y_t^0 = \xi^0 + \int_t^T f_0(s, Y_s^0, Y_s^1, Z_s^0)ds - \int_t^T Z_s^0 dW_s + K_T^0 - K_t^0 \\ Y_t^0 \geq h_{0,1}(t, Y_t^1) \; ; \quad \int_0^T [Y_t^0 - h_{0,1}(t, Y_t^1)]dK_t^0 = 0 \end{cases}$$

$$\begin{cases} Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Y_s^0, Z_s^1)ds - \int_t^T Z_s^1 dW_s + K_T^1 - K_t^1 \\ Y_t^1 \geq h_{1,0}(t, Y_t^0) \; ; \quad \int_0^T [Y_t^1 - h_{1,0}(t, Y_t^0)]dK_t^1 = 0 \end{cases}$$

- Constrained BSDE with jumps :

Random switching regime : $I_t = I_0 + \int_0^t (1 - I_{s-})\mu(ds, 1) \quad t \leq T$,
and the one-dimensional constrained BSDE with jumps on $[0, T]$:

$$\tilde{Y}_t = \xi^{I_T} + \int_t^T f_{I_s}(s, \tilde{Y}_s, \tilde{Y}_s + \tilde{U}_s, \tilde{Z}_s)ds + \tilde{K}_T - \tilde{K}_t - \int_t^T \tilde{Z}_s.dW_s - \int_t^T \tilde{U}_s \mu(ds, 1),$$

constrained by : $\quad \tilde{Y}_{t-} - h_{I_{t-}, 1-I_{t-}}(t, \tilde{Y}_{t-} + \tilde{U}_t) \geq 0, \; a.e.$

Link via $\quad \tilde{Y}_t = Y_t^{I_{t-}}, \quad \tilde{Z}_t = Z_t^{I_{t-}}$ and $\quad \tilde{U}_t = Y_t^{1-I_{t-}} - Y_{t-}^{I_{t-}}$.

Possible extension : optimal Switching with controlled diffusion

Consider the optimal switching problem : $\sup_{a\in\mathcal{A}} J(a)$ with

$$J(a) := \mathbb{E}\Big[g_{a_T}(X^a_T) + \int_0^T f_{a_s}(X^a_s)ds - \sum_{0<\tau_k\leq T} c_{a_{\tau_k^-}, a_{\tau_k}}(X^a_{\tau_k})\Big].$$

where the underlying X^a, is the controlled diffusion defined by

$$X^a_t = X_0 + \int_0^t b_{a_s}(X^a_s)ds + \int_0^t \sigma_{a_s}(X^a_s)dW_s, \quad t \geq 0 .$$

Representation in terms of constrained BSDE with jumps ?

- Introduce the forward process (I, X^I) defined by

$$I_t = i_0 + \int_0^t \int_{\mathcal{I}} (i - I_{t-})\mu(dt, di) , \quad X^I_t = x_0 + \int_0^t \mathbf{b}_{\mathbf{I_s}}(X^I_s)ds + \int_0^t \sigma_{\mathbf{I_s}}(X^I_s)dWs$$

- Consider the constrained BSDE with jumps :

$$\widetilde{Y}_t = g_{I_T}(X^I_T) + \int_t^T f_{I_s}(X^I_s)ds - \int_t^T \widetilde{Z}_s.dW_s - \int_t^T \int_{\mathcal{I}} \widetilde{U}_s(i)\mu(ds, di) + \tilde{K}_T - \tilde{K}_t,$$

on $[0, T]$, with the constraint : $\widetilde{U}_t(i) \leq c_{I_{t^-}, i}(X^I_t)$, $d\mathbb{P} \otimes dt \otimes \lambda(di)$ a.e.

- Y_0 is the solution of the **switching problem** starting in mode i_0 at time 0.

Related systems of variational inequalities

- Bi-dimensional forward process

$$I_t = i_0 + \int_0^t \int_{\mathcal{I}} (i - I_{t-})\mu(dt, di) \,, \quad X^I_t = x_0 + \int_0^t b_{I_s}(X^I_s)ds + \int_0^t \sigma_{I_s}(X^I_s)dW_s$$

- General Constrained BSDE with jumps

$$\widetilde{Y}_t = g_{I_T}(X_T) + \int_t^T f_{I_s}(X_s, \widetilde{Y}_s + \widetilde{U}_s, \widetilde{Z}_s)ds + \tilde{K}_T - \tilde{K}_t - \int_t^T \widetilde{Z}_s \cdot dW_s - \int_t^T \int_{\mathcal{I}} \widetilde{U}_s(j)\mu(ds, dj)$$

together with the constraint

$$h_{I_{s-},j}(X_s, \widetilde{Y}_{s-}, \widetilde{Y}_{s-} + \widetilde{U}_s(j), \widetilde{Z}_s) \geq 0 \,, \qquad j \in \mathcal{I}, \;\; t \leq s \leq T \,.$$

$\Longrightarrow$ We have $\widetilde{Y}_t := v_{I_t}(t, X^I_t)$ where v interprets as the unique viscosity solution of the following coupled system of variational inequalities

$$\Big[-\frac{\partial v_i}{\partial t} - \mathcal{L}^i v_i - f_i(., (v_k)_{1\leq k\leq m}, \sigma_i^\top D_x v_i) \Big] \wedge \min_{1\leq j\leq m} h_{i,j}(., v_i, v_j, \sigma_i^\top D_x v_i) = 0,$$

on $\mathcal{I} \times [0, T) \times \mathbb{R}^d$, with terminal condition $v(T, .) = g$ on $\mathbb{R}^d$,

Numerical approximation

- Bi-dimensional forward process

$$I_t = i_0 + \int_0^t \int_{\mathcal{I}} (i - I_{t-})\mu(dt, di) \ , \quad X_t^I = x_0 + \int_0^t b_{I_s}(X_s^I)ds + \int_0^t \sigma_{I_s}(X_s^I)dW_s$$

- General Constrained BSDE with jumps

$$\widetilde{Y}_t = g_{I_T}(X_T) + \int_t^T f_{I_s}(X_s, \widetilde{Y}_s + \widetilde{U}_s, \widetilde{Z}_s)ds + \tilde{K}_T - \tilde{K}_t - \int_t^T \widetilde{Z}_s \cdot dW_s - \int_t^T \int_{\mathcal{I}} \widetilde{U}_s(j)\mu(ds, dj)$$

together with the constraint

$$h_{I_{s-},j}(X_s, \widetilde{Y}_{s-}, \widetilde{Y}_{s-} + \widetilde{U}_s(j), \widetilde{Z}_s) \geq 0 \ , \qquad j \in \mathcal{I}, \ \ t \leq s \leq T \ .$$

- Numerical approximation via :
  - Forward simulation of (I, X^I)
  - Include the constraint in the driver by penalization
  - Use of approximation scheme for BSDEs with jumps, Bouchard & Elie 07
  - Convergence of the scheme
  - Practical influence of the penalization parameter and the jump frequency

Conclusion

- Probabilistic numerical approximation of optimal switching problems.
  - via obliquely reflected BSDE (convergence rate)
  - via constrained BSDE with jumps (possibility of controlled diffusion)
- Constrained BSDEs with jumps unify and generalize
  - Constrained BSDE without jumps, Peng & Xu 07
  - BSDE with diffusion-transmutation process, Pardoux, Pradeilles & Rao 97
  - BSDE with constrained jumps, Kharroubi, Ma, Pham & Zhang 08
  - Multidimensional BSDE with oblique reflections, Hamadène & Zhang 08
- Numerical approximation for coupled systems of variational inequalities :

$$\min \Big[-\frac{\partial v_i}{\partial t} - \mathcal{L}^i v_i - f_i(., v_i, \sigma_i^\top D_x v_i, [v_j - v_i]_{j\in\mathcal{I}}), \min_{j\in\mathcal{I}} h^{i,j}(., v_i, \sigma_i^\top D_x v_i, v_j - v_i) \Big] = 0,$$

with terminal condition $v(T,.) = g$.