Representation of G-martingales as stochastic integrals with respect to G-Brownian motion

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This talk is based on the following paper:

- Lin, Q., Representation of G-martingales as stochastic integrals with respect to G-Brownian motion, 2009, preprint.

Contents

Introduction

- Peng [2006] introduced G-expectation, G-normal distribution and G-Brownian motion. Moreover, Peng developed an Itô calculus for the G-Brownian motion.
- Xu [2009] obtained the martingale characterization of the G-Brownian motion .

The objective of the present paper is to investigate a representation of G-martingales as stochastic integrals with respect to the G-Brownian motion in the framework of sublinear expectation spaces. In this paper, we

- study stochastic integrals with respect to G-martingale;
- study representation theorem of G-martingales.

Preliminaries

We briefly recall some basic results about G-stochastic analysis in the following papers:

- Peng, S., G-expectation, G-Brownian motion and related stochastic calculus of Itô type. Stochastic analysis and applications, 541–567, Abel Symp., 2, Springer, Berlin,(2007).
- Peng, S. , Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation, Stochastic Processes and their Applications, 118 (12),(2008), 2223-2253.

Let Ω be a given set and $\mathcal{H}$ be a linear space of real functions defined on Ω such that if $x_1, \cdots, x_n \in \mathcal{H}$ then $\varphi(x_1, \cdots, x_n) \in \mathcal{H}$, for each $\varphi \in C_{l,lip}(\mathbb{R}^m)$. Here $C_{l,lip}(\mathbb{R}^m)$ denotes the linear space of functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^n + |y|^n)|x - y|, \text{ for all } x, y \in \mathbb{R}^m,$$

for some $C > 0$ and $n \in \mathbb{N}$, both depending on φ. The space $\mathcal{H}$ is considered as a set of random variables.

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all real valued continuous functions $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i}\Big[\big(\max_{t\in[0,i]} |\omega_t^1 - \omega_t^2|\big) \wedge 1\Big], \ \omega^1, \omega^2 \in \Omega.$$

For each $T > 0$, we consider the following space of random variables:

$$\begin{aligned} L_{ip}^0(\mathcal{F}_T) : \ &= \ \Big\{X(\omega) = \varphi(\omega_{t_1} \cdots, \omega_{t_m}) \mid t_1, \cdots, t_m \in [0, T], \\ &\quad \text{for all } \varphi \in C_{l,lip}(\mathbb{R}^m), \ m \geq 1\Big\}, \end{aligned}$$

$$L_{ip}^0(\mathcal{F}) = \bigcup_{n=1}^{\infty} L_{ip}^0(\mathcal{F}_n).$$

Sublinear expectations

Definition

A Sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}} : \mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(i) **Monotonicity:** If $X \geq Y$, then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$.

(ii) **Constant preserving:** $\hat{\mathbb{E}}[c] = c$, for all $c \in \mathbb{R}$.

(iii) **Self-dominated property:** $\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y]$.

(iv) **Positive homogeneity:** $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

Remark

The sublinear expectation space can be regarded as a generalization of the classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *endowed with the linear expectation associated with* $\mathbb{P}$.

Coherent risk measures and sublinear expectations

Let

$$\rho(X) \doteq \hat{\mathbb{E}}[-X], \ \ X \in \mathcal{H}.$$

Then $\rho(\cdot)$ is a coherent risk measure, namely

1. **Monotonicity:** If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
2. **Constant preserving:** $\rho(c) = -c$, for all $c \in \mathbb{R}$.
3. **Self-dominated property:** $\rho(X) - \rho(Y) \leq \rho(X - Y)$.
4. **Positive homogeneity:** $\rho(\lambda X) = \lambda\rho(X)$, for all $\lambda \geq 0$.

Conversely, for every coherent risk measure ρ, let

$$\hat{\mathbb{E}}[X] \doteq \rho(-X) \ \ X \in \mathcal{H}.$$

Then $\hat{\mathbb{E}}[\cdot]$ is a sublinear expectation.

- For $p \geq 1$, $\|X\|_p = \hat{\mathbb{E}}^{\frac{1}{p}}[|X|^p]$, $X \in L^0_{ip}(\mathcal{F})$.
- Let $\mathcal{H} = L^p_G(\mathcal{F})$ (resp. $\mathcal{H}_t = L^p_G(\mathcal{F}_t)$) be the completion of $L^0_{ip}(\mathcal{F})$ (resp. $L^0_{ip}(\mathcal{F}_t)$) under the norm $\|\cdot\|_p$.
- $(L^p_G(\mathcal{F}), \|\cdot\|_p)$ is a Banach space.
- $L^p_G(\mathcal{F}_t) \subset L^p_G(\mathcal{F}_T) \subset L^p_G(\mathcal{F})$, for all $0 \leq t \leq T < \infty$.

Remark

Bounded and measurable random variables in general are not in $L^p_G(\mathcal{F})$ (e.g. I_A). Thus, the powerful techniques of stopping times in classical situations cannot be applied to G-stochastic analysis. This is a main difficulty faced in the calculus.

Independence

Definition

In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \cdots, Y_n), Y_i \in \mathcal{H}$, is said to be **independent** of another random vector $X = (X_1, \cdots, X_m), X_i \in \mathcal{H}$, if for each test function $\varphi \in C_{l,lip}(\mathbb{R}^{m+n})$ we have

$$\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}].$$

Remark

Independence means the distribution of Y does not change the realization of $X (X = x)$.

Remark

Y is independent of X does not imply that X is independent of Y.

Example

$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[-X] = 0, \hat{\mathbb{E}}[X^+] > 0$, $\hat{\mathbb{E}}[Y^2] > -\hat{\mathbb{E}}[-Y^2] > 0$.

- If X is independent of Y, then $\hat{\mathbb{E}}[XY^2] = 0$.
- But if Y is independent of X, then $\hat{\mathbb{E}}[XY^2] > 0$.

G-normal distribution

Definition

G-normal distribution:

$\xi \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$, if for all $\varphi \in C_{l,lip}(\mathbb{R})$,

$$u(t,x) := \hat{\mathbb{E}}[\varphi(x + \sqrt{t}\xi)], \quad (t,x) \in [0,\infty) \times \mathbb{R}$$

is the solution of the following PDE:

$$\partial_t u = G(\partial^2_{xx} u), \ u|_{t=0} = \varphi,$$

where $G(\alpha) = \frac{1}{2} \sup_{\sigma_1 \le \sigma \le \sigma_2} \alpha\sigma^2$, $0 \le \sigma_1 \le \sigma_2$.

Remark

In the case where $\sigma_1 = \sigma_2 > 0$*, then* $\mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$ *is just the classical normal distribution* $\mathcal{N}(0, \sigma_2^2)$*.*

Remark

If $X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$ *and* φ *is convex, then*

$$\hat{\mathbb{E}}[\varphi(X)] = \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} \varphi(y) \exp(-\frac{(x-y)^2}{2\sigma_2^2 t}) dy.$$

Remark

Let $X \sim \mathcal{N}(0, [\sigma_1^2, \sigma_2^2])$*. If* φ *is concave and* $\sigma_1^2 > 0$*, then*

$$\hat{\mathbb{E}}[\varphi(X)] = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} \varphi(y) \exp(-\frac{(x-y)^2}{2\sigma_1^2 t}) dy.$$

G-Brownian motion

For simplicity, we assume $0 \leq \sigma_1 = \sigma \leq 1, \sigma_2 = 1$ in the following.

Definition

A process B in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called **G-Brownian motion** if for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n < \infty$, $B_{t_1}, \cdots, B_{t_n} \in \mathcal{H}$, the following properties are satisfied:

(i) $B_0 = 0$;

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t \sim \mathcal{N}(0, [\sigma^2 s, s])$;

(iii) For each $t, s \geq 0$, $B_{t+s} - B_t$ is independent of $(B_{t_1}, \cdots, B_{t_n})$, for each $n \in \mathbb{N}$ and $t_n \leq t$.

Hu and Peng [2009] obtained presentation theorem of G-expectation.

Theorem

Let $\hat{\mathbb{E}}$ be a G-expectation. Then there exists a weekly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}} E_P[X], \text{ for all } X \in \mathcal{H},$$

where $E_P[\cdot]$ is the linear expectation with respect to $P \in \mathcal{P}$.

Definition

- Choquet capacity: $c(A) = \sup_{P \in \mathcal{P}} P(A), A \in \mathcal{B}(\Omega)$.
- A set A is called polar if $c(A) = 0$ and a property holds quasi-surely (q.s.) if it holds outside a polar set.

As in the classical stochastic analysis, the definition of a modification of a process plays an important role.

Definition

Let I be a set of indexes, and $\{X_t\}_{t\in I}$ and $\{Y_t\}_{t\in I}$ two processes indexed by I. We say that Y is a modification of X if for all $t \in I$, $X_t = Y_t \; q.s.$

Finally, we recall the definition of a G-martingale introduced by Peng [2006].

Definition

A process $M = \{M_t, t \geq 0\}$ is called a G-martingale (respectively, G-supermartingale, and G-submartingale) if for each $t \in [0, \infty), M_t \in L_G^1(\mathcal{F}_t)$ and for each $s \in [0, t]$, we have

$$\hat{\mathbb{E}}[M_t|\mathcal{H}_s] = M_s, \text{ (respectively } \leq M_s, \text{and } \geq M_s) \; q.s.$$

Definition

A process $M = \{M_t, t \geq 0\}$ is called a symmetric G-martingale, if M and $-M$ are G-martingales.

Remark

B_t is symmetric G-martingale, but $B_t^2 - t$ is not symmetric G-martingale.

Representation theorem for G-martingale

- Our objective: representation theorem for G-martingales
- Recall: classical representation theorem for martingales

Theorem

Let M be a square integrable continuous martingale. $M_t^2 - \int_0^t f_s^2 ds$ is a martingale, for some adapted process f such that $\int_0^T f_s^2 ds < \infty, a.s.,$. Then there exists a Brownian motion B such that

$$M_t = \int_0^t f_s dB_s.$$

Stochastic integral of G-martingales

- Peng [2006] introduced stochastic integrals with respect to G-Brownian motion.
- Xu [2009] introduced stochastic integrals with respect to symmetric G-martingales M, with $\{M_t^2 - t\}_{t\in[0,T]}$ being a G-martingale.

In order to obtain representation of G-martingale, it is necessary to extend the notion of G-stochastic integrals.

Let $p \geq 1$ and $T > 0$. Let $\{A_t, t \in [0,T]\}$ be a continuous and increasing process such that for all $t \in [0,T]$, $A_t \in \mathcal{H}_t$, $A_0 = 0$ and $\hat{\mathbb{E}}[A_T] < \infty$. We first consider the following space of step processes:

$$M_G^{p,0}(0,T) = \Big\{ \eta : \eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j, t_{j+1})}, 0 = t_0 < t_1 < \cdots < t_n = T, \\ \xi_{t_j} \in L_G^p(\mathcal{F}_{t_j}), j = 0, \cdots, n-1, \text{for all } n \geq 1 \Big\},$$

and we define the following norm in $M_G^{p,0}(0,T)$:

$$\| \eta \|_p = \left(\hat{\mathbb{E}}\Big[\int_0^T |\eta_t|^p dA_t \Big] \right)^{\frac{1}{p}} = \left(\hat{\mathbb{E}}\Big[\sum_{j=0}^{n-1} |\xi_{t_j}|^p (A_{t_{j+1}} - A_{t_j}) \Big] \right)^{\frac{1}{p}}.$$

We denote by $M_{G,A}^{p}(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $\|\cdot\|_p$. If $A_t = t$, then we denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $\|\cdot\|_p$.

$$\mathcal{N} = \Big\{ M | M \text{ is a continuous symmetric G-martingale such that } M^2 - A \text{ is a G-supermartingale} \Big\}.$$

Definition

For any $M \in \mathcal{N}$ and $\eta \in M_G^{2,0}(0,T)$ of the form $\eta_t = \sum_{j=0}^{n-1} \xi_{t_j} I_{[t_j, t_{j+1})}(t)$, we define

$$I(\eta) = \int_0^T \eta_t dM_t = \sum_{j=0}^{n-1} \xi_{t_j} (M_{t_{j+1}} - M_{t_j}).$$

Proposition

For all $M \in \mathcal{N}$, the mapping $I : M_G^{2,0}(0,T) \to L_G^2(\mathcal{F}_T)$ is a linear continuous mapping and, thus, can be continuously extended to $I : M_{G,A}^2(0,T) \to L_G^2(\mathcal{F}_T)$. Moreover, for all $\eta \in M_{G,A}^2(0,T)$, the process $\left\{ \int_0^t \eta_s dM_s \right\}_{t\in[0,T]}$ is a symmetric G-martingale and

$$\hat{\mathbb{E}}\Big[\big|\int_0^T \eta_t dM_t\big|^2\Big] \leq \hat{\mathbb{E}}\Big[\int_0^T |\eta_t|^2 dA_t\Big]. \tag{1}$$

For $0 \leq s \leq t \leq T$ and $\eta \in M^2_{G,A}(0,T)$, we denote

$$\int_s^t \eta_u dM_u = \int_0^T I_{[s,t]}(u)\eta_u dM_u.$$

It is now straightforward to see that we have the following properties of the stochastic integral of G-martingales.

Proposition

Let $0 \leq s < r \leq t \leq T$. For all $M \in \mathcal{N}$ and $\theta, \eta \in M^2_{G,A}(0,T)$, we have

(i) $\int_s^t \eta_u dM_u = \int_s^r \eta_u dM_u + \int_r^t \eta_u dM_u$;

(ii) $\int_s^t (\eta_u + \alpha\theta_u) dM_u = \int_s^t \eta_u dM_u + \alpha \int_s^t \theta_u dM_u$, *for all* α *bounded random variable in* $L^p_G(\mathcal{F}_s)$;

(iii) $\hat{\mathbb{E}}[X + \int_r^T \eta_u dM_u | \mathcal{H}_s] = \hat{\mathbb{E}}[X|\mathcal{H}_s]$, *for all* $X \in L^p_G(\mathcal{F})$.

For proving the continuity of the stochastic integral regarded as a process, we need the following Doob inequality for symmetric G-martingale.

Theorem

If X is a right-continuous symmetric G-martingale running over an interval $[0,T]$ of $\mathbb{R}$, then for every $p>1$ such that $X_T \in L_G^p(\mathcal{F})$,

$$\hat{\mathbb{E}}[\sup_{0\le t\le T} |X_t|^p] \le (\frac{p}{p-1})^p \hat{\mathbb{E}}[|X_T|^p].$$

Theorem

For all $M \in \mathcal{N}$ and $\eta \in M^2_{G,A}(0,T)$, there exists a continuous modification of stochastic integral

$$\int_0^t \eta_s dM_s, \quad 0 \leq t \leq T.$$

Now we give the Burkholder-Davis-Gundy inequality for the stochastic integral with respect to G-martingales.

Theorem

For every $q > 0$, there exist a positive constant C_q such that, for all $M \in \mathcal{N}$ and all $\eta \in M^2_{G,A}(0,T)$,

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \Big|\int_0^t \eta_s dM_s\Big|^{2q}\Big] \leq C_q \hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_s^2 dA_s\Big)^q\Big].$$

Assumptions:

- $\hat{\mathbb{E}}[A_T^2] < \infty$
- For all $\{\pi^n\}_{n\geq 1}$ sequence of partitions $\pi^n = \{0 = t_0^n < t_1^n \cdots < t_n^n = T\}$ of $[0, T]$ such that $|\pi^n| \to 0$, as $n \to \infty$, $\hat{\mathbb{E}}[\sum_{i=0}^{n-1}(A_{t_{i+1}^n} - A_{t_i^n})^2] \to 0, n \to \infty$.

Proposition

Let $M \in \mathcal{N}$. Then the quadratic variation of M exists and

$$\langle M \rangle_t = M_t^2 - 2\int_0^t M_s dM_s, \text{ for all } t \geq 0.$$

Remark

The quadratic variation of M is increasing and continuous.

Now we can give another kind of the Burkholder-Davis-Gundy inequalities for the stochastic integral with respect to G-martingales.

Theorem

For every $p > 0$, there exist two positive constants c_p and C_p such that, for all $M \in \mathcal{N}$ and all $\eta \in M^2_{G,A}(0,T)$,

$$c_p \hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_s^2 d\langle M\rangle_s\Big)^p\Big] \leq \hat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \Big|\int_0^t \eta_s dM_s\Big|^{2p}\Big] \leq C_p \hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_s^2 d\langle M\rangle_s\Big)^p\Big].$$

Proposition

For a fixed $T \geq 0$, M is a symmetric G-martingale such that $M^2 - A$ and $-M^2 + \sigma_0^2 A$ be G-martingales. If $f \in M^1_{G,A}(0,T)$, then

$$X_t := \int_0^t f_s d\langle M\rangle_s - 2\int_0^t G(f_s)dA_s,\ t \in [0,T]$$

is a decreasing G-martingale.

Recall $G(\alpha) = \frac{1}{2}(\alpha^+ - \sigma^2\alpha^-), \quad \alpha \in \mathbb{R}$.

Corollary

$\int_0^t f_s d\langle B\rangle_s - 2\int_0^t G(f_s)ds,\ t \in [0,T],$ *is a G-martingale.*

- With respect to a linear expectation, if X is a continuous martingale with finite variation, then X is a constant.
- But it is not true in G-stochastic analysis.

Example

$\langle B\rangle_t - t$ is a continuous G-martingale with finite variation.
But $\langle B\rangle_t - t$ is not a constant. It is a decreasing stochastic process.

Representation theorem of G-martingales

Special case of the martingale representation is the Lévy characterization theorem of Brownian motion.

- Recall: Lévy characterization theorem of Brownian motion.

  With respect to a linear expectation we have

Lemma

A process M is a Brownian motion if

1. *M is continuous and $M_0 = 0$;*
2. *M is a local martingale;*
3. *$M_t^2 - t$ is a local martingale.*

Lévy characterization theorem of G-Brownian motion

Xu [2009] obtained a Lévy characterization theorem for the G-Brownian motion.

Lemma

A process $M \in M_G^2(0,T)$ is a G-Brownian motion with a parameter $0 < \sigma \leq 1$ if

1. M *is continuous and* $M_0 = 0$;
2. M *is a symmetric G-martingale;*
3. *For any* $t \geq 0$, $M_t^2 - t$ *is a G-martingale;*
4. *For any* $t \geq 0$, $\hat{\mathbb{E}}[-M_t^2] = -\sigma^2 t$.

Remark

In our framework, we do not need the assumption $M \in M_G^2(0,T)$.

Main Results —-Representation of G-martingales

The following representation of G-martingales as stochastic integrals with respect to G-Brownian motion is the main result in this section.

Theorem

Let $0 < \sigma \leq 1$ *and* $f \in M_G^2(0,T)$ *be such that* $\hat{\mathbb{E}}[\int_0^T |f_s|^4 ds] < \infty$. *Moreover, if there exists a constant* C *(small enough) such that* $0 < C \leq |f|$ *and the following hold*

1. M *is a symmetric G-martingale and* $M_0 = 0$*;*
2. $M_t^2 - \int_0^t f_s^2 ds$ *and* $-M_t^2 + \sigma^2 \int_0^t f_s^2 ds$ *are G-martingales, for* $t \in [0,T]$*,*

then there exists a G-Brownian motion B *such that* $M_t = \int_0^t f_s dB_s$*, for all* $t \in [0,T]$.

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Thanks for your attention!